Sums of exponentials of random walks

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Abstract

This paper shows that the sum of the exponential of an oscillating random walk converges in distribution, after rescaling by the exponential of the maximum value of the random walk.

1 Introduction

This paper shows the convergence in distribution of statistics of the form

$$R_n = \sum_{t=1}^{n} f(S_t - M_n)$$

where S_t, $t = 1, \ldots, n$ is an oscillating random walk, $S_0 = 0$, $M_n = \max(0, \max_{1 \leq t \leq n} S_t)$, and $f : (-\infty, 0] \rightarrow [0, \infty)$ is a Borel measurable function such that $|f(x)| \leq C|x + 1|^{-2-\varepsilon}$ for all $x \leq 0$ and some $\varepsilon > 0$. It follows from an slight modification of an argument in Davies and Krämer (2003) (see Lemma 2 below) that for this choice of $f(\cdot)$, $\sup_{n \geq 1} ER_n < \infty$ under the regularity conditions below. Note that the notation conventions used in this paper follow those of the probability literature, in order to preserve notation compatibility with the references from this literature that are referred to in this paper.

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The case $f(x) = \exp(x)$ is central to this paper, as it will allow to show the convergence in distribution of statistics of the form

$$\exp(-M_n) \sum_{t=1}^{n} \exp(S_t).$$

This is the statistic for which Davies and Krämer (2003) showed the property $\sup_{n \geq 1} ER_n < \infty$ under regularity conditions. Earlier, Park and Phillips (1999) established that under these conditions,

$$n^{-1/2} \sum_{t=1}^{n} \exp(S_t - M_n) \xrightarrow{p} 0;$$

see also Davies and Krämer (2003, p.867). Below, I will use a Laplace transform argument and show that $\lim_{n \to \infty} E \exp(-r \sum_{t=1}^{n} f(S_t - M_n))$ exists for all $r \geq 0$; and from that result, it is then easy to prove the convergence in distribution of R_n.

In the econometrics literature, papers such as Granger and Hallmann (1991), Ermini and Granger (1993), and Corradi (1995) attempt to define the I(1) property (loosely defined here as the property that a series displays some type of fading memory property after differencing) in such a way that under some transformations the property is preserved. A related literature seeks to find unit root tests whose null distribution is robust to monotonic transformations; see Granger and Hallmann (1991), Burridge and Guerre (1996), Gourieroux and Breitung (1997) for work along these lines. Box-Cox approaches can be found in Franses and McAleer (1998), Franses and Koop (1998) and Kobayashi and McAleer (1999). However, such papers lack the mathematical precision and conciseness of the work of Borodin and Ibragimov (1995) and Park and Phillips (1999), which seeks to characterize the limit behavior (after rescaling) of sums of the form

$$\sum_{t=1}^{n} f(S_t).$$

For functions $f(\cdot)$ that are "asymptotically homogeneous" the limit behavior of the rescaled statistic can be derived because it is asymptotically equivalent to a sum of a function of $S_t/\sqrt{T}$, and at that point an appeal to the functional central limit theorem can be used to derive the limit distribution. Trivial examples of such functions are the identity and the square. See Park and Phillips (1999) and Borodin and Ibragimov (1995). For integrable functions $f(\cdot)$, the convergence in distribution of $n^{-1/2} \sum_{t=1}^{n} f(S_t)$ has been derived in Borodin and Ibragimov (1995) and Park and Phillips (1999). Borodin and Ibragimov (1995) also derive a central limit theorem type result for periodic functions $f(\cdot)$. To the best of my knowledge however, no result for the exponential is known. Such a result has potential applications to situations where it is unknown whether the level of a series or its exponential can be viewed as a random walk.

It is straightforward to extend the results of this paper to statistics of the type

$$\sum_{t=1}^{n} f(S_0 + S_t - \max(0, S_0 + \max_{1\leq t\leq n} S_t))$$

and

$$\sum_{t=1}^{n} f(S_t - \max_{1\leq t\leq n} S_t))$$

where S_0 is an arbitrary random variable. This is because under the conditions used in this paper a functional central limit theorem holds, which implies that $P(S_0 + \max_{1\leq t\leq n} S_t > 0) \to 1$.

