A Direct Approach to Evaluating Technical and Allocative Efficiency in Marketing

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Abstract

Assessing the efficiency of expenditures for promotion, distribution and other marketing variables requires models of consumer response that are often not well represented by standard economic production functions. Marketing production functions may not have simple dual cost representations used by existing estimation methods for dealing endogenous expenditures. Moreover, prices may not be available for some input variables. This paper proposes a direct approach to evaluation that does not rely on a dual cost specification, dealing directly with the issue of simultaneity without the use of instrumental variables. We illustrate our approach using data from a services company operating in multiple geographic regions.

Keywords: Bayesian analysis, Marketing Resource Allocation

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1. Introduction

Marketing expenditures in support of product, promotion, and channel activities are made to maximize return on investment by responding to individual wants, attracting new customers and increasing sales. Expenditures are effective when they are allocated across decision options in a manner that leads to maximum value of these criteria, subject to various constraints placed by the firm. Decision options might take the form of different product development efforts, geographic allocation of promotional dollars, and channel-related decisions such as the deployment of a sales force. While, intuitively, it makes sense to allocate greater expenditure to options that are more responsive, the true responsiveness of the options are not known until the allocation has been made. Analysis of marketing expenditures must therefore deal with the simultaneous nature of observed allocations that are not independently determined, but made to generate large returns.

Evaluating the accuracy of these judgments requires two things. The first is a measure of the responsiveness of the decision option to levels of expenditure. Response measures are obtained by modeling the association of a dependent (output) variable, y, such as sales, to the variables being allocated (x), also known as the input variables. The second is a norm used to assess whether the inputs were made in an optimal manner, i.e., in agreement with what we know about the response coefficients. This second item involves an analysis where the inputs become the dependent variable, and the response coefficients are explanatory.

The challenge in proceeding with such an analysis is the simultaneity present when the inputs are treated as explanatory in the response equation, and as dependent variables in the allocation equations, both related to the response coefficients (β). In addition, there often exists relatively limited data for each decision option from which to estimate the response coefficient. We propose a general approach to modeling the simultaneous relationship between demand-side responses to marketing expenditures, and their supply-side allocation.

Our approach extends current methods of modeling technical and allocative efficiencies in the economics literature to response functions encountered in marketing applications. Technical efficiency measures the relative ability of a production unit to convert a given set of inputs to outputs (see Koop, Osiewalski and Steel 1004), and allocative efficiency measures the optimality of the input levels. We deal directly with the simultaneous relationship between inputs and outputs that are linked by common coefficients, rather than solving a simultaneous system of cost functions (Kumbhakar and Tsionas, 2005). The advantage of our approach is that it facilitates analysis of a wider range of response functions, and does not require knowledge of all input prices. It can easily accommodate different allocations for capital and expense budget items, deals with simultaneity without relying on instrumental variables, and accommodates multiple input variables that give rise to multiple supply-side allocation equations. Our approach is implemented within a hierarchical Bayes framework that effectively deals with heterogeneous response coefficients (for assessing technical efficiency), simultaneity (for assessing allocative efficiency) and small sample sizes.

The organization of the paper is as follows. We introduce our model in the next section and discuss its application to evaluating various marketing expenditures. Measures of technical and allocative efficiency are derived and compared to existing approaches. Section 3 describes data from a service firm used to illustrate our approach. The firm allocates promotional expenditures to 21 geographic areas to generate new business in these markets, with each allocation spent over a 14 month period. Alternative response models and associated supply-side allocation equations are derived and fit to the data. Implications for technical and allocative efficiencies are then discussed in section 4, and concluding remarks are offered in section 5.

2. Assessing Technical and Allocative Efficiencies

Marketing expenditures are an investment that usually result in a positive return. The return can be in the form of increased sales, or customers, or some form of infra-structure that makes acquiring these items easier in the future – i.e., some expenditures produce an immediate return, while others make it easier to produce future returns. Price promotions and advertising, for example, may have an immediate impact on sales, while some forms of channel investment (e.g., new distribution outlets) may work more on an interactive level to support promotional spending. It is useful to think of marketing investment expenditures as inputs to a hierarchical production function:

$$y = f(x,\beta) = f(x_1, \beta_1 = g(x_2, \beta_2 = h(x_3, \beta_3 = ...))) \quad (1)$$

where x_1 are expenditures whose variations lead to direct effects on y, x_2 are expenditures associated with β_1 (the response coefficient for x_1), and x_3 are expenditures associated with β_2, (the response coefficient for x_2). For example, consider the case where x_1 denotes expenditures related to the formulation of the good in terms of attributes and benefits, x_2 denotes advertising expenditures resulting in greater exposure (GRPs), and x_3 reflects the intensity of distribution (ACV). Equation (1) implies that distribution expenditures (x_3) serve to make an offering more readily available to a market target, so that advertising expenditures (x_2) will have a greater impact. In turn, by alerting prospects to the attributes and benefits of an offering, the effects of product enhancements will be larger.

The form of the response functions, f, g and h, is general but needs to have specific properties for globally optimal conditions to exist (see Varian, 1992). These conditions are used to create a norm, or standard, by which current expenditures are evaluated. In particular, they must be i) positively valued, ii) have positive marginal returns that iii) diminish as the allocation increases (i.e., diminishing marginal returns). Response functions without these properties can still be used to assess technical efficiency, but cannot be used to measure allocative efficiency. Moreover, response functions without a global optima can only be used to derive directional results, such as whether it is better to marginally increase or decrease a specific input variable.

Researchers in marketing often use variants of the Cobb-Douglas production function to assess technical and allocative efficiency. Examples include Misra (2005) who posits a hierarchical model structure to account for district- and regional-effects in decomposing salesforce technical effects, Horsky and Nelson (1995) who proposes an approach for benchmarking sales response, Carroll, Green and DeSarbo (1979) who investigate preferences for leisure time, Morey and McCann (1983) who study optimal lead generation from advertising, and Mantrala, Sinha and Zoltners (1992) who compare allocation rules for various concave functions to top-down budgeting practices at firms. None of this literature, however, acknowledges the possibility that if a firm allocates inputs with even an implicit understanding of market response, then both input and output variables are dependent and determined from within the system of study. Thus, these previous approaches that condition on the inputs are prone to misspecification due to the presence of simultaneity.

The description of our model begins with a Bayesian approach to assessing technical efficiency using a single output response function. We then expand our discussion to include the supply-side allocation equations associated with the response functions. A Bayesian estimation algorithm that deals directly with the simultaneously related equations is then discussed.

Technical Efficiency

Models of technical efficiency augment equation (1) with a unit-specific parameter, ν_i, that measures the efficiency of the unit: $y_i = f(x_i,\beta,\nu_i)$, where ν_i is constrained to be greater than zero. In the economics literature, ν_i is usually specified as $y_i = f\left(x_i e^{-\nu_i}, \beta\right)$ and ν_i is interpreted as the amount that a technically inefficient producer overuses the inputs compared with an efficient producer producing the same output (Kumbhakar and Tsionas 2005). In marketing, the inefficiency parameter is sometimes introduced as a scalar $y_i = f\left(x_i, \beta\right) e^{-\nu_i}$ and interpreted as the reduction in output for the same level of input (Dutta, Kamakura and Ratchford 2004). The difference between these measures amounts to whether efficiency is measured in terms of inputs for a given level of output (i.e., $y_i = f\left(x_i e^{-\nu_i}, \beta\right)$), where ν_i is a vector of technical inefficiency parameters associated with each element of the input vector x_i, or a scalar associated with the output for given levels of input (i.e., $y_i = f\left(x_i, \beta\right) e^{-\nu_i}$). When x_i is a vector of inputs, the former is more diagnostic.

An alternative approach to assessing technical efficiency recognizes that the efficiency parameter simply serves to allow for heterogeneity in the response parameters. Associating ν_i with a specific input alters the response coefficients to be unit-specific, β_i, and associating ν_i with the overall level of output alters the model intercept for each unit of analysis. Moreover, the restriction that $\nu_i > 0$ is done for the basis of interpretation – i.e.., so that one unit is identified as the most technically efficient (ν_i=0) and others contrasted to it.

