Further results on the asymptotics for nonlinear transformations of integrated time series

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Abstract

This paper establishes various results involving functions of integrated processes. Two theorems - that improve similar results by Park and Phillips - are proven for averages of functions of an integrated process that has not been rescaled by the square root of sample size. In addition, two results are given that characterize asymptotic behavior of averages of non-integrable functions of rescaled integrated processes; the observations close to the pole take over asymptotic behavior in that case. Throughout, we make the assumption that the innovations of the integrated process are a linear process.

1 Introduction

This paper proves three results about functions of integrated processes. Our first result is an extension of a result in Park and Phillips (1999), where it is proven that for integrable functions $T(.)$ and for I(1) processes x_t,

$$n^{-1/2}\sum_{t=1}^{n}T(x_t)\xrightarrow{d}(\int_{-\infty}^{\infty}T(s)ds)L(1,0),$$

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where $L(t,s)$ is a two-parameter stochastic process called (Brownian) *local time*. The remarkable thing about this result is that it establishes limit theory for a function of an I(1) process that has not been rescaled by $n^{-1/2}$. Park and Phillips establish the above result under some regularity conditions on the I(1) process x_t and the integrable function $T(.)$. In this paper, we show that Park and Phillips' regularity conditions for the above result can be relaxed and also that their result can be extended to yield, for $0 \leq \alpha < 1/2$,

$$n^{-1/2-\alpha}\sum_{t=1}^{n} T(n^{-\alpha}x_t) \xrightarrow{d} (\int_{-\infty}^{\infty} T(s)ds)L(1,0).$$

A central tool for the proof of this first result is a lemma that was recently established in de Jong (2004). Also in Park and Phillips (1999), it is shown that for functions $T(.)$ that satisfy

$$T(\lambda x) = \nu(\lambda)H(x) + R(x,\lambda)$$

under conditions on $R(.,.)$ that basically serve to ensure asymptotic negligibility of

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n} R(x_t, n^{1/2}),$$

we have

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n} T(x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r))dr,$$

where $\sigma^2 = \lim_{n\to\infty} n^{-1}Ex_n^2$. Again the interesting aspect of the above result is the fact that it considers integrated processes that have not been rescaled by $n^{-1/2}$. Functions $T(.)$ that satisfy the appropriate condition are coined *asymptotically homogeneous* by Park and Phillips. The asymptotically homogeneous condition is trivially satisfied for $T(x) = |x|^a$ for $a \geq 0$, but is general enough to also deal with functions such as $T(x) = |x|^a \log|x|$ for all $a \geq 0$. In this paper, we show the more general result that whenever for functions $H(.)$ and $\nu(.)$ we have

$$\nu(\lambda)^{-1}T(\lambda x) \to H(x) \quad \text{as } \lambda \to \infty$$

in L_1 sense, we have for $0 \leq \alpha < 1/2$, under regularity conditions,

$$\nu(n^{1/2-\alpha})^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-\alpha}x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r))dr.$$

Therefore, we show that Park and Phillips' class of asymptotically homogeneous functions can be extended, and we consider $n^{-\alpha}x_t$ for $0 \leq \alpha < 1/2$ instead of x_t as the argument for $T(.)$.
A third result that is proven in this paper concerns averages of the type

$$n^{-1}\sum_{t=1}^{n}|n^{-1/2}x_t|^{-m}I(n^{-1/2}x_t > c_n)$$

and

$$n^{-1}\sum_{t=1}^{n}|n^{-1/2}x_t|^{-m}I(|n^{-1/2}x_t| > c_n),$$

where $m > 1$. While it has been shown in de Jong (2004) and Pötscher (2004) that under regularity conditions for locally integrable functions $T(.)$ we have

$$n^{-1}\sum_{t=1}^{n}T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))dr,$$

it is yet unknown what happens to functions $T(.)$ that are not integrable. Using a "clipping device" involving a deterministic sequence c_n that converges to 0 with n, it will be proven that for $m > 1$,

$$(m-1)c_n^{1-m}n^{-1}\sum_{t=1}^{n}|\sigma^{-1}n^{-1/2}x_t|^{-m}I(\sigma^{-1}n^{-1/2}x_t > c_n) \xrightarrow{d} L(1,0),$$

and also that

$$(1/2)(m-1)c_n^{1-m}n^{-1}\sum_{t=1}^{n}|\sigma^{-1}n^{-1/2}x_t|^{-m}I(|\sigma^{-1}n^{-1/2}x_t| > c_n) \xrightarrow{d} L(1,0).$$

2 Assumptions and result for integrable functions

Identically to Park and Phillips (1999), linear process conditions for x_t are assumed and

$$x_t = x_{t-1} + w_t,$$

where w_t is generated according to

$$w_t = \sum_{k=0}^{\infty}\phi_k\varepsilon_{t-k}$$

where ε_t is assumed to be a sequence of i.i.d. random variables with mean zero, and where it is assumed that $\sum_{k=0}^{\infty} \phi_k \neq 0$. In addition, we will assume that x_0 is an arbitrary random variable that is independent of all w_t, $t \geq 1$. The main assumptions used in this paper are Assumption 2.1 and 2.2 from Park and Phillips (1999):

Assumption 1 $\sum_{k=0}^{\infty} k^{1/2}\phi_k < \infty$ *and* $E\varepsilon_t^2 < \infty$.

