Exponential Functionals of Integrated Processes

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Abstract

This paper derives a limit distribution result involving exponential functionals of integrated processes. This implies the availability of an additional class of functions for which the limit behavior of the average of a function of an integrated process is well established.

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1 Introduction

Let \( x_t \) denote an integrated process that satisfies

\[
n^{-1/2} x_{[nr]} \Rightarrow \Omega^{1/2} B(r),
\]

where \( x_t \in \mathbb{R}^k \), \([nr]\) denotes the largest integer not exceeding \( nr \), “\( \Rightarrow \)” signifies weak convergence, \( B(\cdot) \) denotes a \( k \)-vector standard Brownian motion on \([0, 1]\), and \( \Omega \) is some positive definite \( k \times k \) matrix. Let \( X(r) = \Omega^{1/2} B(r) \); that is, \( X(\cdot) \) is a \( k \)-vector Brownian motion with variance matrix \( \Omega \); \( \Omega \) has only an unimportant scaling effect in our analysis below. It is well-known that objects of the form \( n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) \) satisfy

\[
n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) \xrightarrow{d} \int_{0}^{1} T(X(r))dr;
\]

in the case where \( T(\cdot) \) is continuous on \( \mathbb{R}^k \), this follows from the Continuous Mapping Theorem. De Jong (2004) and Pötscher (2004) have explored the limits of such results in

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settings where \( T(\cdot) \) is no longer continuous. For objects of the form

\[
c_n n^{-1} \sum_{t=1}^{n} T(x_t)
\]

for some rate \( c_n \), i.e., the case where no scaling with \( n^{1/2} \) has been applied, several results are known from the work of Park and Philips (1999) and de Jong and Wang (2005). In the case where \( T(\cdot) \) is absolutely integrable, under some regularity conditions

\[
n^{-1/2} \sum_{t=1}^{n} T(x_t) \xrightarrow{d} L(1,0) \int_{-\infty}^{\infty} T(s)ds,
\]

where \( L(1,0) \) denotes Brownian local time. Also, if \( T(\lambda x) \approx \kappa(\lambda) h(x) \) in the sense that is made precise in Park and Phillips (1999) or de Jong and Wang (2005), we have

\[
\kappa(n^{1/2})^{-1} n^{-1} \sum_{t=1}^{n} T(x_t) \xrightarrow{d} \int_{0}^{1} h(X(r))dr.
\]

These results are the basis for a number of papers, of which Park and Phillips (2000) and Park and Phillips (2001) are probably the most impressive. For the case of the exponential function \( T(x) = \exp(x) \) or similar functions however, the only result that is known in the literature is that on page 278 of Park and Phillips (1999), and using the observation that \( L(1, s_{\text{max}}) = 0 \), it follows that the limit established there is degenerate.

In this note, we will derive a limit characterization for averages of exponential-type functions of integrated processes.

## 2 Main result

We consider the asymptotic behavior of the average of exponential functionals of a possibly nonlinear transformation of an integrated time series \( x_t \). Let \( f(\cdot) : \mathbb{R}^k \rightarrow \mathbb{R} \) be a Borel measurable transformation of the integrated process. The following lemma is key to our investigation.

**Lemma 1** Let \( a_n \) be a scaling factor such that \( a_n \log n \to 0 \) as \( n \to \infty \). If

\[
a_n \max_{1 \leq t \leq n} f(x_t) \xrightarrow{d} Z
\]

as \( n \to \infty \) for some random variable \( Z \), then

\[
a_n \log \left( \sum_{t=1}^{n} \exp\left(f(x_t)\right) \right) \xrightarrow{d} Z.
\]
The proofs of this paper can be found in Appendix. Note that the summation can also be replaced by the average in the above lemma, given the assumption that $a_n \log n \to 0$.

Next, we set forth conditions on $f(\cdot)$ that guarantee the convergence condition of Equation (1).

**Assumption 1** The function $f(\cdot)$ satisfies

$$f(\lambda x) = \kappa(\lambda)h(x) + r(x, \lambda),$$

where

(a) $h(\cdot)$ is continuous on $\mathbb{R}^k$, or $x_t \in \mathbb{R}$ and $h(\cdot)$ is monotone$^1$ on $\mathbb{R}$;

(b) Either

1. $|r(x, \lambda)| \leq c\nu(\lambda)g(x)$ for all $\lambda$ sufficiently large and for all $x$ over any compact set $C$, where $c$ is a constant which may depend on $C$, $\nu(\lambda)/\kappa(\lambda) \to 0$ as $\lambda \to \infty$, and $g(\cdot)$ is bounded on $C$; or

2. $|r(x, \lambda)| \leq c\nu(\lambda)g(\lambda x)$, $\nu(\lambda)/\kappa(\lambda) \to 0$ as $\lambda \to \infty$, and $\sup_{x \in \mathbb{R}} |g(x)| < \infty$.

