Addendum to "Asymptotics for nonlinear transformations of integrated time series"*

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Abstract

Typically in time series econometrics, for many statistics, a rescaled integrated process is replaced with Brownian motion in order to find the limit distribution. For averages of functions of a rescaled integrated process, Park and Phillips have shown that this remains true for functions with poles, as long as a sample-size dependent region around the poles is excluded from consideration and the function is locally integrable. In this addendum, I show that there is no need for such a sample size dependent region around the pole in Park and Phillips' theorem, as long as the function under consideration is locally integrable.

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1 Introduction

The continuous mapping theorem ensures that for I(1) processes x_t,

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))dr \tag{1}$$

for continuous functions $T(.)$, where $\xrightarrow{d}$ denotes convergence in distribution, $n^{-1/2}x_{[rn]} \Rightarrow \sigma W(r)$ where $W(.)$ denotes Brownian motion and "$\Rightarrow$" denotes weak convergence, and $\sigma^2 = \lim_{n\to\infty} n^{-1}Ex_n^2$. The continuity of $T(.)$ results in a mapping $\Pi_T : f \mapsto \int_0^1 T(f(r))dr$ from $C[0,1]$ to $\mathbb{R}$ that is continuous, which implies that the continuous mapping theorem can be applied to obtain the result of Equation (1). But for functions $T(.)$ with poles such as $T(x) = |x|^\alpha$ for $-1 < \alpha < 0$, the mapping Π is not continuous. Here and everywhere in this paper, we define $T(.)$ to equal 0 at every pole. Theorem 3.2 of Park and Phillips (1999) shows that for functions that are "regular" (as defined in that paper) and locally integrable, the result of Equation (1) still holds. Among the functions that are dealt with by that theorem is, for example, $T(x) = I(x > 0)$. Park and Phillips' Theorem 3.2 however fails to show that

$$Z_n = n^{-1}\sum_{t=1}^{n} |n^{-1/2}x_t|^\alpha \xrightarrow{d} \int_0^1 |\sigma W(r)|^\alpha dr \tag{2}$$

for all $-1 < \alpha < 0$, and also fails to establish the result of Equation (1) for $T(x) = \log|x|$. For $\alpha \geq 0$, the above result follows from the continuous mapping theorem in a straightforward way, because $|x|^\alpha$ is continuous on $\mathbb{R}$ for $\alpha \geq 0$. But for $-1 < \alpha < 0$, the pole at 0 will make it impossible to apply the continuous mapping theorem directly, and this paper shows that under regularity conditions, the result of Equation (2) still holds. Note that since the right-hand side of Equation (2) is undefined for $\alpha \leq -1$, there is no hope of proving the result of Equation (2) for such values of α. Instead of the result of Equation (2), Park and Phillips (1999) showed that

$$n^{-1}\sum_{t=1}^{n} |n^{-1/2}x_t|^\alpha I(|n^{-1/2}x_t| > \delta_n) \xrightarrow{d} \int_0^1 |\sigma W(r)|^\alpha dr \tag{3}$$

for $\alpha > -1$, for a sequence δ_n that converges to 0 as $n \to \infty$. In this paper, I prove the result of Equation (1) for a function class that is different from that of Park and Phillips, and I show that the result of Equation (2) holds under the exact same assumptions that were needed in Park and Phillips (1999) for the proof of the result of Equation (3). Among those assumptions is one that guarantees that the increments w_t of x_t have an absolutely continuous distribution. In the main theorem, I show that as long as it is possible to "integrate over

the poles" of the function $T(.)$, the result of Equation (1) can still be proven. It should be noted that the result of Equation (3) and a similar result for the logarithm function were established in Park and Phillips' (1999) Remark 3.5(b), and also that the result of Equation (2) appears to be anticipated in Park and Phillips' (1999) Remark 3.5 (c) and was suggested to me in personal communication with Professor Phillips.
If $x_0 = 0$ and w_t is i.i.d. and equals 1 or -1 each with probability 0.5, then it can be shown that the result of Equation (2) does not hold, as was pointed out to me by Professor Phillips. This failure is because in this situation, $P(x_2 = 0) = 1/2$, implying that with probability exceeding 0.5, the statistic Z_n is not properly defined. The assumptions made in this note (which are identical to the assumptions of Park and Phillips (1999)) guarantee that w_t has an absolutely continuous distribution, thereby ruling out x_t such as above.

