Pathria + Beale, Chapter 3

Problem 3.24

The Lagrangean for a relativistic particle is

$$L(\dot{\vec{r}}) = -mc^2\sqrt{1-\dot{\vec{r}}^2/c^2}$$

The momentum conjugate to r_i is

$$p_i = \frac{\partial L}{\partial \dot{r}_i} = \frac{m\dot{r}_i}{\sqrt{1-\dot{\vec{r}}^2/c^2}}$$

This can be inverted to give $\dot{\vec{r}}$ as a function of $\vec{p}$:

$$\dot{r}_i = \frac{p_i/m}{\sqrt{1+\vec{p}^2/m^2c^2}}$$

The Hamiltonian is

$$H = \sum_{i=1}^{3} p_i\dot{r}_i - L$$

$$= mc^2\sqrt{1+\vec{p}^2/m^2c^2}$$

The derivative with respect to p_i is

$$\frac{\partial H}{\partial p_i} = \frac{p_i/m}{\sqrt{1+\vec{p}^2/m^2c^2}}$$

The equipartition theorem implies

$$\left\langle p_i \frac{\partial H}{\partial p_j} \right\rangle = \delta_{ij}\, kT$$

Contracting with δ_{ij}, we get

$$\left\langle \sum_{i=1}^{3} p_i \frac{p_i/m}{\sqrt{1+\vec{p}^2/m^2c^2}} \right\rangle = 3kT$$

$$\left\langle \frac{\vec{p}^2/m}{\sqrt{1+\vec{p}^2/m^2c^2}} \right\rangle = 3kT$$

Setting $\vec{p}^2 = \frac{m^2u^2}{1-u^2/c^2}$, where $u = |\dot{\vec{r}}|$ is the speed, the reduces to

$$\left\langle \frac{mu^2}{\sqrt{1-u^2/c^2}} \right\rangle = 3kT$$

The mean energy of a particle is

$$\langle H \rangle = \left\langle mc^2 \sqrt{1+\vec{p}^2/m^2c^2} \right\rangle$$

In the ultrarelativistic limit $m \to 0$, the reduces to

$$\langle H \rangle \longrightarrow \langle |\vec{p}|c \rangle$$

The ultrarelativistic limit of the equipartition theorem is

$$\langle |\vec{p}|c \rangle \longrightarrow 3kT$$

By comparison, we find $\langle H \rangle \to 3kT$, which is twice the nonrelativistic value.

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Problem 3.31

The energy levels of a Fermi oscillator are $\varepsilon_n = n\varepsilon$, $n = 0, 1$.
The single-oscillator partition function is

$$Z_1 = \sum_{n=0}^{1} e^{-\beta\epsilon_n}$$

$$= 1 + e^{-\beta\varepsilon}$$

The N-oscillator partition function is

$$Z_N = Z_1^{N}$$

$$= \left(1 + e^{-\beta\varepsilon}\right)^{N}$$

The Helmholtz free energy is

$$F = -\frac{1}{\beta}\log Z_N$$

$$= -\frac{1}{\beta}\cdot N\log\left(1 + e^{-\beta\varepsilon}\right)$$

The average energy is

$$U = -\frac{\partial}{\partial\beta}\log Z_N$$

$$= -\frac{\partial}{\partial\beta}N\log\left(1 + e^{-\beta\varepsilon}\right)$$

$$= N \frac{\epsilon}{1+e^{-\beta\epsilon}}$$

The thermodynamic relation for F is

$$dF = -S\,dT + \mu\,dN$$

The entropy is

$$S = -\left(\frac{\partial F}{\partial T}\right)_N = \beta^2\left(\frac{\partial F}{\partial \beta}\right)_N$$

$$= \beta^2(-N)\frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\log\left(1+e^{-\beta\epsilon}\right)\right)$$

$$= -N\beta^2\left[-\frac{1}{\beta^2}\log\left(1+e^{-\beta\epsilon}\right) + \frac{1}{\beta}\frac{1}{1+e^{-\beta\epsilon}}(-\epsilon)\right]$$

$$= N\left[\log\left(1+e^{-\beta\epsilon}\right) + \frac{\beta\epsilon}{1+e^{-\beta\epsilon}}\right]$$

The chemical potential is

$$\mu = \left(\frac{\partial F}{\partial N}\right)_T$$

$$= -\frac{1}{\beta}\log\left(1+e^{-\beta\epsilon}\right)$$

The specific heat is

$$C = T\left(\frac{\partial S}{\partial T}\right)_N = \left(\frac{\partial U}{\partial T}\right)_N = -\beta^2\left(\frac{\partial U}{\partial \beta}\right)_N$$

$$= -\beta^2\cdot N\left(-\frac{\epsilon}{(1+e^{-\beta\epsilon})^2}e^{-\beta\epsilon}(-\epsilon)\right) = -N\frac{(\beta\epsilon)^2 e^{-\beta\epsilon}}{(1+e^{-\beta\epsilon})^2}$$

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Problem 3.41

A system of N spins with a negative temperature T has a positive energy given by

$$E = -N\varepsilon \tanh \frac{\varepsilon}{kT}$$

An ideal gas consisting of N' atoms at temperature T' has energy

$$E' = \frac{3}{2} N' k T'$$

If the two systems are in thermal contact, then total energy E + E' will be conserved. They will come to equilibrium at a temperature T_{eq} that satisfies

$$-N\varepsilon \tanh \frac{\varepsilon}{kT} + \frac{3}{2} N' k T' = -N\varepsilon \tanh \frac{\varepsilon}{kT_{eq}} + \frac{3}{2} N' k T_{eq}$$

$$N\varepsilon \left(-\tanh \frac{\varepsilon}{kT} + \tanh \frac{\varepsilon}{kT_{eq}} \right) = \frac{3}{2} N' k \left(T_{eq} - T' \right)$$

The equilibrium temperature T_{eq} will be positive and greater than T'. The difference $T_{eq} - T'$ will be between 0 and $\left(\frac{4\varepsilon}{3k}\right) N/N'$.

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Problem 3.43

A constant magnetic field $\vec{H} = H\hat{z}$ requires a vector potential $\vec{A}(\vec{r})$ that satisfies $\vec{H} = \nabla \times \vec{A}$. One possible choice is

$$\vec{A}(\vec{r}) = \tfrac{1}{2} H \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

The Hamiltonian for a system of charged particles is

$$H = \sum_{n=1}^{N} \frac{1}{2m}\left(\vec{p}_n + \frac{e}{c}\vec{A}(\vec{r}_n)\right)^2$$

The partition function for N particles is

$$Z_N = \frac{1}{N!} Z_1^N$$

The partition function for 1 particle is

$$Z_1 = \frac{1}{(2\pi\hbar)^3}\int d^3r \int d^3p \, \exp\left(-\beta \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}(\vec{r})\right)^2\right)$$

After shifting the momentum integral by $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}(\vec{r})$, we obtain a result independent of H:

$$Z_1 = \frac{1}{(2\pi\hbar)^3}\int d^3r \int d^3p \, \exp\left(-\beta \frac{1}{2m}\vec{p}^2\right)$$