For a function $f(\cdot)$ that satisfies Assumption 1, we may write $f(x_t)$ as

$$f(x_t) = \kappa(n^{1/2})h(n^{-1/2}x_t) + r(n^{-1/2}x_t, n^{1/2}),$$

where $n^{1/2}$ corresponds to $\lambda$ in (2) of Assumption 1. Obviously, for $f(x) = x$, we can set $\kappa(\lambda) = \lambda$, $h(x) = x$, and $r(x, \lambda) = 0$. Assumption 1.(b) is similar to the “asymptotically homogenous” assumption of Park and Phillips (1999).

Our main result now is the following:

**Theorem 1** Let $f(\cdot)$ given by (3) satisfy Assumption 1 and assume that $n^{-1/2}x_{[nr]} \Rightarrow X(r)$, where $x_t \in \mathbb{R}^k$. Then if $\log n/\kappa(n^{1/2}) \to 0$ as $n \to \infty$, we have

$$\frac{1}{\kappa(n^{1/2})} \log \left( \sum_{i=1}^{n} \exp(f(x_t)) \right) \xrightarrow{d} \sup_{r \in [0,1]} h(X(r)).$$

Note that analogous results to the one above can also be derived in cases where a rescaled version of $x_{[nr]}$ converges weakly to a limit process different from Brownian motion. For example, if $x_t$ is demeaned and $n^{-1/2}x_{[nr]}$ converges to Brownian bridge, or if a rescaled version of $x_{[nr]}$ converges weakly to a Lévy process or a fractional Brownian motion process, deriving a result analogous to Theorem 1 should be straightforward, following the line of proof in this paper.

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$^1$A function is said to be monotone if it is nondecreasing or nonincreasing.
3 Examples and discussion

3.1 The exponential function

Setting \( f(x) = x \), it follows from our results that

\[
    n^{-1/2} \log \left( \sum_{t=1}^{n} \exp(x_{t}) \right) \xrightarrow{d} \sup_{r \in [0,1]} X(r).
\]

This result seems to be unknown up to this point.

3.2 An exponential-type functional

Consider

\[
    f(x) = x \Phi(x),
\]

where \( \Phi(\cdot) \) stands for a distribution function. We can write

\[
    f(\lambda x) = \lambda x I(x > 0) + \lambda x (\Phi(\lambda x) - I(\lambda x > 0)),
\]

implying that we can set \( \kappa(\lambda) = \lambda \) and \( h(x) = xI(x > 0) \) and that

\[
    |r(x, \lambda)| \leq g(\lambda x)
\]

where

\[
    g(x) = |x| |\Phi(x) - I(x > 0)|.
\]

Since \( xI(x > 0) \) is nondecreasing and \( g(\cdot) \) is bounded, (b)-2 of Assumption 1 holds. Therefore, by Theorem 1

\[
    n^{-1/2} \log \left( \sum_{t=1}^{n} \exp(x_{t} \Phi(x_{t})) \right) \xrightarrow{d} \sup_{r \in [0,1]} X(r) I(X(r) > 0).
\]

3.3 Discussion

While the above result opens up a new function class for which the limit behavior of sums of functions of an integrated process is well-established, it is unfortunately not powerful enough to fully characterize the behavior of unit root tests if logarithms of the data have been taken while this should not have been done. For example, suppose the data-generating process for \( x_{t} \) follows a simple random walk and consider the case where a unit root test is based on a regression of \( \exp(x_{t}) \) on \( \exp(x_{t-1}) \). Along the lines of Theorem 1 it is possible to show that

\[
    n^{-1/2} \log \hat{\rho} \xrightarrow{p} 0
\]
and 

\[ n^{-1/2} \log \hat{t} \overset{p}{\to} 0, \]

where \( \hat{\rho} \) denotes the least squares coefficient of the linear regression of \( \exp(x_t) \) on \( \exp(x_{t-1}) \), and \( \hat{t} \) denotes the corresponding \( t \)-type Dickey-Fuller test statistic. However, since it does not appear possible to derive the exact rate and a nondegenerate limit distribution in either case using Theorem 1, the above result might be more of theoretical interest than of practical importance.