Assumption 2

(a) $\sum_{k=0}^{\infty} k|\phi_k| < \infty$ *and* $E|\varepsilon_t|^p < \infty$ *for some* $p > 2$.

(b) *The distribution of* ε_t *is absolutely continuous with respect to the Lebesgue measure and has characteristic function* $\psi(s)$ *for which* $\lim_{s\to\infty} s^\eta \psi(s) = 0$ *for some* $\eta > 0$.

Assumption 1 guarantees that $n^{-1/2}x_{[rn]} \Rightarrow \sigma W(r)$ where "$\Rightarrow$" denotes weak convergence in $C[0,1]$, i.e. the space of functions that are continuous on $[0,1]$. Define $W_n^0 = n^{-1/2}x_{[rn]}$. Then by the Skorokhod representation there exists a $W_n(.)$ such that $W_n \stackrel{d}{=} W_n^0$ and $\sup_{r\in[0,1]} |W_n(r) - W(r)| \stackrel{as}{\longrightarrow} 0$. Assumption 2 in addition also guarantees a convergence rate for $\sup_{r\in[0,1]} |W_n(r) - W(r)|$; see Park and Phillips (1999, Lemma 2.3). This result justifies that proving our results for $W_n(.)$ suffices. Several of the manipulations in the proofs of the results in this paper require the use of local time $L(.,.)$. Local time is a random function satisfying

$$L(t,s) = \lim_{\varepsilon\to 0}(2\varepsilon)^{-1}\int_0^t I(|W(r) - s| < \varepsilon)dr.$$

See Park and Phillips (1999, p. 271-272) and Chung and Williams (1990, Ch. 7) for more details regarding local time.
Park and Phillips (1999) establish the following result for integrable functions of integrated random variables:

Theorem 1 *Suppose that* $T(.)$ *is integrable and Assumption 2 holds with* $p > 4$. *If* $T(.)$ *is square integrable and satisfies the Lipschitz condition*

$$|T(x) - T(y)| \leq c|x - y|^l$$

over its support for some constants c *and* $l > 6/(p-2)$, *then*

$$n^{-1/2}\sum_{t=1}^{n} T(x_t) \stackrel{d}{\longrightarrow} \left(\int_{-\infty}^{\infty} T(s)ds\right)L(1,0).$$

For differentiable functions $T(.)$, we need to set $l = 1$, implying that we need $p > 8$ in order for the theorem to work. In order to improve the above result, we needed the following useful lemma, that was established in de Jong (2004):

Lemma 1 *Under Assumption 2, for all $y \in \mathbb{R}$, $\delta > 0$, and $n \geq M$ for some value of M,*

$$P(y \leq n^{-1/2}x_n \leq y + \delta) \leq C\delta, \tag{1}$$

where C and M do not depend on y, δ, or n.

Using this lemma, we were able to improve Park and Phillips' result and show the following quite general result:

Theorem 2 *Suppose Assumption 2 holds. Also assume that $|T(x)| \leq R(x)$, and assume that $R(.)$ is integrable, continuous on $\mathbb{R}$, and monotone on $(0, \infty)$ and $(-\infty, 0)$. If $T(.)$ is continuous, then for $0 \leq \alpha < 1/2$,*

$$n^{-1/2-\alpha}\sum_{t=1}^{n} T(n^{-\alpha}x_t) \xrightarrow{d} (\int_{-\infty}^{\infty} T(s)ds)L(1,0).$$

Compared to Park and Phillips' theorem, we have completely removed their Lipschitz-continuity condition and weakened it to continuity, and in addition, their requirement on p has been removed. Also, weights $n^{-\alpha}$ for $0 \leq \alpha < 1/2$ are allowed for. While no $R(.)$ function such as present in Theorem 2 is explicitly used in their Theorem 1, from Park and Phillips' proof it is clear that existence of such a function is implied. Therefore, Theorem 2 is a "clean" improvement to Park and Phillips' Theorem 1. A simple example of a function $T(.)$ that satisfies our conditions but violates those of Park and Phillips (1999) is $T(x) = (1 - x^2)^{1/2} I(-1 \leq x \leq 1)$.

3 Asymptotically homogeneous functions

In this section, we improve Park and Phillips' (1999) result for asymptotically homogeneous functions. Park and Phillips assume that

$$T(\lambda x) = \nu(\lambda)H(x) + R(x, \lambda)$$

and they show that

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n} T(x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r))dr$$

if either

a. $|R(x,\lambda)| \leq a(\lambda)P(x)$, where $\limsup_{\lambda\to\infty} a(\lambda)/\nu(\lambda) = 0$ and P is locally integrable, or

b. $|R(x,\lambda)| \leq b(\lambda)Q(\lambda x)$, where $\limsup_{\lambda\to\infty} b(\lambda)/\nu(\lambda) < \infty$ and Q is locally integrable and vanishes at infinity, i.e. $Q(x) \to 0$ as $|x| \to \infty$.

