Logarithmic spurious regressions

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Abstract

Spurious regressions, i.e. regressions in which an integrated process is regressed on another integrated process while there is no cointegration, are well understood in contemporary time series econometrics. In this paper, I investigate the properties of regressions in which the logarithm of an integrated process is regressed on the logarithm of another integrated process. It is shown that, exactly as in spurious regressions, in this setup too, the estimated slope coefficient is asymptotically random and the t-value for the slope coefficient is of a stochastic order equal to the square root of sample size. Therefore, spurious regressions results can alternatively be explained as an artifact of regressing the logarithm of an integrated process on the logarithm of another integrated process.

1 Introduction

In this paper, an alternative type of spurious regressions is considered. The issue of spurious regressions is well-documented; it was considered by Granger and Newbold (1974) and a first full mathematical analysis appeared in Phillips (1986). For an accessible explanation of the mathematics of spurious regressions the reader is referred to Hamilton (1994), paragraph 18.3. Spurious regressions occur when two (possibly independent) integrated processes x_t and y_t are regressed on each other. In that case, we find that (i) the estimated slope coefficient is asymptotically random; (ii) the t-value for the slope coefficient $\hat{t}_2$ of the regression of y_t on x_t

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with intercept satisfies $n^{-1/2}\hat{t}_2 \xrightarrow{d} T$ for a nondegenerate random variable T. Therefore, we may spuriously reject the null hypothesis of a zero slope coefficient in a regression of y_t on x_t. Here, I consider the analysis of regressions in which $\log|y_t|$ is regressed on $\log|x_t|$, where x_t and y_t are integrated processes. Given the tendency of applied time series econometricians to apply logarithmic transformations in many settings, this analysis may quite well approximate econometric practice in some instances.

An important tool that is used is a result of de Jong (2001), where it is established that for general integrated processes x_t, and functions $T(.)$ that are explicitly allowed to have a pole, we have under some regularity conditions

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma_x W_x(r))dr \tag{1}$$

where $\sigma_x^2 = \lim_{n\to\infty} E(n^{-1/2}x_n)^2$ and $W_x(.)$ denotes the Brownian motion process associated with the x_t, as long as $\int_{-K}^{K} |T(x)|dx < \infty$ for all $K > 0$. Note that this result is not as straightforward as it may seem at first sight; the possible pole in $T(.)$ makes that the continuous mapping theorem cannot be applied to obtain the above result. Examples of the above result are $T(x) = \log|x|$ and $T(x) = (\log|x|)^2$; clearly $\int_{-K}^{K} |\log|x||dx < \infty$ and $\int_{-K}^{K} |\log|x||^2 dx < \infty$ for all $K > 0$, and therefore it follows that

$$n^{-1}\sum_{t=1}^{n} \log|n^{-1/2}x_t| \xrightarrow{d} \int_0^1 \log|\sigma_x W_x(r)|dr \tag{2}$$

and

$$n^{-1}\sum_{t=1}^{n} (\log|n^{-1/2}x_t|)^2 \xrightarrow{d} \int_0^1 (\log|\sigma_x W_x(r)|)^2 dr. \tag{3}$$

Therefore, the usual type of result that is typically obtained through an application of the functional central limit theorem and the continuous mapping theorem will remain valid for the logarithmic function and its square, even though the functional central limit theorem cannot be applied directly. Note that for a function such as $T(x) = |x|^{-2}$, this type of reasoning is incorrect, because for such a function $T(.)$, $\int_0^1 |W(r)|^{-2} dr = \infty$ almost surely. Similar results extending those of de Jong (2001) have recently been obtained by Pötscher (2001).

2 Main results

Similarly to Park and Phillips (1999), it is assumed that

$$x_t = x_{t-1} + w_t \quad \text{and} \quad y_t = y_{t-1} + v_t, \tag{4}$$

where w_t and v_t are generated according to

$$w_t = \sum_{k=0}^{\infty} \phi_k^x \varepsilon_{t-k} \tag{5}$$

and

$$v_t = \sum_{k=0}^{\infty} \phi_k^y \eta_{t-k} \tag{6}$$

where (ε_t, η_t) is assumed to be a sequence of i.i.d. random vectors with mean zero and a nonsingular covariance matrix Σ, and where it is assumed that $\sum_{k=0}^{\infty} \phi_k^x \neq 0$ and $\sum_{k=0}^{\infty} \phi_k^y \neq 0$. In addition, I will assume that (x_0, y_0) is an arbitrary random vector that is independent of all (w_t, v_t), $t \geq 1$. The main assumption in this paper is similar to Assumption 2.2 from Park and Phillips (1999):

Assumption 1

(a) $\sum_{k=0}^{\infty} k(|\phi_k^x| + |\phi_k^y|) < \infty$ *and* $E|\varepsilon_t|^p + E|\eta_t|^p < \infty$ *for some* $p > 2$.

