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Problem 3.24

The Lagrangian for a relativistic particle is

$$ L(\dot{\mathbf{r}}) = -mc^2 \sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}} $$

The momentum conjugate to $r_i$ is

$$ p_i = \frac{\delta L}{\delta \dot{r}_i} = \frac{m \dot{r}_i}{\sqrt{1 + \dot{r}_i^2/c^2}} $$

This can be inverted to give $\dot{r}_i$ as a function of $p_i$.

$$ \dot{r}_i = \frac{p_i/m}{\sqrt{1 + p_i^2/m^2 c^2}} $$

The Hamiltonian is

$$ H = \sum_{i=1}^{3} p_i \dot{r}_i - L $$

$$ = mc^2 \sqrt{1 + \dot{\mathbf{r}}^2/m^2 c^2} $$

The derivative with respect to $p_i$ is

$$ \frac{\delta H}{\delta p_i} = \frac{p_i/m}{\sqrt{1 + p_i^2/m^2 c^2}} $$
The equipartition theorem implies

\[ \langle \frac{\partial H}{\partial p_i} \rangle = S_{ij} kT \]

Contracting with \( S_{ij} \) we get

\[ \langle \sum_i \frac{p_i^2}{2m} \rangle = 3kT \]

\[ \langle \frac{p_i}{\sqrt{1 + p_i^2/m^2}} \rangle = 3kT \]

Setting \( \beta^2 = \frac{m^2 - u^2}{1 - u^2/c^2} \), where \( u = v/c \) is the speed, the

reduces to

\[ \langle \frac{mu^2}{\sqrt{1 - u^2/c^2}} \rangle = 3kT \]

The mean energy of a particle is

\[ \langle H \rangle = \langle mc^2 \sqrt{1 + \beta^2/m^2} \rangle \]

In the ultrarelativistic limit \( m \to 0 \), the reduce to

\[ \langle H \rangle \to \langle \beta^2 c \rangle \]

The ultrarelativistic limit of the equipartition theorem is

\[ \langle \beta^2 c \rangle \to 3kT \]

By comparison, we find \( \langle H \rangle \to 3kT \), which is twice the non-relativistic value.
The energy levels of a Fermi oscillator are $E_n = n\varepsilon, n = 0, 1$. The single-oscillator partition function is

$$Z_1 = \sum_{n=0}^{\infty} e^{-\beta E_n} = 1 + e^{-\beta\varepsilon}$$

The $N$-oscillator partition function is

$$Z_N = Z_1^N = (1 + e^{-\beta\varepsilon})^N$$

The Helmholtz free energy is

$$F = -\frac{1}{\beta} \log Z_N = -\frac{1}{\beta} \cdot N \log (1 + e^{-\beta\varepsilon})$$

The average energy is

$$U = -\frac{2}{\beta} \log Z_N = -\frac{2}{\beta} \cdot N \log (1 + e^{-\beta\varepsilon})$$
\[ = N \frac{e^{\beta e}}{1+e^{-\beta e}} \]

The thermodynamic relation for \( F \) is:

\[ dF = -SdT + \mu dN \]

The entropy is:

\[ S = -\frac{\partial F}{\partial \beta N} = \beta^2 \left( \frac{\partial F}{\partial \beta} \right)_N \]

\[ = \beta^2 (-N) \frac{\partial}{\partial \beta} \left( \frac{1}{\beta} \log (1+e^{-\beta e}) \right) \]

\[ = -N \beta^2 \left[ -\frac{1}{\beta^2} \log (1+e^{-\beta e}) + \frac{1}{\beta} \frac{1}{1+e^{-\beta e}} (-e) \right] \]

\[ = N \left[ \log (1+e^{-\beta e}) + \frac{e}{1+e^{-\beta e}} \right] \]

The chemical potential is:

\[ \mu = \left( \frac{\partial F}{\partial N} \right)_T \]

\[ = -\frac{1}{\beta} \log (1+e^{-\beta e}) \]

The specific heat is:

\[ C = T \left( \frac{\partial S}{\partial T} \right)_N = \left( \frac{\partial U}{\partial T} \right)_N = -\beta^2 \left( \frac{\partial U}{\partial \beta} \right)_N \]

\[ = -\beta^2 \cdot N \left( -\frac{e}{(1+e^{-\beta e})^2} \frac{e^{-\beta e}}{e^{-\beta e}} (-\beta e) \right) = -N \frac{(\beta e)^2 e^{-\beta e}}{(1+e^{-\beta e})^2} \]
Problem 3.41

A system of $N$ spins with a negative temperature $T$ has a positive energy given by

$$E = -Nc \tanh \frac{c}{kT}$$

An ideal gas consisting of $N'$ atoms at temperature $T'$ has energy

$$E' = \frac{3}{2} N' kT'$$

If the two systems are in thermal contact, their total energy $E + E'$ will be conserved. They will come to equilibrium at a temperature $T_{eq}$ that satisfies

$$-Nc \tanh \frac{c}{kT} + \frac{3}{2} N' kT' = -Nc \tanh \frac{c}{kT_{eq}} + \frac{3}{2} N' kT_{eq}$$

$$Nc \left( -\tanh \frac{c}{kT} + \tanh \frac{c}{kT_{eq}} \right) = \frac{3}{2} N' k \left( T_{eq} - T' \right)$$

The equilibrium temperature $T_{eq}$ will be positive and greater than $T'$. The difference $T_{eq} - T'$ will be between 0 and $\left( \frac{cE}{2k} \right) N/N'$. 

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Problem 3.43

A constant magnetic field $\vec{H} = H \hat{z}$ requires a vector potential $\vec{A}(\vec{r})$ that satisfies $\vec{H} = \nabla \times \vec{A}$.

One possible choice is

$$\vec{A}(\vec{r}) = \frac{1}{2} H \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

The Hamiltonian for a system of charged particles is

$$H = \sum_{n=1}^{N} \frac{\vec{p}_n}{2m} \left( \vec{p}_n + \frac{e}{\beta} \vec{A}(\vec{r}_n) \right)^2$$

The partition function for $N$ particles is

$$Z_N = \frac{1}{N!} Z$$

The partition function for 1 particle is

$$Z_1 = \frac{1}{(2\pi \hbar^2)^3} \int d^3 r \int d^3 p \exp \left( -\beta \frac{1}{2m} (\vec{p} + \frac{e}{\beta} \vec{A}(\vec{r}))^2 \right)$$

After shifting the momentum integral by $\vec{p} \to \vec{p} - \frac{e}{\beta} \vec{A}(\vec{r})$, we obtain a result independent of $H$:

$$Z_1 = \frac{1}{(2\pi \hbar^2)^3} \int d^3 r \int d^3 p \exp \left( -\beta \frac{1}{2m} \vec{p}^2 \right)$$