In this paper, we redefine their notion of an asymptotically homogeneous function, as follows:

Definition 1 *A function $T(.)$ is called* asymptotically homogeneous *if for all $K > 0$ and some function $H(.)$,*

$$\lim_{\lambda\to\infty} \int_{-K}^{K} |\nu(\lambda)^{-1}T(\lambda x) - H(x)|dx = 0.$$

Obviously from the dominated convergence theorem it follows that if for some $\nu(.)$ and $H(.)$, pointwise in x,

$$\nu(\lambda)^{-1}T(\lambda x) \to H(x) \quad \text{as } \lambda \to \infty$$

and $|\nu(\lambda)^{-1}T(\lambda x)| \leq G(x)$ for a locally integrable function $G(.)$, then $T(.)$ is asymptotically homogeneous. Below, we will call a function *monotone regular* if for some $\{a_1, \ldots, a_q\}$, $T(.)$ is monotone on (a_j, a_{j+1}) for $j = 0, \ldots, q$ (setting $a_0 = -\infty$ and $a_{q+1} = \infty$).
The main result of this section is the following:

Theorem 3 *Suppose Assumption 1 holds. Also assume that $T(.)$ is asymptotically homogeneous. In addition, assume that $H(.)$ is continuous and $T(.)$ is monotone regular. Then, for $0 \leq \alpha < 1/2$,*

$$\nu(n^{1/2-\alpha})^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-\alpha}x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r))dr = \int_{-\infty}^{\infty} H(\sigma s)L(1,s)ds.$$

It is also possible to show that our definition of an asymptotically homogeneous function is more general than Park and Phillips'. Under Assumption a. above,

$$\int_{-K}^{K} |\nu(\lambda)^{-1}T(\lambda x) - H(x)|dx = \nu(\lambda)^{-1}\int_{-K}^{K} |R(x,\lambda)|dx$$

$$\leq a(\lambda)\nu(\lambda)^{-1}\int_{-K}^{K} P(x)dx \to 0$$

as $\lambda \to \infty$ if $P(.)$ is locally integrable. Under Assumption b. above,

$$\int_{-K}^{K} |\nu(\lambda)^{-1}T(\lambda x) - H(x)|dx = \nu(\lambda)^{-1} \int_{-K}^{K} |R(x,\lambda)|dx$$

$$\leq b(\lambda)\nu(\lambda)^{-1} \int_{-K}^{K} Q(\lambda x)dx \to 0$$

as $\lambda \to \infty$, because $\limsup_{\lambda\to\infty} b(\lambda)\nu(\lambda)^{-1} < \infty$ and $\lim_{\lambda\to\infty} \int_{-K}^{K} Q(\lambda x)dx = 0$ by boundedness of $Q(.)$ (which is also assumed in Park and Phillips (1999)). Therefore, obviously the set of functions that is “asymptotically homogeneous” in this paper is wider than in Park and Phillips (1999). But clearly, most functions that one may expect to be useful for applications should be expected to already be in Park and Phillips' class of asymptotically homogeneous functions, and the main achievement of our analysis is the redefinition of the class of asymptotically homogeneous functions to as large as possible a collection of functions. It appears to us that the above result should be close to the limits of what should be possible in this setting, and for the authors of this paper, it is hard to see how the above definition of the class of asymptotically homogeneous functions can be relaxed further to yield an even larger function class that generates similar behavior.

4 Nonintegrable functions

In de Jong (2004) and Pötscher (2004) it is proven that under regularity conditions, in spite of possible poles in $T(.)$, as long as $\int_{-K}^{K} |T(x)|dx < \infty$ for all $K > 0$, we have

$$n^{-1} \sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))dr.$$

These results raise the question as to what will happen if a nonintegrable function of an integrated process is used for $T(.)$ in statistics of the form

$$n^{-1} \sum_{t=1}^{n} T(n^{-1/2}x_t).$$

This issue appears to have never been tackled before in either the statistics or the econometrics literature. This section explores this issue for functions

$$T(x) = |x|^{-m} I(x > 0)$$

and

$$T(x) = |x|^{-m},$$

for $m > 1$. As it turns out and is perhaps to be expected, the observations "close to zero" take over the limit behavior of the statistic in this case. We will need a "clipping device" and we construct statistics similar to those constructed in Park and Phillips (1999) for integrable functions. Our first result is the following:

Theorem 4 *Let $c_n = n^{-(2p+1)/3p+\eta}$ for some $\eta > 0$ such that $-(2p+1)/3p + \eta < 0$. In addition, assume that*

$$T(x) = |x|^{-m}$$

for some $m > 1$. Let $d_n = \int_{c_n}^{1} T(x)dx$. Then under Assumption 2,

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(\sigma^{-1} n^{-1/2} x_t) I(\sigma^{-1} n^{-1/2} x_t > c_n) \xrightarrow{d} L(1, 0).$$

Clearly, in the above theorem $d_n = (m-1)^{-1}(c_n^{1-m} - 1)$, but we choose the above formulation to bring out better where our rescaling factor d_n originates from.
The proof of the following "two-sided" version of the above theorem is analogous and therefore omitted:

Theorem 5 *Let $c_n = n^{-(2p+1)/3p+\eta}$ for some $\eta > 0$ such that $-(2p+1)/3p+\eta < 0$. Assume that*

$$T(x) = |x|^{-m}$$

for some $m > 1$. Let $d_n = 2\int_{c_n}^{1} T(x)dx$. Then under Assumption 2,

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(\sigma^{-1} n^{-1/2} x_t) I(|\sigma^{-1} n^{-1/2} x_t| > c_n) \xrightarrow{d} L(1, 0).$$

The above theorems leave the issue wide open to what function class the above theorem can be extended. The line of proof employed in the Appendix may allow for some generalization, but it is not clear to the authors what the outer limits are for which a result as the above might hold.