Introducing heterogeneous intercept and response coefficients provides an alternative approach for assessing technical efficiency. When implemented within a Bayesian model, statistics associated with all possible contrasts of heterogeneous intercepts and slope coefficients have known finite-sample properties. Markov chain Monte Carlo (MCMC) methods facilitate exploration of functions of all model parameters, including contrasts used to assess technical efficiency. Thus, allowing for heterogeneity in intercept and slope parameters, and employing Bayesian estimation methods facilitates the analysis of technical efficiency through exploration of functions of model parameters for comparing the output levels of alternative units for given levels of inputs.

Allocative Efficiency

The optimal allocation of inputs occurs when the marginal effect of an additional input unit per unit cost (e.g., dollar of expenditure) is equal across the decision units. Optimal levels (x*) can be determined by forming the auxiliary production function ($\mathcal{L}$) that includes the budget constraint and Lagrangian multiplier (λ), taking derivatives, and solving the system of equations arising from the identities $\partial\mathcal{L}/\partial x_i = \partial\mathcal{L}/\partial x_k = \partial\mathcal{L}/\partial\lambda = 0$ for x*. These equations lead to the allocation rule that marginal output divided by input prices (p) should be the same across inputs:

$$\frac{\partial y / \partial x_j}{\partial y / \partial x_k} = \frac{p_j}{p_k} \quad for\ all\ \ j\ and\ k \tag{2}$$

where the subscripts "j" and "k" refer to different input variables, and p_j denotes the price of input j. Equations (1) and (2) can be used to specify a system of equations where outputs (y) and inputs (x) are simultaneously determined within the system of study. Observed deviations from the optimal allocations are accommodated by introducing errors (ζ) into the price ratio, i.e., $p_j^* = p_j e^{\zeta_j}$, and assuming that equation (2) holds exactly for p_j replaced by p^*_j. Estimates of error realizations (ζ_i) can then be used to infer allocative inefficiencies.

An alternative explanation to the presence of allocative inefficiencies is that management incorrectly judges the response coefficients, β, instead of the input prices. That is, they allocate expenditures using $\beta_j^* = \beta_j e^{\varsigma_j}$. While this alternative specification can result in a different model specification for the effects of the allocative errors (ζ_i), the net effect is an observed departure from optimal allocation levels as described below.

Schmidt and Lovell (1979) first used the system consisting of the production function and the first order conditions of cost minimization in the context of a Cobb-Douglas production function. However, the generalization to more flexible productions functions like the translog have proven to be difficult (see Kumbhakar 1989). An alternative approach employs the dual cost function formulation. Maximizing output subject to a budget constraint can be shown to be dual to the minimizing costs subject to an output constraint (see Deaton and Muelbauer 1980). When costs and input prices are observed, estimation is simplified for some production functions such as the translog function (see Kumbhakar and Tsionas, 2005).

Our approach deviates from this literature by directly estimating the system of equations formed by the production function and associated first order conditions. An advantage of this approach is that it does not rely on cost functions, and, as demonstrated below, can accommodate a wider range of production functions than those considered in the economics literature. To illustrate our direct approach, consider a standard Cobb-Douglas production function and associated first-order conditions:

$$\begin{aligned} y &= x_1^{\beta_1} x_2^{\beta_2} \\ \lambda &= \frac{\partial y / \partial x_1}{p_1} = \frac{\beta_1}{p_1} x_1^{\beta_1 - 1} x_2^{\beta_2} \\ \lambda &= \frac{\partial y / \partial x_2}{p_2} = \frac{\beta_2}{p_2} x_1^{\beta_1} x_2^{\beta_2 - 1} \end{aligned} \tag{3}$$

Taking logs and rearranging terms we have:

$$\ln(y) = \beta_1 \ln(x_1) + \beta_2 \ln(x_2)$$
$$\ln(x_1) = \left(\frac{\ln(\beta_1) - \ln(\lambda)}{1-\beta_1}\right) - \left(\frac{1}{1-\beta_1}\right)\ln(p_1) + \left(\frac{\beta_2}{1-\beta_1}\right)\ln(x_2) \tag{4}$$
$$\ln(x_2) = \left(\frac{\ln(\beta_2) - \ln(\lambda)}{1-\beta_2}\right) - \left(\frac{1}{1-\beta_2}\right)\ln(p_2) + \left(\frac{\beta_1}{1-\beta_2}\right)\ln(x_1)$$

Adding technical inefficiency ($\beta_{0i}=\nu_i$) and observation error (ε_i) to the output equation, and assuming inputs are allocated based on $p_j^* = p_j e^{\zeta_j}$ rather than p_j leads to estimation equations with additive errors:

$$\ln(y_i) = \beta_{0i} + \beta_1 \ln(x_{1i}) + \beta_2 \ln(x_{2i}) + \varepsilon_i$$
$$\ln(x_{1i}) = \left(\frac{\ln(\beta_1) - \ln(\lambda)}{1-\beta_1}\right) - \left(\frac{1}{1-\beta_1}\right)\ln(p_{1i}) + \left(\frac{\beta_2}{1-\beta_1}\right)\ln(x_{2i}) + \zeta_{1i}^* \tag{5}$$
$$\ln(x_{2i}) = \left(\frac{\ln(\beta_2) - \ln(\lambda)}{1-\beta_2}\right) - \left(\frac{1}{1-\beta_2}\right)\ln(p_{2i}) + \left(\frac{\beta_1}{1-\beta_2}\right)\ln(x_{1i}) + \zeta_{2i}^*$$

where β_{0i} can be interpreted as a unit-specific intercept term. ζ^* measures the resulting deviation from optimal allocation, with $\zeta_{ki}^* = \zeta_{ki} / (\beta_k - 1)$, and the scaling $1/(\beta_k-1)$ is important only if these errors are assumed distributed with unit scale parameters. When the scale parameters are not equal to one, the system of equations in (5) can be used as the basis, or starting point, for assessing departures from optimal allocation through ζ_i^*.

When inputs are misallocated because of management misjudging the response coefficients (β) instead of the input prices (p), equation (5) can still serve as the starting point for analysis. While the allocation equations for this case involve non-linear functions of the response errors (ζ_i), analysis can proceed with ζ_i^* that measures the conditional misallocation of the logarithm of input i, given the other inputs and prices. In reality, the true misallocation most likely comes from a combination of errors in assessing input prices ($p_j^* = p_j e^{\zeta_j}$) and response coefficients ($\beta_j^* = \beta_j e^{\varsigma_j}$), making precise statements about the origins of the misallocation difficult to identify. Equation (5) therefore takes a practical approach to measuring allocative efficiency by making distributional assumptions about conditional misallocation (ζ_i^*) directly, rather than deriving the misallocation from errors in assessing input prices and/or response coefficients. An advantage of this direct approach is that it does not rely on a dual cost function, and can therefore accommodate a wider range of production functions. Moreover, if variables such as input prices are missing, the conditional (on available information) allocation equations in (5) are still valid for those data that are available, with the intercept and allocative error term absorbing the missing variables.

Our approach is related to the empirical industrial organization literature that examines the simultaneity of prices in models of supply and demand (Hausman 1996, Villas-Boas and Winer 1999, Nevo 2000, Yang, Chen and Allenby 2003). In these models, the supply-side equation has prices as dependent variables in models that assume optimal behavior of the firm, where deviation from optimality is due entirely to common supply-side shocks that affect marginal costs. These assumptions are made so that alternative competitive assumptions (e.g., Nash equilibrium) can be tested. In contrast, our production function specification allows for departures from optimality that facilitate assessment of the degree that expenditures are sub-optimally allocated, and allows for multiple input variables allocated across multiple geographic regions.

Our model also builds on recent approaches to modeling strategically determined input variables (Manchanda, Rossi and Chintagunta 2004) that employ descriptive functions, as opposed to production functions such as equation (1). Descriptive response functions are not necessarily associated with globally optimal allocation rules, such as those derived in equations (3) – (5) for the Cobb-Douglas model, and therefore cannot be used to provide allocation norms. However, descriptive functions can be used to assess the effects of incremental changes to the input variables, i.e., whether an increases in an input lead to an increased output. Our approach allows for the simultaneous assessment of allocation across all units and ensures the presence of an optimal solution by using response functions that have a global maximum. Thus, it is better suited for making global assessments of input allocation.