2 Main result

The random walk S_t is assumed to satisfy the following:

Assumption 1. *$S_t = S_{t-1} + X_t$ for $t = 1, ..., n$, $S_0 = 0$, X_t is i.i.d., $EX_t = 0$ and $E|X_t|^3 < \infty$. In addition, X_t is continuously distributed.*

The condition of a continuous distribution allows me to apply results by Ritter (1981) and Bertoin and Doney (1994) freely, without being concerned with the possibility of partial sums equalling exactly zero at some point. The third moment condition on X_t allows the application of a result by Davies and Krämer (2003).

To understand the development of the result, define T_n as the first index at which the maximum of S_t is attained, with the provision that $T_n = 0$ if $\max_{1\leq k\leq n} S_k \leq 0$. This definition implies that $S_{T_n} = M_n$ and that $P(T_n = k) = p_k q_{n-k}$ where

$$p_k = P(\min_{1\leq t\leq k} S_t > 0), \quad q_k = P(\max_{1\leq t\leq k} S_t < 0)$$

for $k = 1, ..., n$, and $p_0 = q_0 = 1$; see Spitzer (1956, p. 334). Since $\sum_{k=0}^{n} I(T_n = k) = 1$, we can now write

$$E\exp(-r\sum_{t=1}^{n} f(S_t - M_n))$$

$$= E\sum_{k=0}^{n} I(T_n = k)E(\exp(-r\sum_{t=1}^{n} f(S_t - S_k))|T_n = k)$$

$$= \sum_{k=0}^{n} p_k q_{n-k} E(\exp(-r\sum_{t=1}^{n} f(S_t - S_k))|T_n = k)$$

$$= \sum_{k=0}^{n} p_k q_{n-k} E(\exp(-r\sum_{t=1}^{n} f(S_t - S_k))|\min_{1\le s\le k-1}(S_k - S_s) > 0, \min_{k+1\le s\le n}(S_k - S_s) \ge 0)$$

$$= \exp(-r)\sum_{k=0}^{n} p_k q_{n-k} E(\exp(-r\sum_{t=1}^{k-1} f(S_t - S_k))|\min_{1\le s\le k-1}(S_k - S_s) > 0)$$

$$\times E(\exp(-r\sum_{t=k+1}^{n} f(S_t - S_k))|\min_{k+1\le s\le n}(S_k - S_s) \ge 0)$$

$$= \exp(-r)\sum_{k=0}^{n} p_k q_{n-k} E(\exp(-r\sum_{t=1}^{k-1} f(-S_t))|\min_{1\le s\le k-1} S_s > 0)E(\exp(-r\sum_{t=1}^{n-k} f(S_t))|\max_{1\le s\le n-k} S_s \le 0)$$

$$= \exp(-r)\sum_{k=0}^{n} p_k q_{n-k} \psi_{k-1}(r)\tilde{\psi}_{n-k}(r)$$

where I defined, for $k \ge 1$

$$\psi_k(r) = E(\exp(-r\sum_{t=1}^{k} f(-S_t))|\min_{1\le s\le k} S_s > 0)$$

and

$$\tilde{\psi}_k(r) = E(\exp(-r\sum_{t=1}^{k} f(S_t))|\max_{1\le s\le k} S_s \le 0).$$

The third equality here follows because as noted in Spitzer (1956) and Feller (1968, p. 573), T_n is characterized by the conditions that $S_{T_n} - S_s > 0$ for $s = 1, ..T_n - 1$ and $S_{T_n} - S_s \ge 0$ for $s = T_n, ..., n$. The fourth equality follows because for independent random vectors X and Y and events A and B of positive probability,

$$E(f_1(X)f_2(Y)|X \in A, Y \in B) = E(f_1(X)|X \in A)E(f_2(Y)|Y \in B).$$

The fifth equality follows because

$$h(X_k, X_k + X_{k-1}, ..., X_k + X_{k-1} + .. + X_2) \stackrel{d}{=} h(X_1, X_1 + X_2, ..., X_1 + X_2 + ... + X_{k-1})$$

and

$$h(X_{k+1}, X_{k+1} + X_{k+2}, ..., X_{k+1} + X_{k+2} + ... + X_n) \stackrel{d}{=} h(X_1, X_1 + X_2, ..., X_1 + X_2 + ... + X_{n-k}).$$

The expressions $\psi_k(r)$ and $\tilde{\psi}_k(r)$ play a crucial role in the proof, and a proof of their pointwise convergence as $k \to \infty$ will be constructed below using the literature on "random walks conditioned to be positive". As explained in Biggins (2003), probabilities involving random walks conditioned to be positive can be related, via an h-transform, to a Markov chain which is "random walk conditioned to stay positive", and this random walk conditioned to be positive can be viewed as a discrete version of a Bessel-3 process.