As a referee pointed out, it may also be of interest to attempt to show convergence towards

$\int_0^1 W(r)^{-1}dr$. Note that this is expression is well-defined while the limit as ε approaches zero from above of $\int_\varepsilon^1 |W(r)|^{-1}dr$ does not exist; the former result can be seen to follow from the occupation time formula and a modulus of continuity type result for the Brownian local time. The techniques used in this paper however seem not suited towards proving a result of this type.

Proofs

Throughout this section, to improve readability, we will assume for every proof that $\sigma^2 = 1$. Below we use the following definitions, which are identically to Park and Phillips (1999):

$$N_n(\nu_n; a, b) = \int_0^1 I(a \leq \nu_n W_n([rn]) \leq b)dr = n^{-1} \sum_{t=1}^{n} I(a \leq \nu_n W_n(t/n) \leq b),$$

and

$$N(\nu_n; a, b) = \int_0^1 I(a \leq \nu_n n^{-1/2} W(r) \leq b)dr.$$

In the proofs below, M and C are the constants from Lemma 1. The following lemma from Park and Phillips (1999) was needed in order to prove our results.

Lemma 2 *Under Assumption 2, as $n \to \infty$,*

$$E(N_n(\nu_n; 0, \delta) - N_n(\nu_n; k\delta, (k+1)\delta))^2 \leq c(\delta n^{-1}\nu_n^{-1})(1 + k\delta^2 n \log(n)\nu_n^{-2})$$

and

$$N_n(\nu_n; 0, \pi_n) = N(\nu_n; 0, \pi_n) + o_p(n^{-(2p-1)/3p+\varepsilon})$$

for $\pi_n \geq \nu_n n^{-2(p+1)/3p}$ and any $\varepsilon > 0$.

Proof:

See Park and Phillips (1999). □

We are now in a position to prove the main theorems of this paper.

Proof of Theorem 2:

Define $T_K(x) = T(x)I(|x| \leq K)$, $T'_K(x) = T(x)I(x > K)$, and $T''_K(x) = T(x)I(x < -K)$. We will show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} E|n^{-1/2-\alpha} \sum_{t=1}^{n} T'_K(n^{-\alpha+1/2}W_n(t/n))| = 0 \tag{2}$$

and the same argument, mutatis mutandis, will hold for $n^{-1/2-\alpha} \sum_{t=1}^{n} T''_K(n^{-\alpha+1/2}W_n(t/n))$. Then, we will show that for all $K > 0$,

$$n^{-1/2-\alpha} \sum_{t=1}^{n} T_K(n^{-\alpha+1/2}W_n(t/n)) \xrightarrow{d} (\int_{-K}^{K} T(s)ds)L(1,0), \tag{3}$$

and the result then follows (for a formal proof that this is sufficient, see for example the start of the proof of Theorem 1 of de Jong (2004)). Let M be as defined in Lemma 1. To show the result of Equation (2), note that for all $K > 0$,

$$|n^{-1/2-\alpha} \sum_{t=1}^{M} T(n^{-\alpha+1/2}W_n(t/n))I(n^{-\alpha+1/2}W_n(t/n) > K)| \leq Mn^{-1/2-\alpha}R(K) \to 0$$

as $n \to \infty$, and

$$E|n^{-1/2-\alpha} \sum_{t=M+1}^{n} T(n^{-\alpha+1/2}W_n(t/n))I(n^{-\alpha+1/2}W_n(t/n) > K)|$$

$$= E|\sum_{j=1}^{\infty} n^{-1/2-\alpha} \sum_{t=M+1}^{n} T(n^{-\alpha+1/2}W_n(t/n))I(Kj < n^{-\alpha+1/2}W_n(t/n) \leq K(j+1))|$$

$$\leq E\sum_{j=1}^{\infty} n^{-1/2-\alpha} \sum_{t=M+1}^{n} R(Kj)I(Kjt^{-1/2}n^{\alpha} < t^{-1/2}n^{1/2}W_n(t/n) \leq K(j+1)t^{-1/2}n^{\alpha})$$

$$\leq \sum_{j=1}^{\infty} n^{-1/2} \sum_{t=1}^{n} R(Kj)CKt^{-1/2}$$

$$\leq C(\sup_{n\geq 1} n^{-1/2} \sum_{t=1}^{n} t^{-1/2})K \sum_{j=1}^{\infty} R(Kj)$$

$$= C' \int_1^\infty R(K[j])d(Kj)$$

$$= C' \int_K^\infty R(K[x/K])dx = C' \int_K^{2K} R(K[x/K])dx + C' \int_{2K}^\infty R(K[x/K])dx$$

$$\leq C'(KR(K) + \int_K^\infty R(x)dx) \to 0$$

as $K \to \infty$, where $C' = C \sup_{n \geq 1} n^{-1/2} \sum_{t=1}^n t^{-1/2}$, and $KR(K) \to 0$ under the assumptions of the theorem because

$$R(2K)K \leq \int_K^{2K} R(x)dx \leq \int_K^\infty R(x)dx \to 0$$

as $K \to \infty$. The first inequality follows from the assumed boundedness of $|T(.)|$ by $R(.)$ and the assumed monotonicity of $R(.)$, and the second is an application of Lemma 1. This completes the proof of the result of Equation (2). The remainder of the proof follows the line of proof of Park and Phillips (1999, proof of Theorem 5.1), but some modifications will be made. In order to show the result of Equation (3) and thereby make the proof of Theorem 2 complete, define for $\delta > 0$

