Pathria & Beale, Chapter 2

Problem 2.2

(a) The transformation from the Cartesian coordinates (x, y, z) to the spherical coordinates (r, θ, ϕ) is

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

The Jacobian matrix of partial derivatives is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\phi \\ 0 & -r\sin\theta & 0 \end{pmatrix}$$

Its determinant is $r^2 \sin\theta$.

The Cartesian components of the momentum can be expressed as linear combination of its spherical components:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial \dot{r}}\frac{\partial r}{\partial x_i} + \frac{\partial L}{\partial \dot{\theta}}\frac{\partial \theta}{\partial x_i} + \frac{\partial L}{\partial \dot{\phi}}\frac{\partial \phi}{\partial x_i}$$

$$= \frac{\partial r}{\partial x_i} p_r + \frac{\partial \theta}{\partial x_i} p_\theta + \frac{\partial \phi}{\partial x_i} p_\phi$$

The Jacobian matrix of partial derivative is

$$\frac{\partial(p_x, p_y, p_z)}{\partial(p_r, p_\theta, p_\phi)} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{pmatrix}$$

The expression for the spherical coordinate as function of the Cartesian coordinate are

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\frac{\sqrt{x^2+y^2}}{z} \qquad \phi = \arctan\frac{y}{x}$$

The Jacobian matrix is therefore

$$\frac{\partial(p_x, p_y, p_z)}{\partial(p_r, p_\theta, p_\phi)} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{zx}{r^2\sqrt{x^2+y^2}} & \frac{zy}{r^2\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{r^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\phi & \frac{1}{r}\cos\theta\sin\theta & -\frac{1}{r}\sin\theta \\ -\frac{1}{r}\frac{\sin\phi}{\sin\theta} & \frac{1}{r}\frac{\cos\phi}{\sin\theta} & 0 \end{pmatrix}$$

Its determinant is $\frac{1}{r^2\sin\theta}$.

The phase space volume element is therefore

$$d\omega = dx\,dy\,dz\,dp_x\,dp_y\,dp_z$$

$$= \left|\det\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| dr\,d\theta\,d\phi \cdot \left|\det\frac{\partial(p_x,p_y,p_\theta)}{\partial(p_r\,p_\theta\,p_\phi)}\right| dp_r\,dp_\theta\,dp_\phi$$

$$= r^2\sin\theta\,dr\,d\theta\,d\phi \cdot \frac{1}{r^2\sin\theta}dp_r\,dp_\theta\,dp_\phi = dr\,d\theta\,d\phi\,dp_r\,dp_\theta\,dp_\phi$$

(b) The kinetic energy is proportional to the sum of the squares of the Cartesian component of the momentum:

$$p_x^2+p_y^2+p_z^2 = \left(\sin\theta\cos\phi\, p_r + \frac{1}{r}\cos\theta\cos\phi\, p_\theta - \frac{\sin\phi}{r\sin\theta}\, p_\phi\right)^2$$

$$+\left(\sin\theta\sin\phi\, p_r + \frac{1}{r}\cos\theta\sin\phi\, p_\theta + \frac{\cos\phi}{r\sin\theta}\, p_\phi\right)^2$$

$$+\left(\cos\theta\, p_r - \frac{1}{r}\sin\theta\, p_\theta\right)^2$$

$$= p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}\,p_\phi^2$$

The integral of a function of $\vec{p}^{\,2}$ over the spherical components of the momentum is

$$\int_{-\infty}^{\infty} dp_r \int_{-\infty}^{\infty} dp_\theta \int_{-\infty}^{\infty} dp_\phi\, f\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2\right)$$

change variable to $p_1 = p_r$, $p_2 = \frac{p_\theta}{r}$, $p_3 = \frac{p_\phi}{r\sin\theta}$

$$= r^2\sin\theta \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3\, f(p_1^2+p_2^2+p_3^2)$$

change variable to spherical coordinates:

$p_1 = p\sin\alpha\cos\beta$, $p_2 = p\sin\alpha\sin\beta$, $p_3 = p\cos\alpha$

$$= r^2\sin\theta \int_0^{\infty} p^2 dp \int_0^{\pi} \sin\alpha\, d\alpha \int_0^{2\pi} d\beta\, f(p^2)$$

$$= r^2\sin\theta \cdot 4\pi \int_0^{\infty} p^2 dp\, f(p^2)$$

Pathria + Beale, Chapter 2

Problem 2.3

The coordinate for a classical rotator is an angle θ with range from 0 to 2π. The conjugate momentum p_θ has range from $-\infty$ to $+\infty$. The Hamiltonian is

$$H = \frac{1}{2I} p_\theta^2$$

Lines of constant energy have constant values of p_θ and extend in θ from 0 to 2π.