Bayesian Estimation

Bayesian estimation of the system of equations in (5) proceeds by recursively generating draws from the full conditional distribution of all model parameters (see Rossi, Allenby and McCulloch 2005, chapter 7). The likelihood for the data can be written as:

$$\pi\left(y,\{\ln x_k\} \mid \beta,\lambda,\sigma_\varepsilon^2,\Sigma_{\zeta^*}\right)=\pi\left(y \mid \beta,\{\ln x_k\},\sigma_\varepsilon^2\right)\times\pi\left(\{\ln x_k\} \mid \beta,\lambda,\Sigma_{\zeta^*}\right) \qquad (6)$$

where β includes unit-specific intercepts. The first factor on the right corresponds to the production function likelihood where the input variables (x) are treated as independent variables. Since the inputs are conditioning arguments, transformations from logarithmic form to other functions of the inputs do not require change-of-variable calculus. The second factor corresponds to the system of allocation equations. The likelihood for these equations require a Jocobian term because of simultaneity. For the system of allocation equations corresponding to the Cobb-Douglas system in equation (5), we have:

$$\pi\left(\{\ln x_k\} \mid \beta, \lambda, \Sigma_{\zeta^*}\right) = \pi\left(\zeta^* \mid \Sigma_{\zeta^*}\right) \mid J_{\zeta^* \to \ln x} \mid \tag{7}$$

where

$$\begin{aligned}\zeta_1^* &= \ln x_1 - \left(\frac{\ln \beta_1 - \ln \lambda}{1-\beta_1}\right) + \left(\frac{1}{1-\beta_1}\right)\ln p_1 - \left(\frac{\beta_2}{1-\beta_1}\right)\ln x_2 \\ \zeta_2^* &= \ln x_2 - \left(\frac{\ln \beta_2 - \ln \lambda}{1-\beta_2}\right) + \left(\frac{1}{1-\beta_2}\right)\ln p_2 - \left(\frac{\beta_1}{1-\beta_2}\right)\ln x_1\end{aligned} \tag{8}$$

and

$$J_{\zeta^* \to \ln x} = 1 - \frac{\beta_1 \beta_2}{(1-\beta_1)(1-\beta_2)} \tag{9}$$

Σ_{ζ^*} are parameters associated with the distribution of the allocative errors (e.g., $\zeta^* \sim N\left(0, \Sigma_{\zeta^*}\right)$).

An advantage of conducting Bayesian estimation is that, given the model parameters, the evaluation of these components of the likelihood is straightforward and estimation can proceed, for example, using a Metropolis algorithm. In addition, with panel data, heterogeneity can be easily introduced to assess technical efficiency. Specific algorithms for estimating the joint system associated with our empirical application are provided in the appendix.

3. Evaluating Marketing Expenditures

The efficiency literature in economics typically assumes a production process where current output is distinct from past outputs, or units in stock. In marketing, the process by which output is generated does not generally allow for this distinction. Many marketing actions have an effect on stock (e.g., the likelihood of retaining a current customer) and on production (e.g., the number of new customers attracted). Thus, output for many problems in marketing needs to be defined more broadly, possibly as the net increase in the number of new "products" (e.g., customers) per time period.

Another distinguishing feature of measuring outputs in marketing is that an input variable can take on a value of zero (e.g., no promotional expenditure in a given month). However, zero input expenditures do not necessarily translate into zero outputs. Output may be non-zero for a number of reasons, such as the lagged effect of the inputs and the possibility of omitted variables in the analysis. Output may also be negative because of competitive influences, poor marketplace execution, and competitive practices. Thus, marketing production functions often need to be specified with additive intercepts, lagged input variables and an additive error term.

Finally, market budget allocation decisions are often made on a yearly basis, while expenditures and their effects may take place monthly. Moreover, allocation decisions for some inputs, such as physical plant, may be subject to a different budget constraint than operational expenditures, yet both may be important in generating output. The specification of production functions for marketing problems therefore may need to allow for different levels of temporal aggregation, and possibly different budget constraints (λ) for the different input variables.

Data and Models

We illustrate our model with data from a services organization operating in 21 regions along the eastern seaboard of the United States. The firm maintains multiple branch offices in each of the regions, and makes yearly promotional allocations that are then spent over the course of the year. One of the outputs generated by these inputs is the number of new customers per month, the dependent variable in our analysis. There are 294 observations available for analysis (14 months of data for each of 21 regions).

Promotional expenditures (x_{1i}) are plotted against the number of branches (x_{2i}) and the region's population in logarithmic form in figure 1. The points in each of the plots fall along the 45 degree line, corresponding to a unit slope, or a proportional allocation rule. That is, it appears that promotional expenditure and the number of branch outlets varies in direct proportion to the population in each region. The question to be addressed is whether this allocation is optimal given the efficiency of the geographic regions.

== Figure 1 ==

It is useful to think of the effect of marketing expenditures in per capita terms. Promotional expenditures and the number of branch outlets have diminishing marginal effects that should depend on the population size of the region, with allocations made in a smaller market areas satiating more quickly. We therefore standardize the variables in the analysis per 1000 population, and view each (standardized) geographic region as providing exchangeable information for estimating model parameters. This standardization allows us to test whether the geographic regions act as homogeneous production units by testing if model parameters are equal across regions.

Figure 2 displays the distribution of new customers per month per 1000 population for each of the regions, the dependent variable in the analysis reported below. Despite this standardization, we find variation in the distributions, suggesting that the residual variance in the production functions should not be constrained across regions.

== Figure 2 ==

We specify and test alternative versions of production functions of the form:

$$y_{i,t} = \beta_{0i} + x_{1,i,t-1}^{\beta_1} x_{2,i}^{\beta_2} + \varepsilon_{i,t} \quad ; \quad \varepsilon_{i,t} \sim N\left(0, \sigma_i^2\right) \quad ; \quad i = 1, \ldots, I \; ; \; t = 2, \ldots, T_i \qquad (10)$$

where $y_{i,t}$ denotes the number of new customers produced by unit i in month t, $x_{1,i,t-1}$ is lagged promotional expenditure in region i, and $x_{2,i}$ denotes the number of branch offices in the region. The versions allow for additive versus multiplicative intercepts (β_0) and input factors (x_1, x_2), and heterogeneous input coefficients (β_1, β_2). The alternative models are described below and summarized in table 1. All versions of production functions investigated can be shown to be globally concave.

Figure 3 displays the input variables in the analysis. The top portion of the figure plots promotional expenditures per 1000 population for each of the 21 regions in the analysis. The plot shows that monthly promotional expenditure is often zero, but has sufficient variation to facilitate estimation of the input coefficient β_1. The bottom portion of the figure displays the number of branch outlets per 1000 population in each region. The number of branches in a region does not vary over the length of the data, and therefore are not plotted against time.

== Figure 3 ==

We investigate separate budget allocations for total promotional expenditures and the number of branch offices. The data provided by the firm does not include input prices for these two variables, and we therefore cannot estimate a common budget constraint parameter (i.e., Lagrangian multiplier) λ used to form the auxiliary function. Instead we assume that a separate budget allocation process for these inputs, which is reasonable, for example, when branch offices are considered a capital expense item, and promotional expenditures are considered period expenses. In addition, we assume that the allocation decision for promotional expenditures are made once for all time periods, and that the deployment of these resources is governed by other factors such as the timing of media purchases and other aspects of media buying. This leads to the following system of allocation equations corresponding to the production function in equation (10):

$$\begin{aligned} \ln x_{1,i} &= \ln\left(\sum_{t=1}^{T_i} x_{1,i,t}\right) = \alpha_1 + \left(\frac{\beta_2}{1-\beta_1}\right)\ln x_{2,i} + \zeta_1 \\ \ln x_{2,i} &= \alpha_2 + \left(\frac{\beta_1}{1-\beta_2}\right)\ln x_{1,i} + \zeta_2 \quad ; \quad \begin{pmatrix} \zeta_{1,i} \\ \zeta_{2,i} \end{pmatrix} \sim N\left(0, \Sigma_\zeta\right) \end{aligned} \tag{11}$$

Equations (10) and (11) form a system of equations for analyzing technical and allocative efficiency. Technical efficiency can be measured by introducing heterogeneous response coefficients through a random-effects specification:

$$\left(\beta_{0i}, \beta_{1i}\right)' \sim Normal\left(\bar{\beta}, V_\beta\right) \tag{12}$$

and testing for model fit relative to more homogeneous specifications. The response coefficient for branch offices, β_2, cannot be specified heterogeneously because the number of branches within a region does not vary over the time period of the data. Production units are more efficient if they have larger intercept and slope coefficients. Allocative efficiency can be assessed through the magnitude of the error variance and specific residuals in equation (11). Moreover, the estimated relationship in (11) specifies the optimal conditional relationship among output – e.g., the optimal level of promotional spending given the number of branch outlets in each region.