2 Main result

In this paper, as in Park and Phillips (1999), it is assumed that

$$x_t = x_{t-1} + w_t, \tag{4}$$

where w_t is generated according to

$$w_t = \sum_{k=0}^{\infty} \phi_k \varepsilon_{t-k} \tag{5}$$

where ε_t is assumed to be a sequence of i.i.d. random variables with mean zero, and where it is assumed that $\sum_{k=0}^{\infty} \phi_k \neq 0$. In addition, I will assume that x_0 is an arbitrary random variable that is independent of all w_t, $t \geq 1$. The main assumption in this paper is Assumption 2.2 from Park and Phillips (1999):

Assumption 1

(a) $\sum_{k=0}^{\infty} k|\phi_k| < \infty$ *and* $E|\varepsilon_t|^p < \infty$ *for some* $p > 2$.

(b) *The distribution of* ε_t *is absolutely continuous with respect to the Lebesgue measure and has characteristic function* $\psi(s)$ *for which* $\lim_{s\to\infty} s^{\eta}\psi(s) = 0$ *for some* $\eta > 0$.

Assumption 1 guarantees that $n^{-1/2}x_{[rn]} \Rightarrow \sigma W(r)$. In order to prove the main result, I needed the following key lemma:

Lemma 1 *Under Assumption 1, there exists a constant $C > 0$ and an index N, such that for all $y \in \mathbb{R}$, $\delta > 0$, and $n \geq N$,*

$$P(y \leq n^{-1/2}x_n \leq y + \delta) \leq C\delta. \tag{6}$$

Obviously, a result such as Lemma 1 will fail to hold for the discrete-valued x_t example at the end of the Introduction. A result such as Lemma 1 is called an "integro-local limit theorem". It was pointed out to me by a referee that the absolute continuity of the distribution of ε_t (which is implied by Assumption 1) and the full force of Lemma 1 may not be necessary in order to obtain the result of Theorem 1 below, and that a local limit theorem for lattice random variables (such as the theorem on page 262 in Paragraph 43 of Gnedenko (1967)) may also suffice for the proof of Theorem 1.
The main result of this paper is the following:

Theorem 1 *Assume that $\int_{-K}^{K}|T(x)|dx < \infty$ for all $K > 0$. In addition, assume that for some $q \geq 1$ there exists a grid $\{a_1, \ldots, a_q\}$, where $a_j < a_{j+1}$ for all $j = 1, \ldots, q-1$, such that $T(.)$ is continuous at any $x \in \mathbb{R}\backslash\{a_1, \ldots, a_q\}$, and monotone on (a_{j-1}, a_j) for $j = 1, \ldots, q+1$ (defining $a_0 = -\infty$ and $a_{q+1} = \infty$). Then under Assumption 1,*

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))dr. \tag{7}$$

A monotone function is defined as a function that is either nondecreasing or nonincreasing. The condition $\int_{-K}^{K}|T(x)|dx < \infty$ for all $K > 0$ is exactly the condition that $T(.)$ is locally integrable. A consequence of Theorem 1 are the following results, which render the improvement of the result of Park and Phillips (1999) mentioned in the Introduction.

Remark 1. Under Assumption 1, $n^{-1}\sum_{t=1}^{n}|n^{-1/2}x_t|^{\alpha} \xrightarrow{d} \int_0^1|\sigma W(r)|^{\alpha}dr$ for all $\alpha > -1$. This result follows immediately from Theorem 1 by choosing $q = 1$ and $a_1 = 0$, and noting that $|x|^{\alpha}$ is monotone on $(-\infty, 0)$ and $(0, \infty)$ and that $\int_{-K}^{K}|x|^{-\alpha}dx < \infty$ for all $K > 0$ for $\alpha > -1$.

Remark 2. Under Assumption 1, $n^{-1}\sum_{t=1}^{n}\log|n^{-1/2}x_t| \xrightarrow{d} \int_0^1 \log|\sigma W(r)|dr$. This follows by again choosing $a_1 = 0$ and $q = 1$ and noting that $\int_{-K}^{K}|\log|x||dx < \infty$ for all $K > 0$.

As far as this author is aware, the result of Remark 2 has not been established elsewhere. Remark 2 may be useful for the analysis of inference procedures when a logarithm transformation has been incorrectly applied to a time series process.