$$T^\delta(x) = \int_{-K/\delta}^{K/\delta - 1} T(j\delta) I(j\delta \leq n^{-\alpha+1/2} W_n(t/n) \leq (j+1)\delta)dj,$$

and note that for all $K > 0$,

$$\int_{-K/\delta}^{K/\delta - 1} I(j\delta \leq n^{-\alpha+1/2} W_n(t/n) \leq (j+1)\delta)dj = I(|n^{-\alpha+1/2} W_n(t/n)| \leq K),$$

and therefore

$$E|n^{-1/2-\alpha} \sum_{t=1}^n (T_K(n^{-\alpha} x_t) - T^\delta(n^{-\alpha} x_t))|$$

$$= E| \int_{-K/\delta}^{K/\delta - 1} n^{-1/2-\alpha} \sum_{t=1}^n (T(j\delta) - T(n^{-\alpha} x_t)) I(j\delta \leq n^{-\alpha+1/2} W_n(t/n) \leq (j+1)\delta)dj|$$

$$\leq \sup_{x \in [-K,K]} \sup_{x' \in [-K,K]: |x-x'| \leq \delta} |T(x) - T(x')|$$

$$\times E \int_{-K/\delta}^{K/\delta-1} n^{-1/2-\alpha} \sum_{t=1}^{n} I(j\delta \leq n^{-\alpha+1/2} W_n(t/n) \leq (j+1)\delta) dj$$

$$= \sup_{x \in [-K,K]} \sup_{x' \in [-K,K]:|x-x'| \leq \delta} |T(x) - T(x')| n^{-1/2-\alpha} \sum_{t=1}^{n}$$

$$\times P(-n^{\alpha} t^{-1/2} K \leq t^{-1/2} n^{1/2} W_n(t/n) \leq n^{\alpha} t^{-1/2} K)$$

$$\leq \sup_{x \in [-K,K]} \sup_{x' \in [-K,K]:|x-x'| \leq \delta} |T(x) - T(x')| n^{-1/2} \sum_{t=1}^{n} 2CKt^{-1/2}$$

$$\leq 2C'K \sup_{x \in [-K,K]} \sup_{x' \in [-K,K]:|x-x'| \leq \delta} |T(x) - T(x')| \to 0$$

as $\delta \to 0$ by continuity of $T(.)$, where the second inequality is Lemma 1. Therefore, we can consider $n^{-1/2-\alpha} \sum_{t=1}^{n} T^{\delta}(n^{-\alpha+1/2} W_n(t/n))$ instead of $n^{-1/2-\alpha} \sum_{t=1}^{n} T_K(n^{-\alpha+1/2} W_n(t/n))$. Now

$$n^{-1/2-\alpha} \sum_{t=1}^{n} T^{\delta}(n^{-\alpha+1/2} W_n(t/n))$$

$$= \int_{K/\delta}^{K/\delta-1} T(j\delta) n^{-1/2-\alpha} \sum_{t=1}^{n} I(j\delta \leq n^{-\alpha+1/2} W_n(t/n) \leq (j+1)\delta) dj$$

$$= \sum_{-K/\delta}^{K/\delta-1} T(j\delta) n^{1/2-\alpha} N_n(n^{1/2-\alpha}; j\delta, (j+1)\delta),$$

and

$$| \int_{-K/\delta}^{K/\delta-1} T(j\delta) n^{1/2-\alpha} N_n(n^{1/2-\alpha}; j\delta, (j+1)\delta) dj$$

$$- \int_{-K/\delta}^{K/\delta-1} T(j\delta) dj n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta)| = o_p(1)$$

because by the Cauchy-Schwartz inequality,

$$E(\int_{-K/\delta}^{K/\delta-1} T(j\delta) n^{1/2-\alpha} N_n(n^{1/2-\alpha}; j\delta, (j+1)\delta) dj - \int_{-K/\delta}^{K/\delta-1} T(j\delta) dj n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta))^2$$

$$\leq n^{1-2\alpha} \int_{-K/\delta}^{K/\delta} R(j\delta)^2 dj \int_{-K/\delta}^{K/\delta} E(N_n(n^{1/2-\alpha}; j\delta, (j+1)\delta) - N_n(n^{1/2-\alpha}; 0, \delta))^2 dj$$

$$\leq n^{1-2\alpha} \int_{-K/\delta}^{K/\delta} R(j\delta)^2 dj \int_{-K/\delta}^{K/\delta} c(\delta n^{-3/2+\alpha})(1 + |j|\delta^2 \log(n) n^{2\alpha}) dj$$

$$\leq n^{-1/2-\alpha}(1/\delta)(\int_{-K}^{K} R(s)^2 ds) c 2K(1 + K\delta n^{2\alpha} \log(n)) = o(1),$$

where the second inequality is Lemma 2. Therefore, it suffices to consider

$$\int_{-K/\delta}^{K/\delta-1} T(j\delta) dj n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta) = \delta^{-1} \int_{-K}^{K-\delta} T(s) ds n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta).$$