(b) *The distributions of* ε_t *and* η_t *are absolutely continuous with respect to the Lebesgue measure and have characteristic functions* $\psi_x(s)$ *and* $\psi_y(s)$ *for which* $\lim_{s\to\infty} s^\eta(\psi_x(s) + \psi_y(s)) = 0$ *for some* $\eta > 0$.

Assumption 1 implies that for $n \geq M$ for some fixed value of M, the densities of $n^{-1/2}x_n$ and $n^{-1/2}y_n$ will both exist and be bounded. This assumption is a multivariate version of the assumption used in de Jong (2001) and Park and Phillips (1999). Assumption 1 also implies that

$$(n^{-1/2}x_{[nr_1]}, n^{-1/2}y_{[nr_2]}) \Rightarrow (\sigma_x W_x(r_1), \sigma_y W_y(r_2)),$$

where "$\Rightarrow$" denotes weak convergence and $W_x(.)$ and $W_y(.)$ denote the Brownian motions associated with the x_t and the y_t respectively, and σ_y^2 is defined analogously to σ_x^2.

Consider a regression where $\log|y_t|$ is regressed on $\log|x_t|$ and an intercept. Let $\hat{\beta}_1$ and $\hat{\beta}_2$ denote the estimated intercept and slope coefficient, and define $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)'$. The asymptotic behavior of $\hat{\beta}$ is as follows:

Theorem 1 *Under Assumption 1,*

$$\hat{\beta}_2 \xrightarrow{d} \frac{\int_0^1 (\log|\sigma_x W_x(s)| - \int_0^1 \log|\sigma_x W_x(r)|dr)(\log|\sigma_y W_y(s)| - \int_0^1 \log|\sigma_x W_x(r)|dr)ds}{\int_0^1 (\log|\sigma_x W_x(s)| - \int_0^1 \log|\sigma_x W_x(r)|dr)^2 ds} \tag{7}$$

and denoting the above limit random variable by B_2, we have

$$(\log(n))^{-1}\hat{\beta}_1 \xrightarrow{d} (1/2) - (1/2)B_2. \tag{8}$$

In order to determine the limit behavior of the t-statistic for $\hat{\beta}_2$, the following lemma is needed. Let s^2 denote the usual regression error variance estimator.

Lemma 1 *Under Assumption 1,*

$$s^2 \xrightarrow{d} S, \tag{9}$$

for some nondegenerate random variable S.

Finally, I will establish the limit behavior of the t-statistics. Let $\hat{t}_1$ and $\hat{t}_2$ denote the usual t-statistics for $\hat{\beta}_1$ and $\hat{\beta}_2$.

Theorem 2 *Under Assumption 1,*

$$n^{-1/2}\hat{t}_1 \xrightarrow{d} T_1 \quad \textit{and} \quad n^{-1/2}\hat{t}_2 \xrightarrow{d} T_2, \tag{10}$$

for nondegenerate random variables T_1 and T_2.

3 Conclusions

Theorem 1 and Theorem 2 provide results similar to the results that can be obtained for spurious regressions; the estimated slope coefficient is asymptotically random, and Theorem 2 implies that the t-value will asymptotically tend to infinity as sample size increases at the rate $n^{1/2}$. This result is similar to the situation for spurious regressions, where the t-value for the slope coefficient is also of stochastic order $n^{1/2}$. Therefore, the conclusion has to be that the spurious regression effect that is observed in applied time series scenarios can equally well be explained as an artifact of having regressed the logarithm of an integrated process on the logarithm of another integrated process.