Theorem 1 suggests the theoretically interesting question as to what will be "minimal" conditions, in addition to the integrability condition on $T(.)$, for the result of Equation (1) to hold. After learning about the research of this note, Pötscher (2001) has worked on finding minimal conditions on $T(.)$ and minimal weak dependence conditions that will allow the result of Equation (1) to go through. The proof as presented in the Appendix seems to rely on the assumed monotonicity in order to obtain Theorem 1. In addition, another problem of interest is to establish what happens to the statistic of Equation (1) if $T(.)$ should not be locally integrable (for example, $T(x) = |x|^\alpha$ for $\alpha < -1$).

This note generated some comments and suggestions for extensions and future research by referees and readers. Referees have suggested ways to consider relaxation of the monotonicity condition on $T(.)$, the relaxation of my weak dependence conditions (even such as to include fractionally integrated processes), a line of proof that might be capable of dealing with innovation distributions that are not necessarily continuous, and extensions to multivariate functionals or functionals with more complex singularities, using a generalized function approach to additive functionals (see Skorokhod and Slobodenyuk (1970)). These suggestions and research topics are left to future research.

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Proofs

Proof of Lemma 1:

Note that by Theorem 6.2.1 of Chung (1974)

$$P(y \leq n^{-1/2}x_n \leq y+\delta) = \lim_{T\to\infty}(2\pi)^{-1}\int_{-T}^{T}(is)^{-1}(\exp(-isy)-\exp(-is(y+\delta)))f_n(s)ds \quad (8)$$

where $f_n(.)$ is the characteristic function of $n^{-1/2}x_n$. Now, note that

$$x_n = x_0 + \sum_{t=1}^{n}\sum_{k=0}^{\infty}\phi_k\varepsilon_{t-k} = x_0 + \sum_{t=1}^{n}\sum_{j=-\infty}^{t}\phi_{t-j}\varepsilon_j = x_0 + \sum_{j=-\infty}^{n}\varepsilon_j\sum_{t=1}^{n}\phi_{t-j}I(j\leq t), \quad (9)$$

and therefore

$$f_n(s) = E\exp(isn^{-1/2}x_n) = \prod_{j=-\infty}^{n}E\exp(is(n^{-1/2}x_0 + n^{-1/2}\varepsilon_j\sum_{t=1}^{n}\phi_{t-j}I(j\leq t))). \quad (10)$$

This in turn implies that

$$|f_n(s)| \leq \prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{t=1}^{n}\phi_{t-j}I(j\leq t))| = \prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|. \quad (11)$$

Therefore, noting that $s^\eta\psi(s)\to 0$ as $|s|\to\infty$ for some $\eta>0$, for λ such that $\lambda^{p-2} = (1/6)\sigma^2/E|\varepsilon_t|^p$ we have

$$P(y \leq n^{-1/2}x_n \leq y+\delta)$$

$$\leq (2\pi)^{-1}\int_{-\infty}^{\infty}|s|^{-1}|\exp(-isy)-\exp(-is(y+\delta))|\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$\leq (2\pi)^{-1}\int_{|s|\leq\lambda n^{1/2}/|2\sum_{l=0}^{\infty}\phi_l|}|s|^{-1}|\exp(-isy)-\exp(-is(y+\delta))|\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$+(2\pi)^{-1}\int_{|s|\geq\lambda n^{1/2}/|2\sum_{l=0}^{\infty}\phi_l|}|s|^{-1}|\exp(-isy)-\exp(-is(y+\delta))|\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$\leq (2\pi)^{-1}\delta \int_{|s|\leq \lambda n^{1/2}/|2\sum_{l=0}^{\infty}\phi_l|} \prod_{j=1}^{n} |\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$+(2\pi)^{-1}\delta \int_{|\xi|\geq \lambda/|2\sum_{l=0}^{\infty}\phi_l|} n^{1/2}\prod_{j=1}^{n} |\psi(\xi\sum_{l=0}^{n-j}\phi_l)|d\xi$$

$$= T_1 + T_2, \tag{12}$$

say, where in the last inequality I set $\xi = sn^{-1/2}$ and used the inequality $|\exp(ia)-\exp(ib)| \leq |a-b|$. To deal with T_2, note that because