Five alternative models are investigated. The models are summarized in table 1, along with fit statistics for each corresponding system of equations. Model 1 is the multiplicative specification where inputs, including the model intercept, interact to produce output within each region. Thus, in model 1, there exist unique effects in each market that lead to varying degrees of efficiency of the input variables. Moreover, when the monthly promotional expenditure ($x_{1,i,t}$) is zero, the number of new customers ($y_{i,t}$) is assumed to be equal to a normally distributed error with mean zero. Given that the distribution of the dependent variables in figure 2 are mostly positive, we expect model 1 to be overly restrictive, but include it in the analysis for completeness.

Model 2 differs from Model 1 by allowing for an additive intercept that does not interact with the other input variables. Thus, in model 2 the dependent variable is not expected to equal zero when promotional expenditure is equal to zero. However, model 2 imposes the restriction that the effect of branch outlets in producing new customers is only present when promotional expenditures are non-zero. This model also assumes that the inputs (promotional expenditure per 1000 population and branch outlets per 1000 population) interact in a uniform matter across regions because the input coefficients, β_1 and β_2, are homogenous across regions.

Model 3 is an additive specification that implies no interaction among the inputs. That is, zero promotional expenditure does not affect the influence of the number of branch outlets in generating new customers. Models 4 and 5 differ from Models 2 and 3 by introducing heterogenous coefficients across regions, allowing for the assessment of technical efficiency for the promotion input variable.

== Table 1 ==

Results

The log marginal density reported on the right side of table 1 provides a measure of fit for model selection. The fit statistic indicates support for model 2 where promotional expenditures and branch outlets interact homogenously across regions. The results imply that the regions act as homogenous production units that simply provide access to different populations of individuals. In other words, the regions are equal in their technical efficiency for the input variables, although the intercepts for each region are found to be different. These results are consistent with the adaptation of global marketing strategies that do not consider the peculiarities of local markets when allocating promotional dollars and building additional branch outlets.

Table 2 and Figure 4 report parameter estimates for the best fitting model, model 2. Table 2 reports estimates for production function intercepts (β_{0i}) through the hyper-parameters of the random-effects distribution. The top portion of the table reports parameters of the production function, and the bottom portion of the table reports parameters unique to the system of allocation equations. Figure 4 displays posterior estimates of error variances (σ_i^2) for each of the 21 regions in the analysis. In general, the posterior standard deviations are small relative to the point estimates, indicating that the parameters are accurately estimated. We note that the input coefficient for promotional expenditure per 1000 population, β_1, is estimated to be much smaller ($\hat{\beta}_1$=0.0065) than the corresponding parameter for the number of branch outlets per 1000 population ($\hat{\beta}_2$=0.886). Despite the small magnitude of the estimated promotional response coefficient, all of the mass of its posterior distribution is located above zero, i.e., all of the MCMC draws of β_1 are greater than zero. We comment further on this result below.

== Table 2 and Figure 4 ==

The posterior mass of the distributions of the input coefficients are located entirely within the unit interval ($0 < \beta_1, \beta_2, < 1$) and their sum is equal to 0.89 with a posterior standard deviation of 0.06. These results ensure the existence of a global optimal solution for the production function. However, the corresponding estimate for the production function only model, where estimates are based on equation (10) but not the system of allocation equations in (11), is 1.43 with a posterior standard deviation of 0.26. Thus, using only the production function specification results in estimates that lead to regional production functions that do not have a global maxima.

4. Discussion

Table 3 displays the optimal allocation of the input variables for each of the five models in our analysis. The optimal allocation is derived by either maximizing the production function subject to budget constraints, or minimizing the associated cost function subject to a given level of output. Either approach involves the use of Lagrangian multipliers, and both correspond to the situation where the allocative errors are equal to zero ($\zeta_1^* = \zeta_2^* = 0$). The first column of table 3 identifies the optimal relationship between promotion and expenditure within a region. The second and third column is the optimal allocation of these variables across regions. Since promotional expenditures are measured in dollars and distributional expenditures are measured in terms of the number of branch outlets, the dollar cost of a branch outlet is needed to compare branch allocations to promotional expenditures. The term "p_2" is used in table 3 to denote the marginal cost of a branch outlet, which, consistent with our empirical application, we assume to be the same across regions.

== Table 3 ==

Each of the models in table 3 implies a different optimal allocation of the input variables. In model 1, where the parameter β_{0i} enters multiplicatively, optimal allocation of promotional dollars (x_1^*) and optimal number of branch outlets (x_2^*) among regions depend on the estimated values of this multiplicative factor. For models with an additive intercept, the relationship among optimal inputs does not depend on β_{0i} because it is not present in the expression of the first derivative. Similarly, the specification of additive versus multiplicative relationships among the input variables influences the optimal allocation expressions.

For model 2, the best fitting model, the optimal allocation of promotional and distributional expenditures is, on a per-capita basis, equal across regions. Thus, the existing relationship between promotional expenditure, number of branch outlets and regional population as displayed in figure 1 approximates an optimal allocation. However, to the extent per capita promotional dollars and branch outlets vary across regions, there is an opportunity for improving the effectiveness of expenditure allocation.

Figure 5 displays the existing per capita promotional expenditures and branch outlets for each of the 21 geographic regions in our analysis. Substantial variation across regions exists for both expenditures, implying potential opportunities for improvement. Promotional expenditures range from 38 to 132 dollars per 1000 population, and the number of branch outlets ranges from 0.17 to 0.44 per 1000 population. Since branch outlets are physical properties that are not easy to dispose of, we investigate the optimal conditional relationship of promotional expenditure given the number of branch outlets in each region.

== Figure 5 ==

The solid line in figure 5 reflects the current relationship between promotion and branch outlets, and the dotted line corresponds to the optimal conditional relationship as specified in equation (11). Given the currently existing branch outlets in each market, the firm can optimize output by reallocating promotional expenditure using the dotted line. That is, the optimal level of promotional spending in each region, given the number of branch offices, are indicated by the points along the regression line. An advantage of this regression-approach to reallocation is that, since the allocative errors have expected value zero, the re-distribution of expenditures sums up to the same total as the current, sub-optimal expenditures.

Optimally reallocating promotional expenditures using the dashed line in Figure 5 only raises the expected output from 145,592 new customers to 148,388 new customers across all regions and months of the analysis, an increase of just 1.9 percent. These calculations are based on the assumption that the new monthly promotional spending changes in proportion to existing spending patterns, i.e., the number of months with zero promotional spending stays the same. Given the near optimal relationship exhibited in figure 1, the marginal improvement is not surprising. Also, since branch outlets were held constant and the exponent β_1 is small (0.0065), only relatively small gains can be achieved by adjusting promotional expenditures. The magnitude of this estimate implies that the effects of promotions quickly satiate, and greater returns are achieved by allocating resources to build branch outlets.

The optimal, long-term relationship between promotional spending and branch outlets is displayed in table 3, and involves the ratio of the exponential coefficients (β_1/β_2=0.0065/0.886). Since the magnitude of branch outlet coefficient (β_2) is so much larger than the promotional coefficient, the results imply that promotional spending should be reduced and funds allocated to additional branch outlets. The small estimated coefficient for promotion (β_1) essentially reduces the effect of promotions, x^{β_1}, to a step function that equals zero when x=0 and equals one with x>0. The implication is that large promotional expenditures have the same effect as small promotional expenditures in the range of expenditures investigated.

When promotional expenditures are positive, their effect depends on the number of branch outlets (per 1000 population) in the region. However, when promotional expenditures are zero, the branch outlets do not help to generate new customers. Thus, the results indicate a strong interactive effect that is consistent with an effects hierarchy as discussed in equation (1). In the data examined, promotional expenditures are equal to zero 76% of the time. The model implies that large gains (i.e., more than double the number of new customers) are available by increasing the frequency of low dollar promotions by reducing the magnitude of high dollar promotions so that promotions occur in every period.