**Appendix: Mathematical Proofs**

**Proof of Lemma 1:**

Observe first that we can write 

\[ a_n \log \left( \sum_{t=1}^{n} \exp(f(x_t)) \right) = a_n \max_{1 \leq t \leq n} f(x_t) + a_n \log \left( \sum_{t=1}^{n} \exp(f(x_t) - \max_{1 \leq t \leq n} f(x_t)) \right). \]

But also observe that by construction, because \( f(x_t) \leq \max_{1 \leq t \leq n} f(x_t) \) and because the equality holds for at least one value of \( t \),

\[ 1 \leq \sum_{t=1}^{n} \exp(f(x_t) - \max_{1 \leq t \leq n} f(x_t)) \leq n, \]

implying that

\[ 0 \leq a_n \log \left( \sum_{t=1}^{n} \exp(f(x_t) - \max_{1 \leq t \leq n} f(x_t)) \right) \leq a_n \log n. \]

Since it is assumed that \( a_n \log n = o(1) \), it now follows that 

\[ a_n \log \left( \sum_{t=1}^{n} \exp(f(x_t)) \right) = o_p(1) + a_n \max_{1 \leq t \leq n} f(x_t) \overset{d}{\to} Z. \]

**Proof of Theorem 1:**

We will apply Lemma 1 and set \( a_n = 1/\kappa(n^{1/2}) \). Clearly, \( a_n \log n \to 0 \) by assumption. Let \( \kappa(n^{1/2}) \) and \( r(n^{-1/2}x_t, n^{1/2}) \) be as defined in (3). Then

\[ \frac{1}{\kappa(n^{1/2})} \max_{1 \leq t \leq n} r(n^{-1/2}x_t, n^{1/2}) \overset{p}{\to} 0. \]
under part (b) of Assumption 1. This is because (b)-1 implies that
\[
\frac{1}{\kappa(n^{1/2})} \max_{1 \leq t \leq n} r(n^{-1/2} x_t, n^{1/2}) \leq c(\nu(n^{1/2})/\kappa(n^{1/2})) \max_{1 \leq t \leq n} g(n^{-1/2} x_t) \xrightarrow{p} 0
\]
since \( \nu(\lambda)/\kappa(\lambda) \to 0 \) as \( \lambda \to \infty \) by assumption and \( \max_{1 \leq t \leq n} g(n^{-1/2} x_t) = O_p(1) \) because \( g(\cdot) \) is bounded on any compact set, while (b)-2 implies
\[
\frac{1}{\kappa(n^{1/2})} \max_{1 \leq t \leq n} r(n^{-1/2} x_t, n^{1/2}) \leq c(\nu(n^{1/2})/\kappa(n^{1/2})) \sup_{x \in \mathbb{R}} |g(x)| \to 0.
\]

Now the desired result immediately follows from the result of Lemma 1 since
\[
\frac{1}{\kappa(n^{1/2})} \max_{1 \leq t \leq n} f(x_t) = \max_{1 \leq t \leq n} \left( h(n^{-1/2} x_t) + r(n^{-1/2} x_t, n^{1/2})/\kappa(n^{1/2}) \right) \xrightarrow{d} \sup_{r \in [0,1]} h(X(r)),
\]
where the asserted convergence in distribution follows immediately if \( h(\cdot) \) is continuous. For the case where \( h(\cdot) \) is monotone, we only discuss the case where \( h(\cdot) \) in nondecreasing since the other case is analogous. We can assume without loss of generality that \( x_{\lfloor nr \rfloor} \) is a Skorokhod version satisfying
\[
\sup_{r \in [0,1]} \left| n^{-1/2} x_{\lfloor nr \rfloor} - X(r) \right| \xrightarrow{a.s.} 0.
\]

Then for any \( \delta > 0 \), for \( n \) large enough,
\[
\max_{1 \leq t \leq n} h(n^{-1/2} x_t) = h(\max_{1 \leq t \leq n} n^{-1/2} x_t) \in \left[ h\left( \sup_{r \in [0,1]} X(r) - \delta \right), h\left( \sup_{r \in [0,1]} X(r) + \delta \right) \right] \text{ a.s.}
\]

Because \( \sup_{r \in [0,1]} |X(r)| \leq K \) with arbitrarily large probability for large \( K \), it suffices to show that for all \( K > 0 \),
\[
E \left[ \left( h\left( \sup_{r \in [0,1]} X(r) + \delta \right) - h\left( \sup_{r \in [0,1]} X(r) - \delta \right) \right) I\left( \sup_{r \in [0,1]} X(r) \leq K \right) \right] \to 0.
\]

This however follows easily from the continuity of the density of \( \sup_{r \in [0,1]} X(r) \), which completes the proof.

\section*{References}