$$\sum_{l=0}^{\infty}\phi_l \neq 0, \tag{13}$$

we have

$$|\sum_{l=0}^{k}\phi_l - \sum_{l=0}^{\infty}\phi_l| \leq (1/2)|\sum_{l=0}^{\infty}\phi_l| \tag{14}$$

for $k \geq M$ for some M. Therefore for $n \geq N = M + [2/\eta+1]$, under Assumption 1,

$$T_2 \leq (2\pi)^{-1}\delta \int_{|\xi|\geq \lambda/|2\sum_{l=0}^{\infty}\phi_l|}^{\infty} n^{1/2}\prod_{j=1}^{n-N} |\psi(\xi\sum_{l=0}^{n-j}\phi_l)|d\xi$$

$$\leq C'\delta n^{1/2}(\sup_{|\xi|\geq\lambda/4}|\psi(\xi)|)^{n-N-[2/\eta+1]} \int_{|\xi|\geq \lambda/|2\sum_{l=0}^{\infty}\phi_l|}^{\infty} \prod_{j=n-N-[2/\eta+1]+1}^{n-N} |\xi\sum_{l=0}^{n-j}\phi_l|^{-\eta}d\xi$$

$$\leq C'\delta n^{1/2}(\sup_{|\xi|\geq\lambda/4}|\psi(\xi)|)^{n-N-[2/\eta+1]} \int_{|\xi|\geq \lambda/|2\sum_{l=0}^{\infty}\phi_l|}^{\infty} |\xi|^{-2}|(1/2)\sum_{l=0}^{\infty}\phi_l|^{-2}d\xi$$

$$\leq c_n\delta, \tag{15}$$

say, where $c_n \to 0$ as $n \to \infty$ because

$$\sup_{|\xi|\geq\lambda/4}|\psi(\xi)| < 1, \tag{16}$$

C' denotes some positive constant not depending on y, δ or n, and the second inequality follows from the assumptions made on $\psi(.)$ in Assumption 1 and the fact that for j in the given index range, by the inequality of Equation (14),

$$|\sum_{l=0}^{n-j} \phi_l| \geq (1/2)|\sum_{l=0}^{\infty} \phi_l|. \tag{17}$$

The result of Equation (16) follows because by Theorem 6.4.7 of Chung (1974), characteristic functions of absolutely continuous random variables will not equal 1 at any point other than $\xi = 0$ if the underlying distribution is absolutely continuous.

We now apply Theorem 11.6 of Davidson (1994) in order to deal with T_1. Recalling that $\lambda^{p-2} = (1/6)\sigma^2/E|\varepsilon_t|^p$, it follows that for $|s|n^{-1/2} \leq \lambda$,

$$|\psi(n^{-1/2}s) - (1 - (1/2)n^{-1}\sigma^2 s^2)|$$

$$\leq E\min(n^{-1}s^2\varepsilon_t^2, (1/6)n^{-3/2}|s|^3|\varepsilon_t|^3)$$

$$= E\min(n^{-1}s^2\varepsilon_t^2, (1/6)n^{-3/2}|s|^3|\varepsilon_t|^3)(I(|s\varepsilon_t| \leq n^{1/2}) + I(|s\varepsilon_t| > n^{1/2}))$$

$$\leq n^{-1}s^2 E\varepsilon_t^2 I(|\varepsilon_t| > |s|^{-1}n^{1/2}) + (1/6)n^{-1}\sigma^2 s^2$$

$$\leq n^{-1}s^2(\lambda^{-1})^{2-p}E|\varepsilon_t|^p + (1/6)n^{-1}\sigma^2 s^2$$

$$= n^{-1}s^2(\lambda^{p-2}E|\varepsilon_t|^p + (1/6)\sigma^2)$$

$$= (1/3)n^{-1}s^2\sigma^2. \tag{18}$$

The first inequality here was Theorem 11.6 of Davidson (1994), and the third uses $E|X|^2 I(|X| > K) \leq E|X|^p K^{2-p}$. Therefore, $|\psi(n^{-1/2}s)| \leq 1 - (1/6)n^{-1}s^2\sigma^2$ for $|s|n^{-1/2} \leq \lambda$. Next, again assume that $n \geq N$, where N is defined after Equation (14). Therefore, for $|s|n^{-1/2} \leq \lambda/|2\sum_{l=0}^{\infty}\phi_l|$,

