Dimensional Analysis

1. Using dimensional analysis arguments, determine how an animal’s ability to jump to height h depends on it’s length L.

2. This question is as far removed from classical mechanics as possible, but it does give a very important example of the use of dimensional analysis in physics. The three fundamental constants of nature: Newton’s gravitational constant $G$, the speed of light $c$ and Planck’s constant $h$. Use dimensional analysis to find the fundamental units of length $L_p$, mass $M_p$ and time $T_p$ in terms of $G$, $c$ and $h$. These are called the Planck length, mass and time, after Max Planck who first pointed them out. Make numerical estimates of the three Planck scales in SI units.

Calculus of Variations

3. Consider the problem of a light rays traveling through the atmosphere. The atmosphere has a continuously varying index of refraction which depends on
height above the earth. For small distances* above earth we model this as \( n(y) = n_0 - ay^* \).

In this problem you will find the path of light traveling in a two dimensional plane in this system.

a) First set up the action integral for the system as the travel time of light along this path between two points P1 and P2.

b) Write the Euler-Lagrange equations for the system.

c) Solve the equations to find the path \( y(x) \) that minimizes this time.

*note there is a physical limit on the system as the refractive index of light in vacuum must be 1. However our approximation will be valid as long as the parameter \( a \) is small and the height \( y \) is not extremely large.

4. A paraboloid is a surface of rotation obtained by revolving a parabola. The points on such a paraboloid surface satisfy

\[ z = \alpha (x^2 + y^2) \]

a) find the line element \( ds \) on this space

b) Write the Euler-Lagrange equations for the geodesics. One equation will be quite simple, use this fact to collapse the Euler-Lagrange equations into a single equation

Hint: convert to cylindrical coordinates

\( c \) Now look at the special case of path along fixed angular coordinate. Using the Euler-Lagrange equation, show that these "radial" paths are geodesics.

5. Find the Euler-Lagrange equation describing a brachistochrone curve for a particle moving inside of the Earth, assuming that the earth has uniform density. Show that the desired curve is a hypocycloid (the curve described by a point on a circle rolling on the inside of a larger circle). Obtain an expression for the time of travel along the brachistochrone between 2 points on the Earth’s surface. How long would it take to go from New York to Los Angeles (assumed to be 4800 km apart) and how far below the surface would the deepest point of the tunnel be?

Hint: in solving for the brachistochrone, the conservation of energy (the first integral) may be useful.