5. Conclusion

This paper introduces a framework for assessing technical and allocative efficiency of marketing expenditures. Alternative marketing production functions are examined, each of which are more complex than those typically studied in the economics literature (e.g., Cobb-Douglas, translog). Our analysis addresses both the efficiency of production units (regions in our analysis) to convert inputs to outputs giving the existing allocations, as well as the optimality of current allocations. The analysis recognizes the possibility that firms may implicitly follow an optimal allocation rule, and measures the deviation from optimality through a simultaneous equation specification where the input variables (x) and the output variables (y) are both related to the response coefficients (β).

Our approach does not rely on a dual cost formulation, offering a direct approach to estimation that can accommodate a wide range of marketing production functions where lagged effects, omitted variables, differences exist between budget allocation and spending, and carryover may exist. It also differs from approaches in the economics literature by making direct distributional assumptions about errors in the conditional allocation equations, rather than deriving the errors from more primitive assumptions about management misjudgment of input prices and/or response coefficients. Our view is that the true source of errors is likely to be due to many reasons, and that inferences based on the conditional allocation equations is often sufficient for optimizing expenditures in marketing.

The empirical analysis found that the regional producers of a service provider were equally efficient at using promotional expenditures and branch offices to generate new customers. In addition, we found support for a hierarchy of effects among the input variables where promotions were necessary, but not sufficient, for generating output. Given the presence of a promotion, the number of branch offices were found to be related to new customer production, and not the magnitude of the promotion expenditure. Finally, our analysis demonstrates the flexibility of our approach in assessing efficiencies despite not having access to input prices or other variables that are influential in generating new business. It is doubtful that analysts in marketing ever have a complete set of variables for analysis, and it is critical that analysis can proceed with functional forms that potentially accommodate their absence through additive intercepts and error terms.

The framework and analysis presented in this paper can be extended in a number of ways. First, our investigation involved aspects of promotion and distribution, but did not involve product-related expenditures. Data were also not available on promotional quality, media content, or differing regional costs of branch outlets. The assessment of input prices is often difficult, being dependent on cost drivers and cost allocation rules used by firms, and the development of input price estimators is an interesting extension of our work. Finally, we feel the development and comparison of alternative production functions for marketing would constitute important contributions to the literature.

Appendix
Estimation Algorithms

Estimation proceeds by recursively generating draws from the full conditional distribution of all model parameters. The likelihood of the data can be factored as:

$$\pi\left(y, \ln x_1, \ln x_2 \mid \beta, \alpha, \sigma^2, V_\zeta\right) = \pi\left(y \mid \ln x_1, \ln x_2, \beta, \sigma^2\right) \times \pi\left(\ln x_1, \ln x_2 \mid \alpha, V_\zeta\right)$$

where, for model 2, the first factor to the right of the equal sign corresponds to equation (10), and the second and third factors corresponds to equation (11). Estimation is straightforward except for parameters present in multiple equations due to simultaneity in the last two factors. We employ a random-walk Metropolis-Hastings algorithm for the α and β parameters where the posterior conditional distribution is evaluated as:

$$[\beta \mid else] \propto \prod_{i=1}^{I} \left(\begin{array}{c} \prod_{t=2}^{T_j} \left[y_{i,t} \mid \beta_{0i}, \beta_1, \beta_2, x_{1,i,t-1}, x_{2,i}, \sigma_i^2 \right] \\ \times \left[\ln x_{1,i}, \ln x_{2,i} \mid \alpha, V_\zeta \right] \end{array} \right) [\beta]$$

where [β] denotes the prior distribution, and

$$\pi\left(\{\ln x_k\} \mid \alpha, \beta, \Sigma_{\zeta^*}\right) = \pi\left(\zeta^* \mid \Sigma_{\zeta^*}\right) \mid J_{\zeta^* \to \ln x} \mid$$

$$\pi \left(\begin{array}{l} \zeta_1^* = \ln x_1 - \alpha_1 - \left(\dfrac{\beta_2}{1-\beta_1} \right) \ln x_2 \\ \zeta_2^* = \ln x_2 - \alpha_2 - \left(\dfrac{\beta_1}{1-\beta_2} \right) \ln x_1 \end{array} \right) = N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, V_{\zeta^*} \right),$$

$$\left| J_{\zeta \to \ln x} \right| = \begin{vmatrix} \dfrac{\partial \zeta_1}{\partial \ln x_1} & \dfrac{\partial \zeta_1}{\partial \ln x_2} \\ \dfrac{\partial \zeta_2}{\partial \ln x_1} & \dfrac{\partial \zeta_2}{\partial \ln x_2} \end{vmatrix} = \begin{vmatrix} 1 & \dfrac{-\beta_2}{1-\beta_1} \\ \dfrac{-\beta_1}{1-\beta_2} & 1 \end{vmatrix} = 1 - \frac{\beta_1 \beta_2}{(1-\beta_1)(1-\beta_2)}$$

The error variances σ_i^2 and V_{ζ^*} follow standard conjugate distributions using the inverted chi-squared and inverted Wishart distributions. Prior distributions for all error variances were specified to be inverted chi-squared gamma with 4 degrees of freedom and prior sum of squares equal to 0.25. The prior on V_{ζ^*} is inverted Wishart with 5 degrees of freedom and prior sum of squares equal to 5I. All other priors were specified as normal with mean zero and covariance equal to 100I.

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Figure 1
Regional Dispersion of Promotional Expenditures, Branch Outlets and Population