$$\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)| \leq \prod_{j=1}^{n-N}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|$$

$$\leq \exp(\sum_{j=1}^{n-N}\log(1 - (1/6)n^{-1}s^2\sigma^2(\sum_{l=0}^{n-j}\phi_l)^2))$$

$$\leq \exp(-(1/6)s^2\sigma^2 n^{-1}\sum_{j=1}^{n-N}(\sum_{l=0}^{n-j}\phi_l)^2), \tag{19}$$

where the last inequality uses that $\log(1+x) \leq x$ for $x > -1$, and it is easily verified that for $|s|n^{-1/2} \leq \lambda/|2\sum_{l=0}^{\infty}\phi_l|$ and $j \leq n-N$,

$$(1/6)n^{-1}s^2\sigma^2(\sum_{l=0}^{n-j}\phi_l)^2 \leq (3/32)\lambda^2\sigma^2 \leq 3/32. \tag{20}$$

This implies that

$$T_1 = (2\pi)^{-1}\delta\int_{|s|\leq\lambda n^{1/2}/|2\sum_{l=0}^{\infty}\phi_l|}\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$\leq (2\pi)^{-1}\delta\int_{-\infty}^{\infty}\exp(-(1/6)s^2\sigma^2 n^{-1}\sum_{j=1}^{n-N}(\sum_{l=0}^{n-j}\phi_l)^2)ds, \tag{21}$$

and because by the definition of N,

$$n^{-1}\sum_{j=1}^{n-N}(\sum_{l=0}^{n-j}\phi_l)^2 \geq (1/4)(\sum_{l=0}^{\infty}\phi_l)^2 > 0, \tag{22}$$

it now follows that for n large enough,

$$T_1 \leq C_1\delta. \tag{23}$$

In conclusion, we have now shown that for large enough n,

$$T_1 + T_2 \leq C\delta \tag{24}$$

for some $C > 0$ not depending on y, δ, or n, which establishes the result. □

Proof of Theorem 1:

First note that $Y_n \xrightarrow{d} Y$ if for all $\delta > 0$ and some $Y_{n\delta}$, $Y_{n\delta} \xrightarrow{d} Y_\delta$ as $n \to \infty$, $Y_\delta \xrightarrow{d} Y$ as $\delta \to 0$, and

$$\lim_{\delta\to 0}\lim_{n\to\infty} E|Y_{n\delta} - Y_n| = 0. \tag{25}$$

This is because

$$\begin{aligned}
&\lim_{n\to\infty} |E\exp(i\xi Y_n) - E\exp(i\xi Y)| \\
&\leq \lim_{\delta\to 0}\lim_{n\to\infty} |E\exp(i\xi Y_n) - E\exp(i\xi Y_{n\delta})| \\
&+ \lim_{\delta\to 0}\lim_{n\to\infty} |E\exp(i\xi Y_{n\delta}) - E\exp(i\xi Y_\delta)| \\
&+ \lim_{\delta\to 0} |E\exp(i\xi Y_\delta) - E\exp(i\xi Y)|.
\end{aligned} \tag{26}$$

Note that the second term and third term in the last equation converge to 0 by assumption, and the first term can be bounded by

$$\lim_{\delta\to 0}\lim_{n\to\infty} |\xi| E|Y_{n\delta} - Y_n|, \tag{27}$$

which equals 0 by assumption. For simplicity, I will assume that $\sigma = 1$ throughout this proof.
Next, I will argue that for any $j = 0, \ldots, q$,

$$\begin{aligned}
Y_n &= n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) I(a_j \leq n^{-1/2}x_t \leq a_{j+1}) \\
&\xrightarrow{d} \int_0^1 T(W(r)) I(a_j \leq W(r) \leq a_{j+1}) dr = Y.
\end{aligned} \tag{28}$$