The phase space cells with lowest energy and volume h are from -0 to $+p_1$, where $p_1 \cdot 2\pi = h$, and from $-p_1$ to 0.

The boundaries of the general cell are from p_{n-1} to p_n, where $(p_n - p_{n-1}) 2\pi = h$. The solution for the boundaries of the cells are $p_n = nh/2\pi$.

This is the same as the energy eigenvalue of the quantum rotator.

Pathria + Beale, Chapter 2

Problem 2.7

(i) The energy for a set of N quantum harmonic oscillators with quantum numbers $n_1, n_2, \ldots, n_N$ is

$$E = \sum_{i=1}^{N} (n_i + \tfrac{1}{2}) \hbar\omega$$
$$= \sum_{i=1}^{N} n_i \hbar\omega + N(\tfrac{1}{2}\hbar\omega)$$

We define the sum of the quantum numbers to be $Q = \sum_{i=1}^{N} n_i$. The energy is then $E = (Q + \frac{1}{2}N)\hbar\omega$

The number of ways the energy E can be distributed among N oscillators is the number of ways Q balls can be distributed into N boxes:

$$\Omega = \frac{(Q+N-1)!}{Q!\,(N-1)!}$$

For large Q and N, the logarithm of this number is approximately

$$\log \Omega = \log((Q+N)!) - \log(Q!) - \log(N!)$$

We can obtain an asymptotic expression by using Stirling's formula for $\log(n!)$:

$$\log \Omega = \left[(Q+N)\log(Q+N) - (Q+N)\right]$$

$$- \left[Q \log Q - Q\right] - \left[N \log N - N\right]$$

$$= Q \log \frac{Q+N}{Q} + N \log \frac{Q+N}{N}$$

If we take the limit $Q >> N$, this reduces to

$$\log \Omega = Q \log\left(1+\frac{N}{Q}\right) + N\left[\log \frac{Q}{N} + \log\left(1+\frac{N}{Q}\right)\right]$$

$$= Q\left[\frac{N}{Q} + \cdots\right] + N\left[\log \frac{Q}{N} + \frac{N}{Q} + \cdots\right]$$

$$\simeq N\left[\log \frac{Q}{N} + 1\right]$$

(ii) The Hamiltonian for N classical harmonic oscillators is

$$H = \sum_{n=1}^{N} \left(\frac{1}{2m} p_n^2 + \frac{1}{2} m\omega^2 x_n^2\right)$$

The volume of phase space for the region $H \leq E$ is

$$\omega = \int dx_1 dp_1 \int dx_2 dp_2 \cdots \int dx_n dp_n \, \Theta(E-H)$$

It is convenient to introduce rescaled coordinates and momenta

$$\hat{x}_n = \sqrt{\tfrac{1}{2} m\omega^2}\, x_n \qquad \hat{p}_n = \sqrt{\tfrac{1}{2m}}\, p_n$$

The differential phase space volume for one oscillator is

$$dx_n dp_n = \left(\frac{1}{\sqrt{\frac{1}{2}m\omega^2}} d\hat{x}_n\right)\left(\sqrt{2m}\, d\hat{p}_n\right)$$

$$= \frac{2}{\omega} d\hat{x}_n d\hat{p}_n$$

The phase space volume is then

$$\omega = \left(\frac{2}{\omega}\right)^N \int d\hat{x}_1 d\hat{p}_1 \int d\hat{x}_2 d\hat{p}_2 \cdots \int d\hat{x}_N d\hat{p}_N \, \theta\left(E - \sum_{n=1}^{N}(\hat{p}_n^2 + \hat{x}_n^2)\right)$$

The integral is the volume of a 2N-dimensional ball of radius $\sqrt{E}$

$$\omega = \left(\frac{2}{\omega}\right)^N \frac{\pi^N}{N!}\left(\sqrt{E}\right)^{2N}$$

$$= \frac{1}{N!}\left(\frac{2\pi E}{\omega}\right)^N$$

The phase-space volume for H within a narrow band of width Δ around E is

$$\omega' = \frac{d}{dE}\left(\frac{1}{N!}\left(\frac{2\pi E}{\omega}\right)^N\right)\Delta$$

$$= \frac{1}{N!} N \left(\frac{2\pi E}{\omega}\right)^{N-1} \frac{2\pi\Delta}{\omega}$$

Its logarithm for large Q and N is approximately

$$\log \omega' = (N-1)\log\frac{2\pi E}{\omega} - \log\left((N-1)!\right) + \log\frac{2\pi\Delta}{\omega}$$

$$\simeq N \log \frac{2\pi E}{\omega} - (N \log N - N)$$

$$= N \left[\log \frac{2\pi E}{N\omega} + 1 \right]$$

If we divide the phase space ω' by h^N, the logarithm becomes

$$\log \frac{\omega'}{h^N} = N \left[\log \frac{2\pi E}{Nh\omega} + 1 \right]$$

This agrees with the result for the quantum oscillators if we ignore the zero-point energies and express the total number of quanta as $Q = \frac{E}{\hbar\omega}$, where $\hbar = h/2\pi$.