Appendix: Mathematical Proofs

Proof of Theorem 1:

Note that

$$\hat{\beta}_2 = \frac{\sum_{t=1}^{n}(\log|x_t| - \overline{\log|x_t|})(\log|y_t| - \overline{\log|y_t|})}{\sum_{t=1}^{n}(\log|x_t| - \overline{\log|x_t|})^2}$$

$$= \frac{n^{-1}\sum_{t=1}^{n}(\log|n^{-1/2}x_t| - \overline{\log|n^{-1/2}x_t|})(\log|n^{-1/2}y_t| - \overline{\log|n^{-1/2}y_t|})}{n^{-1}\sum_{t=1}^{n}(\log|n^{-1/2}x_t| - \overline{\log|n^{-1/2}x_t|})^2}, \quad (11)$$

where $\overline{a}_t$ denotes $n^{-1}\sum_{t=1}^{n} a_t$. By applying the results of de Jong (2001) as quoted in Equations (2) and (3) and noting that the convergence in distribution of the various summations is joint, the only result that remains to be proven is

$$n^{-1}\sum_{t=1}^{n} \log|n^{-1/2}x_t| \log|n^{-1/2}y_t| \xrightarrow{d} \int_0^1 \log|\sigma_y W_y(r)| \log|\sigma_x W_x(r)| dr. \quad (12)$$

Note that $\max_{1\leq t\leq n}|n^{-1/2}x_t| + \max_{1\leq t\leq n}|n^{-1/2}y_t| = O_P(1)$, and therefore with arbitrarily large probability, the above statistic equals

$$n^{-}\sum_{t=1}^{n} \log|n^{-1/2}x_t| \log|n^{-1/2}y_t| I(|n^{-1/2}x_t| \leq K) I(|n^{-1/2}y_t| \leq K), \quad (13)$$

and therefore it suffices to show that

$$n^{-1}\sum_{t=1}^{n} \log|n^{-1/2}x_t| \log|n^{-1/2}y_t| I(|n^{-1/2}x_t| \leq K) I(|n^{-1/2}y_t| \leq K)$$

$$\xrightarrow{d} \int_0^1 \log|\sigma_y W_y(r)| \log|\sigma_x W_x(r)| I(|\sigma_x W_x(r)| \leq K) I(|\sigma_y W_y(r)| \leq K) dr. \quad (14)$$

Note that the combination of the functional central limit theorem and the continuous mapping theorem fails to obtain this result, because of the poles of the logarithmic functions at 0. The strategy of the proof will be to apply the functional central limit theorem and the continuous mapping theorem to conclude that

$$n^{-1}\sum_{t=1}^{n} f_\delta(n^{-1/2}x_t) f_\delta(n^{-1/2}y_t) I(|n^{-1/2}x_t| \leq K) I(|n^{-1/2}y_t| \leq K)$$

$$\xrightarrow{d} \int_0^1 f_\delta(\sigma_x W_x(r)) f_\delta(\sigma_y W_y(r)) I(|\sigma_x W_x(r)| \le K) I(|\sigma_y W_y(r)| \le K) dr \tag{15}$$

for a bounded and continuous approximation $f_\delta(.)$ of $\log|x|$, and show that differences are negligible asymptotically. Furthermore note that, for M as defined in the text following Assumption 1,

$$n^{-1} \sum_{t=1}^{M} \log|n^{-1/2}x_t| \log|n^{-1/2}y_t| I(|n^{-1/2}x_t| \le K) I(|n^{-1/2}y_t| \le K) = o_P(1). \tag{16}$$

For any $\delta > 0$, let

$$f_\delta(x) = \log|x| I(|x| > \delta) + \log(\delta) I(|x| \le \delta). \tag{17}$$

I will now show that

$$E|n^{-1} \sum_{t=M+1}^{n} \log|n^{-1/2}x_t| \log|n^{-1/2}y_t| I(|n^{-1/2}x_t| \le K) I(|n^{-1/2}y_t| \le K)$$

$$-n^{-1} \sum_{t=M+1}^{n} f_\delta(n^{-1/2}x_t) f_\delta(n^{-1/2}y_t) I(|n^{-1/2}x_t| \le K) I(|n^{-1/2}y_t| \le K)| \le h(\delta) \tag{18}$$

for some function $h(.)$ that does not depend on n such that $h(\delta) \to 0$ as $\delta \to 0$, and also that

$$\int_0^1 f_\delta(\sigma_x W_x(r)) f_\delta(\sigma_y W_y(r)) dr \xrightarrow{as} \int_0^1 \log|\sigma_y W_y(r)| \log|\sigma_x W_x(r)| dr \quad \text{as} \quad \delta \to 0, \tag{19}$$