The joint convergence in distribution of the $q + 1$ terms can be easily proven using the reasoning as given below for a single term.
Now define, for N as in Lemma 1, small enough $\delta > 0$, and for $j = 1, \ldots, q - 1$,

$$T_\delta(x) = T(x) I(a_j + \delta \leq x \leq a_{j+1} - \delta),$$

$$Y_{n\delta} = n^{-1}\sum_{t=N+1}^{n} T_\delta(n^{-1/2}x_t) + n^{-1}\sum_{t=1}^{N} T(n^{-1/2}x_t), \tag{29}$$

and

$$Y_\delta = \int_0^1 T_\delta(W(r))dr. \tag{30}$$

Note that the argument for $j = 0$ and $j = q$ is analogous (although for the proof, one needs to use that $\max_{1 \le t \le n} n^{-1/2}|x_t| = O_P(1)$). Then by Park and Phillips' (1999) Theorem 3.2, it follows that for all $\delta > 0$, as $n \to \infty$,

$$Y_{n\delta} = o_P(1) + n^{-1} \sum_{t=1}^{n} T_\delta(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T_\delta(W(r))dr = Y_\delta \tag{31}$$

because $T_\delta(x)$ is regular and locally integrable; the $o_P(1)$ assertion follows by noting that by the monotonicity of $T(.)$ on (a_j, a_{j+1}),

$$n^{-1} \sum_{t=1}^{N} T(n^{-1/2}x_t) I(a_j \le n^{-1/2}x_t \le a_{j+1})$$

$$\le n^{-1/2} \sum_{t=1}^{N} \int_{a_j}^{a_{j+1}} |T(x)|dx(\min(|x_t - n^{1/2}a_j|, |x_t - n^{1/2}a_{j+1}|))^{-1} = O_P(n^{-1/2}). \tag{32}$$

Furthermore, as $\delta \to 0$, by the occupation times formula,

$$|Y_\delta - Y| = |\int_0^1 T_\delta(W(r))dr - \int_0^1 T(W(r))I(a_j \le W(r) \le a_{j+1})dr|$$

$$\le |\int_{a_j}^{a_j+\delta} L(1,s)T(s)ds| + |\int_{a_{j+1}-\delta}^{a_{j+1}} L(1,s)T(s)ds|$$

$$\le \sup_s |L(1,s)| \int_{a_j}^{a_j+\delta} |T(s)|ds + \sup_s |L(1,s)| \int_{a_{j+1}-\delta}^{a_{j+1}} |T(s)|ds \to 0 \tag{33}$$

(where $L(t,s)$ denotes the Brownian local time; see Park and Phillips (1999)) because $\int_{a_j}^{a_{j+1}} |T(s)|ds < \infty$ by assumption. Finally, note that because of absolute continuity of the distributions of x_t,

$$|Y_n - Y_{n\delta}| \le |n^{-1} \sum_{t=N+1}^{n} T(n^{-1/2}x_t) I(a_j \le n^{-1/2}x_t \le a_j + \delta)|$$

$$+|n^{-1}\sum_{t=N+1}^{n}T(n^{-1/2}x_t)I(a_{j+1}-\delta\leq n^{-1/2}x_t\leq a_{j+1})|. \tag{34}$$

I will only deal with the second term, since the third term can be dealt with analogously. Defining $\delta_k=\delta 2^{-k}$, the expectation of that term can be bounded by

$$\sum_{k=0}^{\infty}E|n^{-1}\sum_{t=N+1}^{n}T(n^{-1/2}x_t)I(a_j+\delta_{k+1}\leq n^{-1/2}x_t\leq a_j+\delta_k)|$$

$$\leq\sum_{k=0}^{\infty}(|T(a_j+\delta_{k+1})|+|T(a_j+\delta_k)|)n^{-1}\sum_{t=N+1}^{n}EI(a_j+\delta_{k+1}\leq n^{-1/2}x_t\leq a_j+\delta_k)$$

$$\leq\sum_{k=0}^{\infty}(|T(a_j+\delta_{k+1})|+|T(a_j+\delta_k)|)n^{-1}\sum_{t=N+1}^{n}(\delta_k-\delta_{k+1})n^{1/2}t^{-1/2}$$

$$\leq(\sup_{n\geq 1}n^{-1/2}\sum_{t=N+1}^{n}t^{-1/2})\sum_{k=0}^{\infty}|T(a_j+\delta_{k+1})|+|T(a_j+\delta_k)|)\delta_{k+1}$$

$$\leq C\int_{a_j}^{a_j+\delta}|T(x)|dx, \tag{35}$$

for some constant $C>0$, where the first inequality follow from monotonicity of $T(.)$ on (a_{j-1},a_j), the second uses Lemma 1, and the third uses the definition of δ_k. The last expression obviously converges to 0 as $\delta\to 0$ under the assumption of the theorem, which completes the proof of this theorem. □