Results from papers by Ritter (1981) and Bertoin and Doney (1994) on random walk conditioned to be positive will be used to show the following result:

Lemma 1. *There exist functions $\psi(r)$ and $\tilde{\psi}(r)$ such that pointwise for all $r \geq 0$, $\lim_{k\to\infty} \psi_k(r) = \psi(r)$ and $\lim_{k\to\infty} \tilde{\psi}_k(r) = \tilde{\psi}(r)$.*

Using this lemma, the following result now follows:

Theorem 1. *Under Assumption 1, R_n converges in distribution to $1 + Y_1 + Y_2$, where Y_1 is independent of Y_2 and*

$$Y_1 \stackrel{d}{=} \sum_{t=0}^{\infty} f(-\tilde{S}_{1t}), \quad Y_2 \stackrel{d}{=} \sum_{t=0}^{\infty} f(\tilde{S}_{2t}),$$

where $\tilde{S}_{1t}$ and $\tilde{S}_{2t}$ are Markov chains.

The following corollary is now trivial:

Corollary 1. *Under Assumption 1,*

$$\exp(-M_n) \sum_{t=1}^{n} \exp(S_t) \stackrel{d}{\longrightarrow} 1 + \sum_{t=0}^{\infty} \exp(-\tilde{S}_{1t}) + \sum_{t=0}^{\infty} \exp(\tilde{S}_{2t}),$$

where $\tilde{S}_{1t}$ and $\tilde{S}_{2t}$ are Markov chains.

While the above result shows the convergence in distribution, the limit distribution in general depends on the distribution on X_t. Therefore, the limit is not distribution-free; if the above limit distribution result is to be used for testing, the limit distribution will need to be obtained through some resampling method.

Note that the above result and its proof can be interpreted as saying that the limit distribution of R_n is determined by a finite but large number of values of S_t for which t is "close" to T_n. That is, the R_n statistic is asymptotically close to

$$\sum_{t=1}^{n} f(S_t - M_n) I(|t - T_n| \leq L)$$

for large L.

3 Simulations

The above theorem is easily illustrated with a simple simulation for the case $f(x) = \exp(x)$. While Theorem 1 holds for any value of the scaling parameter EX_t^2, we should expect that the approximation will be poor for relatively low values of EX_t^2. In that case after all, $S_t - M_n$ will be relatively small as well, and

$$\sum_{t=1}^{n} f(S_t - M_n) \approx \sum_{t=1}^{n} f(0) = nf(0)$$

and a large value for n will be needed in such a situation in order to achieve a good approximation to the limit distribution.

Similarly, for large values of EX_t^2, $|S_t - M_n|$ will be relatively large for all values of t except those for which $S_t = M_n$. Essentially, the random walk will jump towards its maximum and away from it with a "large" jump, implying that $f(S_t - M_n) \approx 0$ for all t for which $S_t \neq M_n$. This also may be problematic in terms of the quality of the asymptotic approximation for moderate values of n, as the statistic will be close to 1 in that case if we assume X_t to be continuously distributed.

A Fortran simulation program (available from the author upon request) was used to generate simulation results for

$$Q_n(c) = \sum_{t=1}^{n} \exp(c(S_t - M_n))$$

for various values of n and c and various distributions for i.i.d. X_t that had a variance of 1. For the distribution of X_t, I used a standard normal, a uniform$[-\sqrt{3}, \sqrt{3}]$, and a Rademacher distribution (i.e., $P(X_t = -1) = P(X_t = 1) = 0.5$). Note that the Rademacher distribution is obviously not continuous and falls outside of the scope of the formal results of the previous chapter.

In order to observe the convergence in distribution from the simulation, I used the values 50, 100, 500, 1000, 5000 and 10,000 in all situations, and added simulations for $n = 50,000$ and $n = 100,000$ when appropriate. Everywhere, $1,000,000$ replications were used to obtain the results, except for the simulations conducted for $n = 50,000$ and $n = 100,000$, where $100,000$ replications were used.