which suffices to prove the result.
To show the result of Equation (18), note that

$$n^{-1} \sum_{t=M+1}^{n} \log|n^{-1/2}x_t| \log|n^{-1/2}y_t| I(|n^{-1/2}y_t| \le K) I(|n^{-1/2}x_t| \le K)$$

$$= n^{-1} \sum_{t=M+1}^{n} (\log|n^{-1/2}x_t| - f_\delta(n^{-1/2}x_t)) \log|n^{-1/2}y_t| I(|n^{-1/2}x_t| \le K) I(|n^{-1/2}y_t| \le K)$$

$$+n^{-1} \sum_{t=M+1}^{n} f_\delta(n^{-1/2}x_t)(\log|n^{-1/2}y_t| - f_\delta(n^{-1/2}y_t)) I(|n^{-1/2}x_t| \le K) I(|n^{-1/2}y_t| \le K)$$

$$+n^{-1}\sum_{t=M+1}^{n} f_\delta(n^{-1/2}x_t)f_\delta(n^{-1/2}y_t)I(|n^{-1/2}x_t|\le K)I(|n^{-1/2}y_t|\le K). \tag{20}$$

I will only demonstrate the asserted result for the second term, but the proof for the third term is analogous. To deal with the first term, note that by the Cauchy-Schwartz inequality,

$$(E|n^{-1}\sum_{t=M+1}^{n}(\log|n^{-1/2}x_t| - f_\delta(n^{-1/2}x_t))\log|n^{-1/2}y_t|I(|n^{-1/2}x_t|\le K)I(|n^{-1/2}y_t|\le K))^2$$

$$\le En^{-1}\sum_{t=M+1}^{n}(\log|n^{-1/2}x_t| - f_\delta(n^{-1/2}x_t))^2 En^{-1}\sum_{t=M+1}^{n}(\log|n^{-1/2}y_t|)^2 I(|n^{-1/2}y_t|\le K). \tag{21}$$

Noting that the densities of $t^{-1/2}x_t$ and $t^{-1/2}y_t$ are both uniformly bounded over $t \ge M+1$, it follows that

$$En^{-1}\sum_{t=M+1}^{n}(\log|n^{-1/2}y_t|)^2 I(|n^{-1/2}y_t|\le K)$$

$$\le n^{-1}\sum_{t=M+1}^{n}\int_{-\infty}^{\infty}(\log|n^{-1/2}t^{1/2}r|)^2 I(|n^{-1/2}t^{1/2}r|\le K)dr$$

$$\le n^{-1}\sum_{t=1}^{n} n^{1/2}t^{-1/2}\int_{-K}^{K}(\log|s|)^2 ds < C \tag{22}$$

for some constant C. Therefore, the expression of Equation (21) can be bounded by

$$C\int_{-\infty}^{\infty} n^{-1}\sum_{t=M+1}^{n}(\log|n^{-1/2}t^{1/2}r| - f_\delta(n^{-1/2}t^{1/2}r))^2 dr$$

$$\le Cn^{-1}\sum_{t=M+1}^{n} t^{-1/2}n^{1/2}\int_{-\delta}^{\delta}(\log|s| - \log|\delta|)^2 ds$$

$$\le C'\int_{-\delta}^{\delta}(\log|s| - \log|\delta|)^2 ds \to 0 \quad \text{as} \quad \delta \to 0. \tag{23}$$

Now in order to show the result of Equation (19) and complete the proof, note that

$$\int_0^1 f_\delta(\sigma_x W_x(r))f_\delta(\sigma_y W_y(r))I(|\sigma_y W_y(r)|\le K)I(|\sigma_x W_x(r)|\le K)dr$$

$$= \int_0^1 (f_\delta(\sigma_x W_x(r)) - \log|\sigma_x W_x(r)|) f_\delta(\sigma_y W_y(r)) I(|\sigma_x W_x(r)| \le K) I(|\sigma_y W_y(r)| \le K) dr$$

$$+ \int_0^1 \log|\sigma_x W_x(r)| (f_\delta(\sigma_y W_y(r)) - \log|\sigma_y W_y(r)|) I(|\sigma_x W_x(r)| \le K) I(|\sigma_y W_y(r)| \le K) dr$$

$$+ \int_0^1 \log|\sigma_x W_x(r)| \log|\sigma_y W_y(r)| I(|\sigma_x W_x(r)| \le K) I(|\sigma_y W_y(r)| \le K) dr, \tag{24}$$