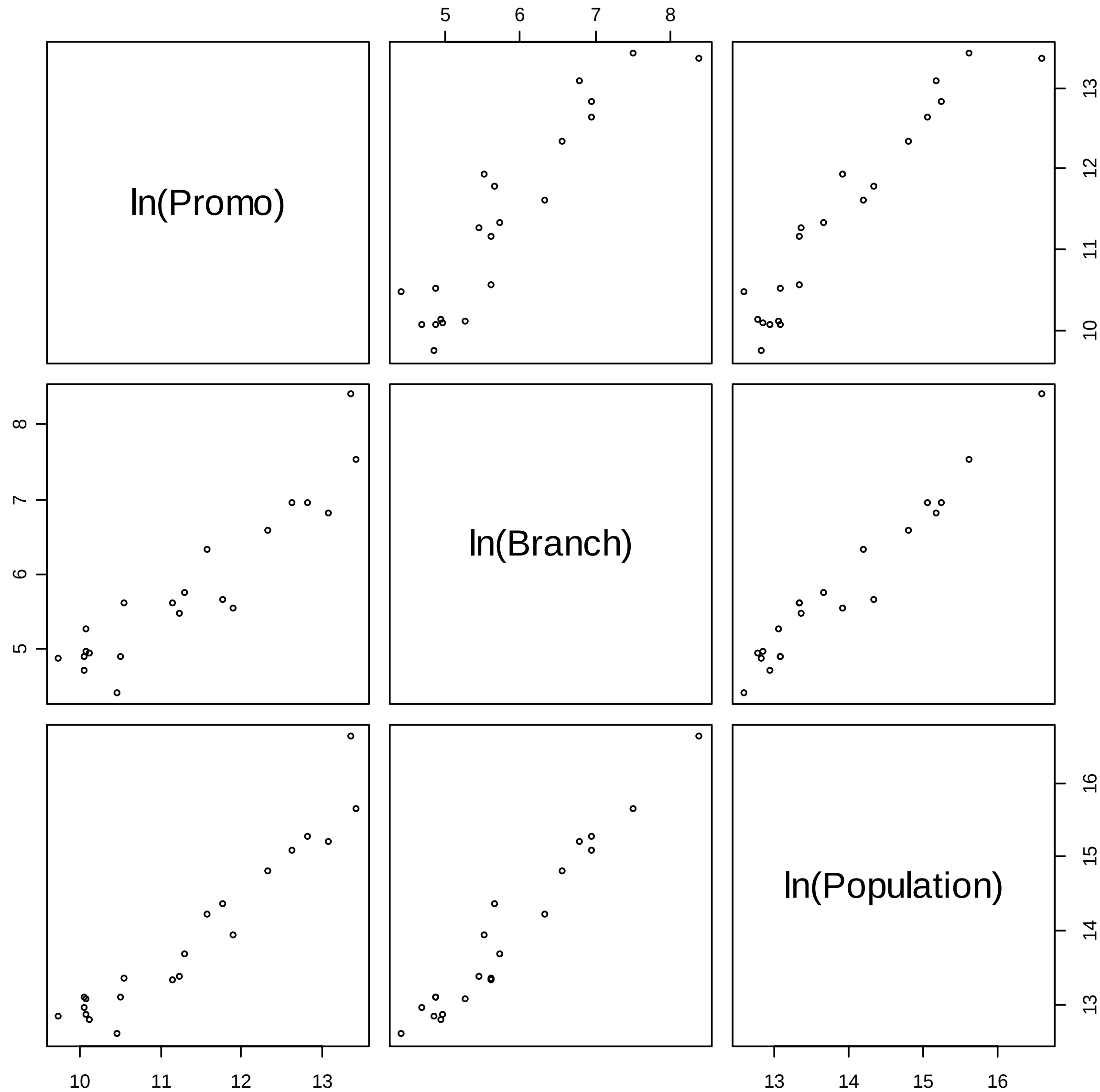


Figure 2
Regional Distributions of New Customers per Month per 1000 Population

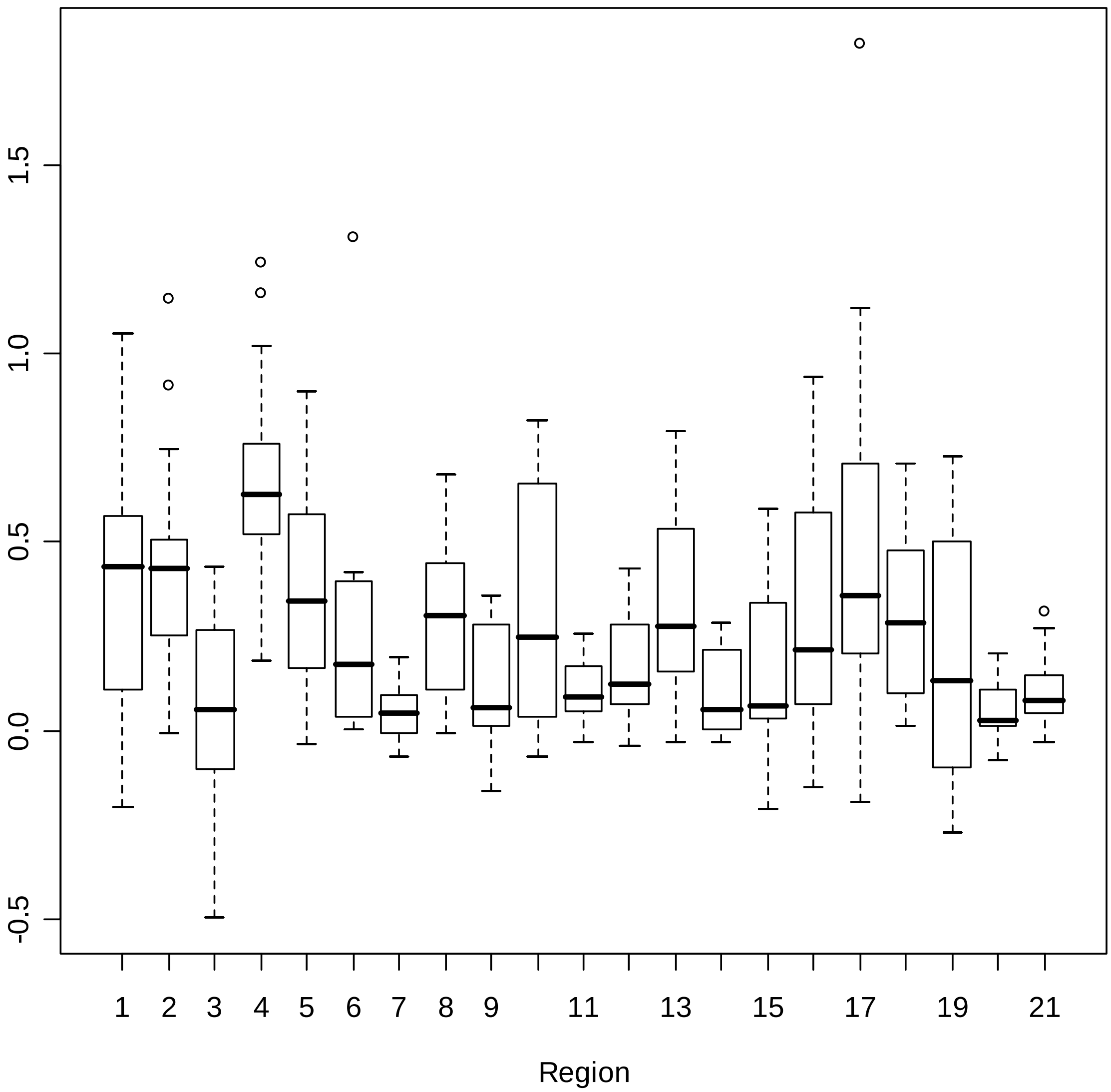
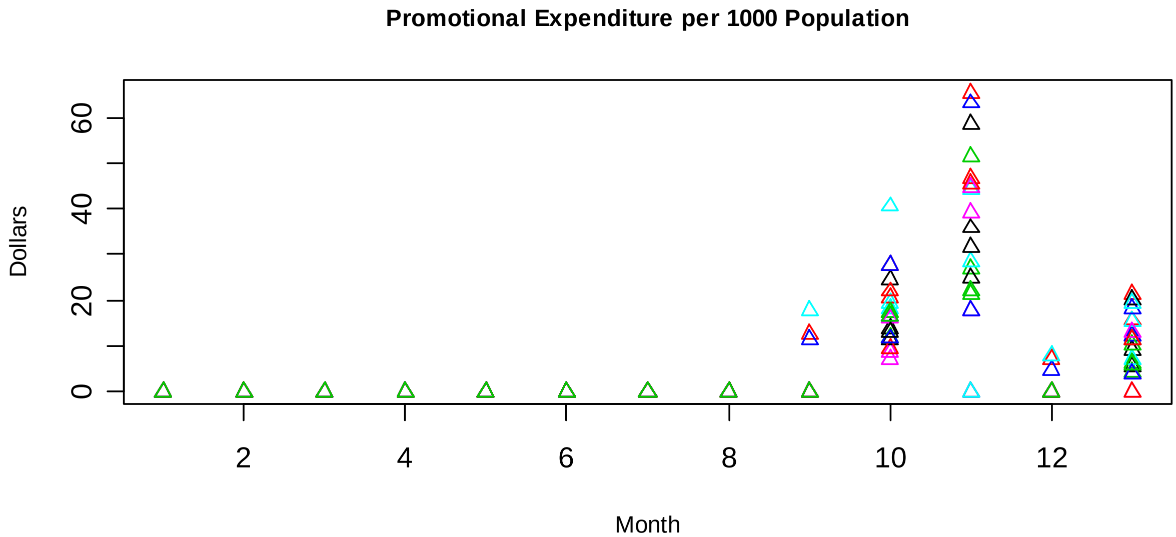


Figure 3
Input Variables



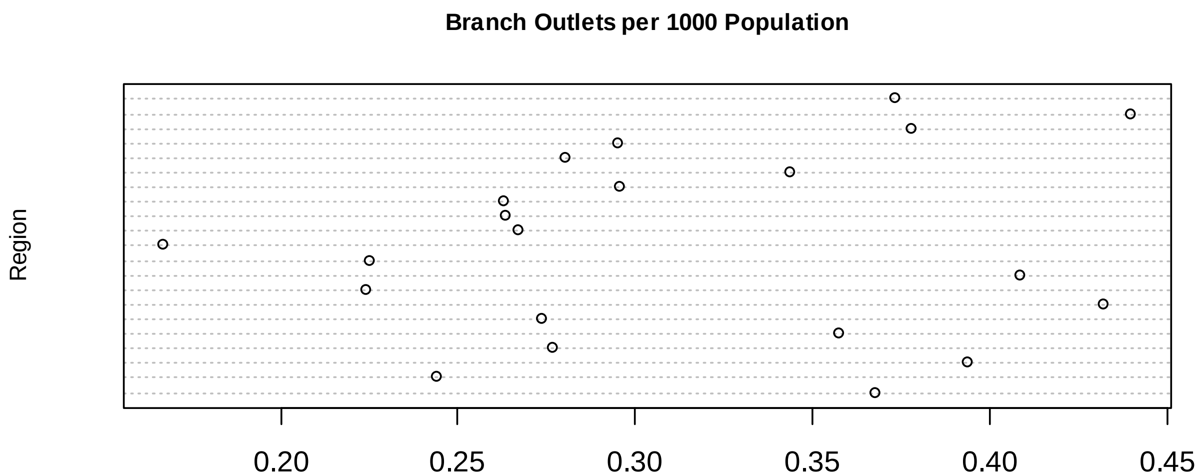


Figure 4
Posterior Distributions of Error Variance (σ_i^2)