Simulation results can be found in the tables of Appendix 2. In the simulations, it can be observed that some quantiles end up below 1. This is possible because M_n equals 0 in the case where the entire random walk is negative, and X_t will not take the value M_n for any t in that situation. However, since under my assumptions the random walk is oscillating, the probability of the entire random walk being negative vanishes asymptotically.

For $c = 1$, the convergence in distribution appears to be relatively slow. For higher values of c such as 5 and 10, the convergence in distribution appears to be rapid, with a good approximation to the limit distribution being reached for $n = 100$ to $n = 500$. For $c = 50$, the convergence is rapid, but the distribution of the $Q_n(50)$ statistic degenerates into excessive closeness to 1.

For Rademacher random variables and high values for c, the $R_n(c)$ statistic is close to the number of times that the maximum is attained. This number is obviously an integer, and we accordingly see in this case that the quantiles are close to integer values. Another striking feature is that the quantiles seem to be increasing with n.

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Appendix 1: Mathematical proofs

The modification of the proof in Davies and Krämer (2003) mentioned in the text is the following:

Lemma 2. *Under Assumption 1,* $\sup_{n\geq 1} ER_n < \infty$.

Proof of Lemma 2: Similarly to Davies and Krämer's argument of page 868,

$$ER_n = E\sum_{t=1}^{n} f(S_t - M_n) = \sum_{t=1}^{n}\int_0^\infty f(-x)dF_{tn}(x)$$

$$= \sum_{t=1}^{n}(f(0)F_{tn}(0) + \int_0^\infty f(-x)F_{tn}(x)dx)$$

where $F_{tn}(\cdot)$ denotes the distribution function of $M_n - S_t$. Since

$$F_{tn}(x) \leq t^{-1/2}(n-t+1)^{-1/2}(a+bx)^2$$

as argued in Davies and Krämer (2003, p. 868), it follows that

$$\sup_{n\geq 1} ER_n \leq C \sup_{n\geq 1}\sum_{t=1}^{n} t^{-1/2}(n-t+1)^{1/2} < \infty.$$

□

Proof of Lemma 1: For any $M \geq 1$ and $k \geq M+1$,

$$|E(\exp(-r\sum_{t=1}^{M} f(-S_t))| \min_{1\leq s\leq k-1} S_s > 0) - E(\exp(-r\sum_{t=1}^{k-1} f(-S_t))| \min_{1\leq s\leq k-1} S_s > 0)|$$

$$\leq 1 - E\exp(-r\sum_{t=M+1}^{k-1} f(-S_t))|\min_{1\leq s\leq k-1} S_s > 0).$$

Pick $\eta \in ((2+\varepsilon)^{-1}, 1/2)$ and note that by Theorem 2 of Ritter (1981), for such η

$$\lim_{\delta\to 0}\liminf_{k\to\infty} P(\inf_{1\leq t\leq k}(S_t - \delta t^\eta) > 0|\min_{1\leq s\leq k-1} S_s > 0) = 1,$$

and therefore

$$1 - E\exp(-r\sum_{t=M+1}^{k-1} f(-S_t))|\min_{1\leq s\leq k-1} S_s > 0)$$

$$\leq 1 - E(I(\inf_{1\leq t\leq k}(S_t - \delta t^\eta) > 0)\exp(-r\sum_{t=M+1}^{k-1} f(-S_t))|\min_{1\leq s\leq k-1} S_s > 0)$$

$$+P(\inf_{1\leq t\leq k}(S_t - \delta t^\eta) \leq 0|\min_{1\leq s\leq k-1} S_s > 0)$$

$$\leq 1 - E\exp(-r\sum_{t=M+1}^{k-1} f(-\delta t^\eta)) + P(\inf_{1\leq t\leq k}(S_t - \delta t^\eta) \leq 0|\min_{1\leq s\leq k-1} S_s > 0)$$

$$\leq 1 - E\exp(-r\sum_{t=M+1}^{\infty} f(-\delta t^\eta)) + P(\inf_{1\leq t\leq k}(S_t - \delta t^\eta) \leq 0|\min_{1\leq s\leq k-1} S_s > 0)$$

$$\leq r\sum_{t=M+1}^{\infty} f(-\delta t^\eta) + P(\inf_{1\leq t\leq k}(S_t - \delta t^\eta) \leq 0|\min_{1\leq s\leq k-1} S_s > 0).$$