Now note that

$$|n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta) - n^{1/2-\alpha} N(n^{1/2-\alpha}; 0, \delta)| = o_p(n^{1/2-\alpha} n^{-(2p-1)/3p})$$

$$= o_p(n^{(1-p/2)/(3p)}) = o_p(1)$$

by the second part of Lemma 2. Therefore,

$$|\int_{-K}^{K-\delta} T(s) ds n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta) - \int_{-K}^{K-\delta} T(s) ds n^{1/2-\alpha} N(n^{1/2-\alpha}; 0, \delta)| = o_p(1),$$

implying that it suffices to analyze

$$(\int_{-K}^{K-\delta} T(s) ds)(\delta^{-1} n^{1/2-\alpha} N(n^{1/2-\alpha}; 0, \delta)).$$

As $n \to \infty$,

$$\delta^{-1} n^{1/2-\alpha} N(n^{1/2-\alpha}; 0, \delta) \to L(1, 0) \quad \text{almost surely,}$$

as explained in the text following Lemma 2.5 of Park and Phillips (1999). In addition, as $\delta \to 0$, by continuity of $T(.)$,

$$\int_{-K}^{K-\delta} T(s) ds \to \int_{-K}^{K} T(s) ds.$$

Therefore,

$$n^{-1/2-\alpha} \sum_{t=1}^{n} T_K(n^{-\alpha} x_t) \xrightarrow{d} (\int_{-K}^{K} T(s) ds) L(1, 0),$$

implying that the condition of Equation (3) is now verified. This completes the proof. $\square$

For the proof of Theorem 3, we need the following lemma:

Lemma 3 *Under Assumption 1,*

$$\sup_{x\in\mathbb{R}} |n^{-1}\sum_{t=1}^{n} I(W_n(t/n) \leq x) - \int_0^1 I(W(r) \leq x)dr| \xrightarrow{as} 0.$$

Proof of Lemma 3:

For n large enough, $\sup_{r\in[0,1]} |W_n(r) - W(r)| \leq \delta$ almost surely for any $\delta > 0$, implying that for n large enough

$$\sup_{x\in\mathbb{R}} |n^{-1}\sum_{t=1}^{n} I(W_n(t/n) \leq x) - \int_0^1 I(W(r) \leq x)dr|$$

$$= \sup_{x\in\mathbb{R}} |\int_0^1 I(W_n([rn]/n) \leq x) - \int_0^1 I(W(r) \leq x)dr|$$

$$\leq \sup_{x\in\mathbb{R}} |\int_0^1 I(x - \delta \leq W(r) \leq x + \delta)dr|$$

$$\leq \sup_{x\in\mathbb{R}} \int_{x-\delta}^{x+\delta} L(1, s)ds \leq 2\delta \sup_{s\in\mathbb{R}} |L(1, s)|$$

where the equality follows from the occupation times formula (see Park and Phillips (1999, Lemma 2.4)) and because $\sup_{s\in\mathbb{R}} |L(1, s)|$ is a well-defined random variable. Since δ can be chosen arbitrarily small, the result now follows. $\square$

Proof of Theorem 3:

Because $\sup_{1\le t\le n}|W_n(t/n)| = O_p(1)$, it now suffices to show that for any $K>0$,

$$\nu(n^{1/2-\alpha})^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-\alpha+1/2}W_n(t/n))I(|W_n(t/n)|\le K)$$

$$\xrightarrow{d} \int_0^1 H(W(r))I(|W(r)|\le K)dr = \int_{-K}^{K} H(s)L(1,s)ds.$$

Now, by Lemma 3, $n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \le x) \Rightarrow \int_0^1 I(W(r)\le x)dr$. By the Skorokhod Representation Theorem, we can assume without loss of generality that $\sup_{x\in\mathbb{R}}|n^{-1}\sum_{t=1}^{n} I(W_n(t/n) \le x) - \int_0^1 I(W(r)\le x)dr| = c_n \xrightarrow{as} 0$. Now for all $\delta>0$, let

$$S_{1n\delta} = S_{1n} = \nu(n^{1/2-\alpha})^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-\alpha+1/2}W_n(t/n))I(|W_n(t/n)|\le K)$$

$$= \nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta-1} n^{-1}\sum_{t=1}^{n} T(n^{-\alpha+1/2}W_n(t/n))I(j\delta \le W_n(t/n) \le (j+1)\delta)dj,$$

$$S_{2n\delta} = \nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta-1} T(n^{1/2-\alpha}j\delta)n^{-1}\sum_{t=1}^{n} I(j\delta \le W_n(t/n) \le (j+1)\delta)dj,$$

$$S_{3n\delta} = \nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta-1} T(n^{1/2-\alpha}j\delta)\int_0^1 I(j\delta \le W(r) \le (j+1)\delta)dr\,dj,$$

$$S_{4n\delta} = \nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta-1} T(n^{1/2-\alpha}j\delta)\delta L(1,j\delta)dj$$

$$= \nu(n^{1/2-\alpha})^{-1}\int_{-K}^{K-\delta} T(n^{1/2-\alpha}s)L(1,s)ds,$$

$$S_{5n\delta} = S_5 = \int_{-K}^{K} H(s)L(1,s)ds = \int_0^1 H(W(r))I(|W(r)|\le K)dr.$$