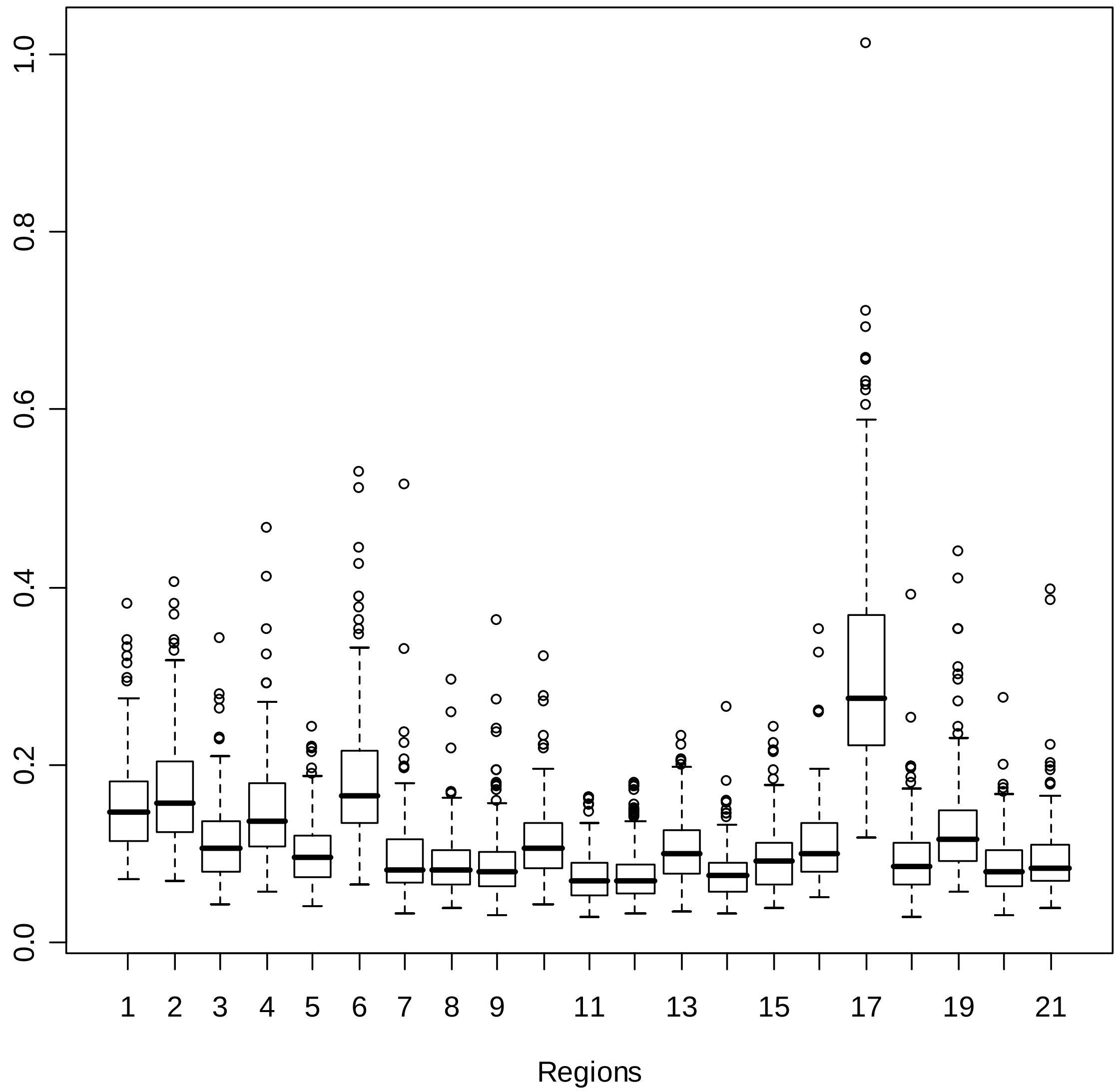


Figure 5
Existing and Optimal Promotional Expenditures Conditional on Branch Outlets