The last inequality follows because $1 - \exp(-|x|) \leq |x|$. Also, there exists a Markov chain $\tilde{S}_t$ such that

$$\lim_{k\to\infty} E(\exp(-r\sum_{t=1}^{M} f(-S_t))|\min_{1\leq s\leq k-1} S_s > 0)$$

$$= E\exp(-r\sum_{t=1}^{M} f(-\tilde{S}_t))$$

by Theorem 1 of Bertoin and Doney (1994, p. 2158). Also, since the last expression is decreasing in M and bounded from below by zero, $\psi(r) = \lim_{M\to\infty} E\exp(-r\sum_{t=1}^{M} f(-\tilde{S}_t))$ exists. Therefore, for all $M \geq 2$,

$$\limsup_{k\to\infty} |E(\exp(-r\sum_{t=1}^{k-1} f(-S_t))|\min_{1\leq s\leq k-1} S_s > 0) - \psi(r)|$$

$$\leq r \sum_{t=M+1}^{\infty} f(-\delta t^{\eta}) + \limsup_{k\to\infty} P(\inf_{1\leq t\leq k}(S_t - \delta t^{\eta}) \leq 0 | \min_{1\leq s\leq k-1} S_s > 0) + |E\exp(-r\sum_{t=1}^{M} f(-\tilde{S}_t)) - \psi(r)|,$$

and by making M approach infinity first and then making δ approach zero, it now follows that

$$\lim_{k\to\infty} E(\exp(-r\sum_{t=1}^{k-1} f(-S_t)) | \min_{1\leq s\leq k-1} S_s > 0) = \psi(r).$$

Similarly,

$$\lim_{k\to\infty} E(\exp(-r\sum_{t=1}^{k-1} f(S_t)) | \max_{1\leq s\leq k-1} S_s \leq 0) = \tilde{\psi}(r).$$

□

Proof of Theorem 1: For $\alpha \in (0,1)$, write

$$E\exp(-rR_n) = \sum_{k=0}^{n} p_k q_{n-k} E\exp(-r\sum_{t=1}^{n} f(S_t - M_n))$$

$$= \sum_{k=[n\alpha]+1}^{[(1-\alpha)n]} p_k q_{n-k} E\exp(-r\sum_{t=1}^{n} f(S_t - S_k)) | T_n = k)$$

$$+ \sum_{k=0}^{[n\alpha]} p_k q_{n-k} E\exp(-r\sum_{t=1}^{n} f(S_t - S_k)) | T_n = k)$$

$$+ \sum_{k=[(1-\alpha)n]+1}^{n} p_k q_{n-k} E\exp(-r\sum_{t=1}^{n} f(S_t - S_k)) | T_n = k).$$

By bounding the exponential term by 1, it can be seen that the last two expressions are both bounded by terms that converge to $\pi^{-1}\int_0^{\alpha} x^{-1/2}(1-x)^{-1/2}dx$ by the arcsine law of Spitzer (1956, p.337, Theorem 7.1), and are therefore asymptotically irrelevant because this expression converges to zero as $\alpha \downarrow 0$. Next, consider

$$|\sum_{k=[n\alpha]+1}^{[(1-\alpha)n]} p_k q_{n-k} (E\exp(-r\sum_{t=1}^{n} f(S_t - S_k)) | T_n = k) - \exp(-r)\psi(r)\tilde{\psi}(r))|$$

$$= |\sum_{k=[n\alpha]+1}^{[(1-\alpha)n]} p_k q_{n-k} \exp(-r)(\psi_k(r)\tilde{\psi}_{n-k}(r) - \psi(r)\tilde{\psi}(r))|$$

and note that the last term is bounded from above by

$$\sum_{k=[n\alpha]+1}^{[(1-\alpha)n]} p_k q_{n-k} \exp(-r)(\sup_{k\geq n\alpha} |\psi_k(r) - \psi(r)| + \sup_{k\geq n\alpha} |\tilde{\psi}_k(r) - \tilde{\psi}(r)|) = o(1)$$

because

$$\sum_{k=[n\alpha]+1}^{[(1-\alpha)n]} p_k q_{n-k} \leq 1.$$

Since α was arbitrary and because

$$\sum_{k=[n\alpha]+1}^{[(1-\alpha)n]} p_k q_{n-k} \exp(-r)\psi(r)\tilde{\psi}(r) \to \pi^{-1} \int_\alpha^{1-\alpha} x^{-1/2}(1-x)^{-1/2} dx \exp(-r)\psi(r)\tilde{\psi}(r)$$