Pathria + Beale, Chapter 2

Problem 2.9(a)

The Hamiltonian for a gas of N massless particle in 1 dimension with length L is

$$H = \sum_{n=1}^{\infty} |p_n| c$$

The phase space volume with $H < E$ is

$$\omega = \int dx_1\, dp_1 \int dx_2\, dp_2 \cdots \int dx_N dp_N\, \Theta(E-H)$$

$$= L^N \int dp_1 \int dp_2 \cdots \int dp_N\, \Theta\left(E/c - \sum_{n=1}^{N} |p_n|\right)$$

The momentum integrals

$$\int_{-E/c}^{+E/c} dp_1 \int_{-(E/c-|p_1|)}^{+(E/c-|p_1|)} dp_2 \cdots \int_{-(E/c-\sum_{n=1}^{N-2}|p_n|)}^{+(E/c-\sum_{n=1}^{N-2}|p_n|)} dp_{N-1} \underbrace{\int_{-(E/c-\sum_{n=1}^{N-1}|p_n|)}^{+(E/c-\sum_{n=1}^{N-1}|p_n|)} dp_N}_{2\left(\frac{E}{c} - \sum_{n=1}^{N-1}|p_n|\right)}$$

$$= \int_{-E/c}^{+E/c} dp_1 \int_{-(E/c-|p_1|)}^{+(E/c-|p_1|)} dp_2 \cdots \underbrace{\int_{-(E/c-\sum_{n=1}^{N-2}|p_n|)}^{+(E/c-\sum_{n=1}^{N-2}|p_n|)} dp_{N-1}\; 2\left(\frac{E}{c} - \sum_{n=1}^{N-2}|p_n| - |p_{N-1}|\right)}_{2\left[\left(\frac{E}{c} - \sum_{n=1}^{N-2}|p_n|\right) \cdot 2\left(\frac{E}{c} - \sum_{n=1}^{N-2}|p_n|\right) - \left(\frac{E}{c} - \sum_{n=1}^{N-2}|p_n|\right)^2\right]}$$

$$= \int_{-E/c}^{+E/c} dp_1 \int_{-(E/c-|p_1|)}^{+(E/c-|p_1|)} dp_2 \cdots \underbrace{\int_{-\left(\frac{E}{c}-\sum_{n=1}^{N-3}|p_n|\right)}^{+\left(\frac{E}{c}-\sum_{n=1}^{N-3}|p_n|\right)} dp_{N-2}\ 2\left(\frac{E}{c}-\sum_{n=1}^{N-3}|p_n| - |p_n|\right)^2}_{2\cdot 2\,\frac{\left(E/c-\sum_{n=1}^{N-3}|p_n|\right)^3}{3}}$$

$$= 2^N \frac{(E/c)^N}{N!}$$

The phase space volume is then

$$\omega = L^N \cdot 2^N \frac{(E/c)^N}{N!}$$

$$= \frac{1}{N!}\left(\frac{2LE}{c}\right)^N$$

In the corresponding quantum problem, we consider wavefunctions that vanish at $x=0$ and $x=L$: $\psi_n(x) = \sin(n\pi x/L)$, $n=1,2,3,\ldots$ The corresponding energy eigenvalue is $p_n c = (n\pi\hbar/L)c$.

The total energy for N particle is

$$E = \frac{\pi\hbar c}{L}\sum_{i=1}^{N} n_i$$

The sum of the quantum numbers is

$$Q = \sum_{i=1}^{N} n_i = \frac{LE}{\pi\hbar c}$$

The number of ways of distributing Q quanta among the N particle is

$$\Omega = \frac{(Q+N-1)!}{Q!\,(N-1)!}$$

For $Q \gg N \gg 1$, its logarithm can be approximated by the logarithm of

$$\frac{1}{N!}\,Q^N = \frac{1}{N!}\left(\frac{LE}{\pi \hbar c}\right)^N$$

This agrees with $\frac{\omega}{h^N}$ if $h = 2\pi\hbar$.