We will show that $\lim_{\delta\to 0}\limsup_{n\to\infty}|S_{jn\delta} - S_{j+1,n\delta}| = 0$ almost surely for $j = 1,\ldots,4$. By the monotone regular condition, we can act as if $T(.)$ is monotone without loss of generality. For $|S_1 - S_{2n\delta}|$ we then have

$$\limsup_{n\to\infty}|S_1 - S_{2n\delta}|$$

$$\leq \limsup_{n\to\infty} \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta-1} n^{-1} \sum_{t=1}^{n} |T(n^{-\alpha+1/2} W_n(t/n)) - T(n^{1/2-\alpha} j\delta)|$$

$$\times I(j\delta \leq W_n(t/n) \leq (j+1)\delta) dj$$

$$\leq \limsup_{n\to\infty} \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta-1} n^{-1} \sum_{t=1}^{n} |T(n^{1/2-\alpha}(j+1)\delta) - T(n^{1/2-\alpha} j\delta)|$$

$$\times I(j\delta \leq W_n(t/n) \leq (j+1)\delta) dj$$

$$\leq \limsup_{n\to\infty} \int_{-K/\delta}^{K/\delta-1} |\nu(n^{1/2-\alpha})^{-1} T(n^{1/2-\alpha}(j+1)\delta)$$

$$-\nu(n^{1/2-\alpha})^{-1} T(n^{1/2-\alpha} j\delta) - H((j+1)\delta) + H(j\delta)| dj$$

$$+ \int_{-K/\delta}^{K/\delta-1} |H((j+1)\delta) - H(j\delta)| dj = \int_{-K}^{K-\delta} |H(x+\delta) - H(x)| dx,$$

and as $\delta \to 0$, the last term disappears because of continuity of $H(.)$, the second inequality follows from monotonicity of $T(.)$, and the third by our definition of an asymptotically homogeneous function. To show that $\lim_{\delta\to 0} \limsup_{n\to\infty} |S_{2n\delta} - S_{3n\delta}| = 0$ almost surely, note that

$$|\nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta-1} T(n^{1/2-\alpha} j\delta)(n^{-1} \sum_{t=1}^{n} I(j\delta \leq W_n(t/n) \leq (j+1)\delta)$$

$$- \int_0^1 I(j\delta \leq W(r) \leq (j+1)\delta) dr) dj|$$

$$\leq 2c_n \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta-1} |T(n^{1/2-\alpha} j\delta)| dj$$

$$\leq 2c_n \delta^{-1} \int_{-K}^{K} |\nu(n^{1/2-\alpha})^{-1} T(n^{1/2-\alpha} x) - H(x)| dx + 2c_n \delta^{-1} \int_{-K}^{K} |H(x)| dx = o(1)$$

almost surely under our assumptions and by the definition of c_n. For $|S_{3n\delta} - S_{4n\delta}|$ we have

$$|S_{3n\delta} - S_{4n\delta}|$$

$$\leq \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta} \delta |T(n^{1/2-\alpha} j\delta)| (\delta^{-1} \int_0^1 I(j\delta \leq W(r) \leq (j+1)\delta) dr - L(1, j\delta)) dj$$

$$\leq \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta} \delta |T(n^{1/2-\alpha} j\delta)| dj \sup_{|x| \leq K} |\delta^{-1} \int_0^1 I(x \leq W(r) \leq x+\delta) dr - L(1, x)|. (4)$$

By the earlier argument,

$$\sup_{n \geq 1} \sup_{\delta > 0} \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta} \delta |T(n^{1/2-\alpha} j\delta)| dj < \infty,$$

and therefore it suffices to show that as $\delta \to 0$,

$$\sup_{|x| \leq K} |\delta^{-1} \int_0^1 I(x \leq W(r) \leq x + \delta) dr - L(1, x)| \to 0.$$

By the occupation times formula, the above expression satisfies

$$\sup_{|x| \leq K} |\delta^{-1} \int_x^{x+\delta} L(1, s) ds - L(1, x)| = \sup_{|x| \leq K} |\delta^{-1} \int_x^{x+\delta} (L(1, s) - L(1, x)) ds|$$

$$\leq \sup_{|x| \leq K} \sup_{s \in [x, x+\delta]} |L(1, s) - L(1, x)| \to 0 \quad \text{as } \delta \to 0$$

by uniform continuity of $L(1, .)$ on $[-K, K]$. Finally, for $|S_{4n\delta} - S_5|$, we have

$$\lim_{n \to \infty} |\int_{-K}^{K} (\nu(n^{1/2-\alpha})^{-1} T(n^{1/2-\alpha} s) - H(s)) L(1, s) ds|$$

$$\leq \sup_{|s| \leq K} |L(1, s)| \lim_{n \to \infty} \int_{-K}^{K} |\nu(n^{1/2-\alpha})^{-1} T(n^{1/2-\alpha} s) - H(s)| ds = 0$$

by the definition of an asymptotically homogeneous function, which completes the proof. □

The following lemma is needed for the proof of Theorem 4.

Lemma 4 *For any sequence b_n such that $c_n = o(b_n)$, under the assumptions of Theorem 4,*

$$\lim_{\delta \to 0} \lim_{n \to \infty} \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) d_n^{-1} \delta c_n = 1.$$

Proof of Lemma 4:

This result follows because

$$\sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) d_n^{-1} \delta c_n$$

$$= \int_{j=0}^{\infty} T([j]\delta c_n) I(([j]+1)\delta c_n > c_n) I([j]\delta c_n \leq b_n) d_n^{-1} \delta c_n dj$$

$$\leq \int_{j=0}^{\infty} T((j-1)\delta c_n) I((j+1)\delta c_n > c_n) I((j-1)\delta c_n \leq b_n) d_n^{-1} \delta c_n dj$$

$$= \int_{x=0}^{\infty} T(x) I(x + 2\delta c_n > c_n) I(x \leq b_n) d_n^{-1} dx$$

$$= (\int_{x=c_n}^{1} T(x)dx)^{-1} \int_{x=c_n(1-2\delta)}^{b_n} T(x)dx.$$

Now because $T(x) = |x|^{-m} I(x > 0)$, the last expression equals

$$(m-1)(c_n^{1-m} - 1)^{-1}(m-1)^{-1}((c_n(1-2\delta))^{1-m} - b_n^{1-m})$$

$$= (c_n^{1-m} - 1)^{-1}((c_n(1-2\delta))^{1-m} - b_n^{1-m}),$$

and because $m > 1$ and $c_n = o(b_n)$, the result now follows. A similar argument will hold for a lower bound, which then completes the proof of the lemma. □

Proof of Theorem 4:

Note that, for $b_n = c_n^{1-1/m-\alpha}$ for some $\alpha > 0$ small enough that $b_n \to 0$ and $d_n^{-1} T(b_n) \to 0$ as $n \to \infty$,

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(W_n(t/n)) I(W_n(t/n) > c_n)$$

$$= d_n^{-1} n^{-1} \sum_{t=1}^{n} T(W_n(t/n)) I(W_n(t/n) > c_n) I(W_n(t/n) \leq b_n)$$

$$+d_n^{-1}n^{-1}\sum_{t=1}^{n}T(W_n(t/n))I(W_n(t/n)>b_n),$$

and the second term is $o_p(1)$ because

$$d_n^{-1}n^{-1}\sum_{t=1}^{n}T(W_n(t/n))I(W_n(t/n)>b_n)$$

$$\leq d_n^{-1}T(b_n)\to 0$$

by assumption. Now note that trivially, for all $\delta>0$,

$$d_n^{-1}n^{-1}\sum_{t=1}^{n}T(W_n(t/n))I(W_n(t/n)>c_n)I(W_n(t/n)\leq b_n)$$

$$=\sum_{j=0}^{\infty}d_n^{-1}\int_0^1 T(W_n(r))I(W_n(r)>c_n)I(W_n(r)\leq b_n)I(j\delta c_n\leq W_n(r)<(j+1)\delta c_n)dr.$$

An upper bound for the last term is

$$\sum_{j=0}^{\infty}T(j\delta c_n)d_n^{-1}\int_0^1 I(W_n(r)>c_n)I(W_n(r)\leq b_n)I(j\delta c_n\leq W_n(r)<(j+1)\delta c_n)dr$$

$$\leq\sum_{j=0}^{\infty}T(j\delta c_n)I((j+1)\delta c_n>c_n)I(j\delta c_n\leq b_n)d_n^{-1}\int_0^1 I(j\delta c_n\leq W_n(r)<(j+1)\delta c_n)dr$$

$$=\sum_{j=0}^{\infty}T(j\delta c_n)I((j+1)\delta c_n>c_n)I(j\delta c_n\leq b_n)d_n^{-1}N_n(1;j\delta c_n,(j+1)\delta c_n).$$

Similarly, a lower bound is

$$\sum_{j=0}^{\infty}T((j+1)\delta c_n)I(j\delta c_n>c_n)I((j+1)\delta c_n\leq b_n)d_n^{-1}N_n(1;j\delta c_n,(j+1)\delta c_n).$$

We will only consider the upper bound and determine its limit, but the argument for the lower bound is identical and renders the same limit. By Lemma 2,

$$E\sum_{j=0}^{\infty}T(j\delta c_n)I((j+1)\delta c_n>c_n)I(j\delta c_n\leq b_n)d_n^{-1}|N_n(1;j\delta c_n,(j+1)\delta c_n)-N_n(1;0,\delta c_n)|$$

$$\leq \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) d_n^{-1}(c(\delta c_n/n)(1 + (j(\delta c_n)^2 n \log(n))))^{1/2}$$

$$\leq (d_n^{-1}\delta c_n \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n))$$

$$\times \delta^{-1} c_n^{-1}(c(\delta c_n/n)(1 + ((b_n/(\delta c_n))(\delta c_n)^2 n \log(n))))^{1/2}. \quad (5)$$

Now, by Lemma 4,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} d_n^{-1} \delta c_n \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) = 1,$$

and therefore the expression of Equation (5) converges to zero in probability if

$$c_n^{-2}((c_n/n) + (c_n/n)((b_n/(c_n))(c_n)^2 n \log(n))) \to 0.$$

First, note that by assumption $c_n^{-1} n^{-1} \to 0$, and that the second part of the above expression is

$$O(b_n \log(n)) = o(1)$$

by assumption. Therefore, it suffices to consider

$$\sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) d_n^{-1} \delta c_n (N_n(1; 0, \delta c_n)/(\delta c_n)).$$

Now by the comment following Lemma 2.5 in Park and Phillips (1999),

$$N_n(1; 0, \delta c_n)/(\delta c_n) = L(1, 0) + o_p(1)$$

if $\delta c_n \geq n^{-(2p-1)/3p+\eta}$ for some $\eta > 0$, which is the case by assumption for n large enough. Therefore, we only need consider

$$L(1, 0) d_n^{-1} \delta c_n \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n).$$

Now by Lemma 4, it follows that by choosing δ arbitrarily small, the limit distribution will be arbitrarily close to $L(1, 0)$; and noting that the same argument will work for the lower bound, this suffices to prove the result. □

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