and

$$\pi^{-1} \int_0^1 x^{-1/2}(1-x)^{-1/2} dx = 1,$$

it now follows that

$$E \exp(-r \sum_{t=1}^{n} f(S_t - M_n)) \to \exp(-r)\psi(r)\tilde{\psi}(r).$$

By Feller's (1968) Theorem 2 of page 408, $\exp(-r)\psi(r)\tilde{\psi}(r)$ is the transform of a possibly defective distribution $F(\cdot)$, and the limit $F(\cdot)$ is not defective if $\exp(-r)\psi(r)\tilde{\psi}(r) \to 1$ as $r \downarrow 0$. Since

$$|\exp(-r)\psi(r)\tilde{\psi}(r) - 1| = \lim_{n\to\infty} |E \exp(-r \sum_{t=1}^{n} f(S_t - M_n)) - 1|$$

$$\leq |r| \sup_{n\geq 1} E \sum_{t=1}^{n} f(S_t - M_n)$$

because $1 - \exp(-|x|) \leq |x|$ and $\sup_{n\geq 1} E \sum_{t=1}^{n} f(S_t - M_n) < \infty$ as was shown in Lemma 2, it follows that the limit distribution is not defective.

□

Appendix 2: Simulation results

Table 1: Simulated quantiles and mean of the distribution of $Q_n(1)$; X_t standardnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.74	2.23	3.17	4.47	6.13	7.93	9.12	4.82
100	1.99	2.50	3.50	4.96	6.88	9.03	10.5	5.43
500	2.40	2.93	4.03	5.71	7.98	10.6	12.5	6.34
1000	2.52	3.05	4.17	5.90	8.24	11.0	12.9	6.56
5000	2.71	3.24	4.37	6.15	8.60	11.5	13.5	6.87
10000	2.76	3.28	4.43	6.22	8.69	11.6	13.7	6.95
50000	2.82	3.34	4.50	6.30	8.80	11.7	13.8	7.04
100000	2.83	3.36	4.51	6.32	8.84	11.8	13.9	7.08

Table 2: Simulated quantiles and mean of the distribution of $Q_n(5)$; X_t standardnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.18	1.00	1.04	1.23	1.68	2.18	2.54	1.38
100	0.82	1.00	1.05	1.27	1.74	2.24	2.62	1.44
500	1.00	1.01	1.07	1.32	1.80	2.33	2.71	1.51
1000	1.00	1.01	1.08	1.33	1.82	2.35	2.74	1.52
5000	1.00	1.02	1.09	1.35	1.84	2.37	2.77	1.55
10000	1.01	1.02	1.09	1.35	1.84	2.37	2.77	1.55

Table 3: Simulated quantiles and mean of the distribution of $Q_n(10)$; X_t standardnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.02	1.00	1.00	1.03	1.24	1.67	1.92	1.12
100	0.37	1.00	1.00	1.04	1.27	1.71	1.95	1.16
500	1.00	1.00	1.00	1.05	1.31	1.75	1.99	1.20
1000	1.00	1.00	1.00	1.05	1.32	1.76	1.99	1.22
5000	1.00	1.00	1.00	1.06	1.33	1.78	2.01	1.23
10000	1.00	1.00	1.00	1.06	1.34	1.78	2.01	1.23

Table 4: Simulated quantiles and mean of the distribution of $Q_n(50)$; X_t standardnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.00	1.00	1.00	1.00	1.00	1.05	1.23	0.96
100	0.00	1.00	1.00	1.00	1.00	1.06	1.26	0.98
500	1.00	1.00	1.00	1.00	1.00	1.07	1.29	1.02
1000	1.00	1.00	1.00	1.00	1.00	1.08	1.29	1.02
5000	1.00	1.00	1.00	1.00	1.00	1.08	1.30	1.03
10000	1.00	1.00	1.00	1.00	1.00	1.08	1.31	1.04

Table 5: Simulated quantiles and mean of the distribution of $Q_n(1)$; X_t uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.74	2.19	3.03	4.23	5.76	7.42	8.53	4.56
100	1.97	2.44	3.35	4.69	6.48	8.47	9.82	5.13
500	2.38	2.85	3.85	5.40	7.51	9.97	11.7	6.00
1000	2.50	2.97	3.99	5.58	7.77	10.3	12.1	6.22
5000	2.68	3.14	4.18	5.83	8.11	10.8	12.7	6.51
10000	2.72	3.18	4.23	5.88	8.20	10.9	12.8	6.58