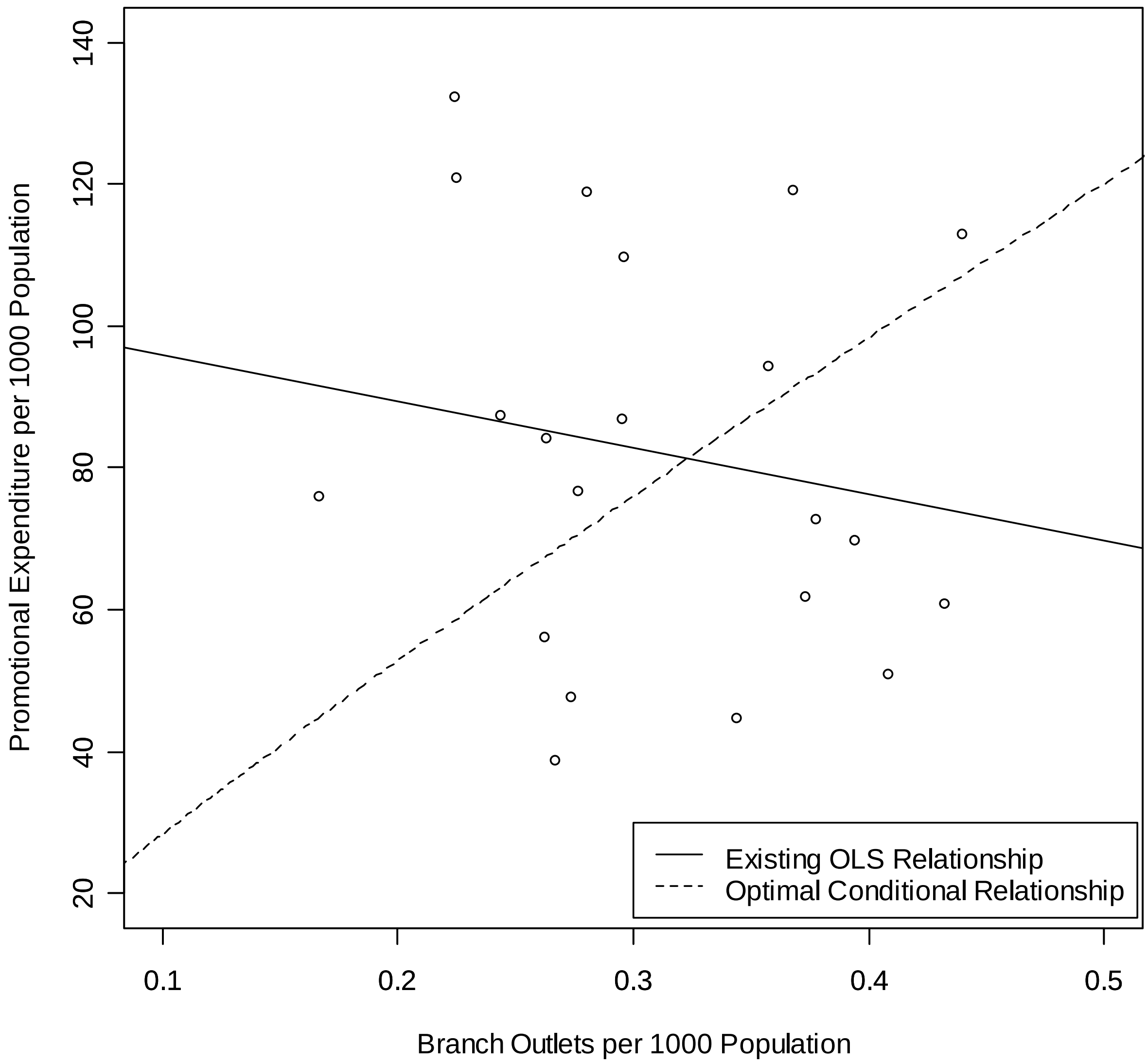


Table 1
Model Description and Fit

	Model Specification	Description	Log Marginal Density
1	$y_{i,t} = \beta_{0i} x_{1,i,t-1}^{\beta_1} x_{2,i}^{\beta_2} + \varepsilon_{i,t}$; $\varepsilon_{i,t} \sim N\left(0,\sigma_i^2\right)$ $\ln(x_{1,i}) = \alpha_1 + \frac{\ln\left(\beta_{0i}\right)}{1-\beta_1} + \frac{\beta_2}{1-\beta_1}\ln(x_{2,i}) + \zeta_{1,i}^*$; $\ln(x_{2,i}) = \alpha_2 + \frac{\ln\left(\beta_{0i}\right)}{1-\beta_2} + \frac{\beta_1}{1-\beta_2}\ln(x_{1,i}) + \zeta_{2,i}^*$; $\left(\zeta_{1,i}^*, \zeta_{2,i}^*\right)' \sim N\left(0,\Sigma_{\zeta^*}\right)$	Multiplicative	-68.5
2	$y_{i,t} = \beta_{0i} + x_{1,i,t-1}^{\beta_1} x_{2,i}^{\beta_2} + \varepsilon_{i,t}$; $\varepsilon_{i,t} \sim N\left(0,\sigma_i^2\right)$ $\ln(x_{1,i}) = \alpha_1 + \frac{\beta_2}{1-\beta_1}\ln(x_{2,i}) + \zeta_{1,i}^*$; $\ln(x_{2,i}) = \alpha_2 + \frac{\beta_1}{1-\beta_2}\ln(x_{1,i}) + \zeta_{2,i}^*$; $\left(\zeta_{1i}^*, \zeta_{2,i}^*\right)' \sim N\left(0,\Sigma_{\zeta^*}\right)$	Additive Intercept with Multiplicative Effects	-32.0
3	$y_{i,t} = \beta_{0i} + x_{1,i,t-1}^{\beta_1} + x_{2,i}^{\beta_2} + \varepsilon_{i,t}$; $\varepsilon_{i,t} \sim N\left(0,\sigma_i^2\right)$ $\ln(x_{1,i}) = \alpha_1 + \zeta_{1,i}^*$; $\ln(x_{2,i}) = \alpha_2 + \zeta_{2,i}^*$; $\left(\zeta_{1,i}^*, \zeta_{2,i}^*\right)' \sim N\left(0,\Sigma_{\zeta^*}\right)$	Additive	-86.9
4	$y_{i,t} = \beta_{0i} + x_{1,i,t-1}^{\beta_{1i}} x_{2,i}^{\beta_2} + \varepsilon_{i,t}$; $\varepsilon_{i,t} \sim N\left(0,\sigma_i^2\right)$ $\ln(x_{1,i}) = \alpha_1 + \frac{\beta_2}{1-\beta_{1i}}\ln(x_{2,i}) + \zeta_{1,i}^*$; $\ln(x_{2,i}) = \alpha_2 + \frac{\beta_{1i}}{1-\beta_2}\ln(x_{1,i}) + \zeta_{2,i}^*$; $\left(\zeta_{1i}^*, \zeta_{2,i}^*\right)' \sim N\left(0,\Sigma_{\zeta^*}\right)$	Additive Intercept with Region-Specific Multiplicative Effects $\beta_{1,j}$	-56.8
5	$y_{i,t} = \beta_{0i} + x_{1,i,t-1}^{\beta_{1i}} + x_{2,i}^{\beta_2} + \varepsilon_{i,t}$; $\varepsilon_{i,t} \sim N\left(0,\sigma_i^2\right)$ $\ln(x_{1,i}) = \alpha_1 + \zeta_{1,i}^*$; $\ln(x_{2,i}) = \alpha_2 + \zeta_{2,i}^*$; $\left(\zeta_{1i}^*, \zeta_{2,i}^*\right)' \sim N\left(0,\Sigma_{\zeta^*}\right)$	Additive with Region-Specific Additive Effects $\beta_{1,j}$	-95.1

Table 2
Parameter Estimates for Model 2
Posterior Mean (Posterior Standard Deviation)

Production Function Parameters	Intercepts: $\beta_{0i} \sim N\left(\bar{\beta}, V_{\beta}\right)$	$\bar{\beta}$ 0.168 (0.105)	V_{β} 0.239 (0.070)
	Promotional Expenditure:	β_1 0.0065 (0.0061)	
	Branch Offices:	β_2 0.886 (0.0615)	
Allocation Parameters	Intercepts:	α_1 5.410 (0.144)	α_2 -1.462 (0.273)
	Allocative Error Variance: $\left(\zeta_1^*, \zeta_2^*\right)' \sim N\left(0, \Sigma_{\zeta^*}\right)$	$\Sigma_{\zeta^*}^{(1,1)}$ 0.341 (0.171)	$\Sigma_{\zeta^*}^{(1,2)}$ -0.117 (0.075)
			$\Sigma_{\zeta^*}^{(2,2)}$ 0.136 (0.056)

Table 3
Optimal Allocation of Input Variables

Model	**Relationship between** $x^*_{1,i}$ **and** $x^*_{2,i}$	**Relationship between** $x^*_{1,i}$ **and** $x^*_{1,k}$	**Relationship between** $x^*_{2,i}$ **and** $x^*_{2,k}$
1) $y_i = \beta_{0i} x_{1,i}^{\beta_1} x_{2,i}^{\beta_2}$	$x^*_{1,i} = \left(\frac{\beta_1}{\beta_2}\right) p_2 x^*_{2,i}$	$x^*_{1,i} = \left(\frac{\beta_{0i}}{\beta_{0k}}\right)^{\frac{1}{1-\beta_1-\beta_2}} x^*_{1,k}$	$x^*_{2,i} = \left(\frac{\beta_{0i}}{\beta_{0k}}\right)^{\frac{1}{1-\beta_1-\beta_2}} x^*_{2,k}$
2) $y_i = \beta_{0i} + x_{1,i}^{\beta_1} x_{2,i}^{\beta_2}$	$x^*_{1,i} = \left(\frac{\beta_1}{\beta_2}\right) p_2 x^*_{2,i}$	$x^*_{1,i} = x^*_{1,k}$	$x^*_{2,i} = x^*_{2,k}$
3) $y_i = \beta_{0i} + x_{1,i}^{\beta_1} + x_{2,i}^{\beta_2}$	$x^*_{1,i} = \left(\frac{p_2\beta_1}{\beta_2}\right)^{\frac{1}{1-\beta_1}} x^{*\,\frac{\beta_2-1}{\beta_1-1}}_{2,i}$	$x^*_{1,i} = x^*_{1,k}$	$x^*_{2,i} = x^*_{2,k}$
4) $y_i = \beta_{0i} + x_{1,i}^{\beta_{1,i}} x_{2,i}^{\beta_2}$	$x^*_{1,i} = \left(\frac{\beta_{1,i}}{\beta_2}\right) p_2 x^*_{2,i}$	$x^*_{1,i} = \left(\frac{\beta_{1.i}}{\beta_{1,k}}\right)^{\frac{1-\beta_2}{1-\beta_{1,i}-\beta_2}} x^{*\,\frac{\beta_2+\beta_{1,k}-1}{\beta_2+\beta_{1,i}-1}}_{1,k}$	$x^*_{2,i} = p_2^{\beta_{1,k}-\beta_{1,i}} \beta_{1,i}^{\frac{-\beta_{1,i}}{\beta_2+\beta_{1,i}-1}} \beta_{1,k}^{\frac{\beta_{1,k}}{\beta_2+\beta_{1,i}-1}} \beta_2^{\frac{\beta_{1,i}-\beta_{1,k}}{\beta_2+\beta_{1,i}-1}} x^{*\,\frac{\beta_2+\beta_{1,k}-1}{\beta_2+\beta_{1,i}-1}}_{2,k}$
5) $y_i = \beta_{0i} + x_{1,i}^{\beta_{1,j}} + x_{2,i}^{\beta_2}$	$x^*_{1,i} = \left(\frac{p_2\beta_{1,i}}{\beta_2}\right)^{\frac{1}{1-\beta_{1,i}}} x^{*\,\frac{\beta_2-1}{\beta_{1,i}-1}}_{2,i}$	$x^*_{1,i} = \left(\frac{\beta_{1.i}}{\beta_{1,k}}\right)^{\frac{1}{1-\alpha_{1,i}}} x^{*\,\frac{\beta_{1,k}-1}{\beta_{1,i}-1}}_{1,k}$	$x^*_{2,i} = x^*_{2,k}$

p_2 is the annual operating expenses for a branch office.