and the first and second term converge to zero almost surely as $\delta \to 0$. I will show this for the first term only. By the Cauchy-Schwartz inequality, the square of the first term can be bounded by

$$\int_0^1 (f_\delta(\sigma_x W_x(r)) - \log|\sigma_x W_x(r)|)^2 I(|\sigma_x W_x(r)| \le K) dr$$

$$\times \int_0^1 (\log|\sigma_y W_y(r)|)^2 I(|\sigma_y W_y(r)| \le K) dr, \tag{25}$$

and by the occupation times formula (see for example Park and Phillips (1999)), letting $L_y(.,.)$ denote the Brownian local time of $W_y(.)$,

$$\int_0^1 (\log|\sigma_y W_y(r)|)^2 I(|\sigma_y W_y(r)| \le K) dr$$

$$= \int_{-K}^{K} L(1,r)(\log|\sigma_y r|)^2 dr \le \sup_{|r| \le K} |L(1,r)| \int_{-K}^{K} (\log|\sigma_y r|)^2 dr = O_P(1), \tag{26}$$

while similarly,

$$\int_0^1 (f_\delta(\sigma_x W_x(r)) - \log|\sigma_x W_x(r)|)^2 I(|\sigma_x W_x(r)| \le K) dr$$

$$= \int_{-\infty}^{\infty} L_x(1,r)(f_\delta(\sigma_x r) - \log|\sigma_x r|)^2 I(|\sigma_x r| \le K) dr$$

$$\le \sup_{|r| \le K/\sigma_x} |L_x(1,r)| \int_{-\delta}^{\delta} (\log|\delta| - \log|r|)^2 dr \quad \text{as} \quad \delta \to 0, \tag{27}$$

which completes the proof of the asserted result for $\hat{\beta}_2$.
Now note that

$$\begin{aligned}\hat{\beta}_1 &= \overline{\log|y_t|} - \hat{\beta}_2\overline{\log|x_t|}\\ &= \overline{\log|n^{-1/2}y_t|} - \hat{\beta}_2\overline{\log|n^{-1/2}x_t|} - \log(n^{-1/2}) + \hat{\beta}_2\log(n^{-1/2}),\end{aligned} \tag{28}$$

and therefore

$$(\log(n))^{-1}\hat{\beta}_1 = o_P(1) + (1/2) - (1/2)\hat{\beta}_2 \xrightarrow{d} (1/2) - (1/2)B_2, \tag{29}$$

which completes the proof. □

Proof of Lemma 1:

This result follows from noting that

$$s^2 = (n-2)^{-1}\sum_{t=1}^{n}((\log|n^{-1/2}y_t| - \overline{\log|n^{-1/2}y_t|}) - \hat{\beta}_2(\log|n^{-1/2}y_t| - \overline{\log|n^{-1/2}y_t|}))^2, \tag{30}$$

and when working out the square in the above expression, all terms can be dealt with by Theorem 1 in combination with the results of Equations (2) and (3) to show convergence in distribution to a random limit S. □

Proof of Theorem 2:

Note that

$$\hat{t}_2 = n^{1/2}\hat{\beta}_2 s^{-1}(n^{-1}\sum_{t=1}^{n}(\log|n^{-1/2}x_t| - \overline{\log|n^{-1/2}x_t|})^2)^{-1/2}, \tag{31}$$

and because $\hat{\beta}_2$, s, and $n^{-1}\sum_{t=1}^{n}(\log|n^{-1/2}x_t| - \overline{\log|n^{-1/2}x_t|})^2$ all converge jointly in distribution, the result now follows. For showing the result for $\hat{t}_1$, note that

$$\hat{t}_1 = n^{1/2}\frac{(\log(n))^{-1}\hat{\beta}_1)(n^{-1}\sum_{t=1}^{n}(\log|n^{-1/2}x_t| - \overline{\log|n^{-1/2}x_t|})^2)^{1/2}}{(s^2(\log(n))^{-2}n^{-1}\sum_{t=1}^{n}(\log|x_t|)^2)^{1/2}}, \tag{32}$$

and because

$$(\log(n))^{-2} n^{-1} \sum_{t=1}^{n} (\log |y_t|)^2 \xrightarrow{p} (1/4) \tag{33}$$

and because all other terms converge in distribution, the result now follows.

□

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