Table 6: Simulated quantiles and mean of the distribution of $Q_n(5)$; X_t uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.14	1.00	1.02	1.17	1.58	2.04	2.38	1.32
100	0.69	1.00	1.03	1.20	1.63	2.10	2.45	1.36
500	1.00	1.01	1.05	1.25	1.69	2.18	2.54	1.43
1000	1.00	1.01	1.06	1.26	1.71	2.20	2.57	1.45
5000	1.00	1.01	1.06	1.27	1.73	2.22	2.59	1.47
10000	1.00	1.01	1.06	1.27	1.73	2.23	2.60	1.48

Table 7: Simulated quantiles and mean of the distribution of $Q_n(10)$; X_t uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.01	1.00	1.00	1.02	1.18	1.60	1.86	1.10
100	1.00	1.00	1.00	1.02	1.21	1.64	1.89	1.13
500	1.00	1.00	1.00	1.03	1.25	1.68	1.95	1.17
1000	1.00	1.00	1.00	1.03	1.26	1.69	1.93	1.19
5000	1.00	1.00	1.00	1.04	1.27	1.71	1.95	1.20
10000	1.00	1.00	1.00	1.04	1.27	1.71	1.95	1.20

Table 8: Simulated quantiles and mean of the distribution of $Q_n(50)$; X_t uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.00	1.00	1.00	1.00	1.00	1.03	1.19	0.95
100	0.00	1.00	1.00	1.00	1.00	1.04	1.21	0.98
500	1.00	1.00	1.00	1.00	1.00	1.05	1.24	1.01
1000	1.00	1.00	1.00	1.00	1.00	1.06	1.25	1.02
5000	1.00	1.00	1.00	1.00	1.00	1.06	1.26	1.03
10000	1.00	1.00	1.00	1.00	1.00	1.06	1.26	1.03

Table 9: Simulated quantiles and mean of the distribution of $Q_n(1)$; X_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.72	2.19	2.89	4.06	5.72	7.69	9.09	4.56
100	2.00	2.40	3.19	4.53	6.46	8.83	10.5	5.16
500	2.39	2.77	3.69	5.24	7.56	10.5	12.6	6.06
1000	2.49	2.87	3.81	5.42	7.82	10.8	13.1	6.29
5000	2.64	3.02	4.00	5.68	8.19	11.4	13.7	6.60
10000	2.68	3.05	4.04	5.73	8.28	11.5	13.9	6.67

Table 10: Simulated quantiles and mean of the distribution of $Q_n(5)$; X_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.02	1.01	1.01	1.03	2.04	3.05	4.05	1.70
100	1.01	1.01	1.02	1.04	2.05	3.07	4.07	1.80
500	1.01	1.01	1.02	1.05	2.06	4.05	5.05	1.92
1000	1.01	1.01	1.02	1.05	2.07	4.05	5.06	1.96
5000	1.01	1.01	1.02	1.07	2.08	4.05	5.07	2.01
10000	1.01	1.01	1.02	1.07	2.08	4.05	5.07	2.02

Table 11: Simulated quantiles and mean of the distribution of $Q_n(10)$; X_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.00	1.00	1.00	1.00	2.00	3.00	4.00	1.67
100	1.00	1.00	1.00	1.00	2.00	3.00	4.00	1.76
500	1.00	1.00	1.00	1.00	2.00	4.00	5.00	1.89
1000	1.00	1.00	1.00	1.00	2.00	4.00	5.00	1.93
5000	1.00	1.00	1.00	1.00	2.00	4.00	5.00	1.96
10000	1.00	1.00	1.00	1.00	2.00	4.00	5.00	1.98

Table 12: Simulated quantiles and mean of the distribution of $Q_n(50)$; X_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.00	1.00	1.00	1.00	2.00	3.00	4.00	1.67
100	1.00	1.00	1.00	1.00	2.00	3.00	4.00	1.76
500	1.00	1.00	1.00	1.00	2.00	4.00	5.00	1.89
1000	1.00	1.00	1.00	1.00	2.00	4.00	5.00	1.93
5000	1.00	1.00	1.00	1.00	2.00	4.00	5.00	1.97
10000	1.00	1.00	1.00	1.00	2.00	4.00	5.00	1.95