

Lecture 4

Propagation of errors

Introduction

- Example: we measure the current (I) and resistance (R) of a resistor.
 - ◆ Ohm's law:
 $V = IR$
 - ◆ If we know the uncertainties (e.g. standard deviations) in I and R, what is the uncertainty in V?
- Given a functional relationship between several measured variables (x, y, z),
 $Q = f(x, y, z)$
 - ◆ What is the uncertainty in Q if the uncertainties in x, y , and z are known?
 - To answer this question we use a technique called Propagation of Errors.
 - ◆ Usually when we talk about uncertainties in a measured variable such as x , we assume:
 - the value of x represents the mean of a Gaussian distribution
 - the uncertainty in x is the standard deviation (σ) of the Gaussian distribution
 - **not all measurements can be represented by Gaussian distributions (more on that later)**

Propagation of Error Formula

- To calculate the variance in Q as a function of the variances in x and y we use the following:
$$\sigma_Q^2 = \sigma_x^2 \left(\frac{\partial Q}{\partial x} \right)^2 + \sigma_y^2 \left(\frac{\partial Q}{\partial y} \right)^2 + 2\sigma_{xy} \left(\frac{\partial Q}{\partial x} \right) \left(\frac{\partial Q}{\partial y} \right)$$
- ◆ If the variables x and y are uncorrelated ($\sigma_{xy} = 0$), the last term in the above equation is zero.
- ◆ Assume we have several measurement of the quantities x (e.g. $x_1, x_2 \dots x_N$) and y (e.g. $y_1, y_2 \dots y_N$).
 - The average of x and y :

$$\mu_x = \frac{1}{N} \sum_{i=1}^N x_i \quad \text{and} \quad \mu_y = \frac{1}{N} \sum_{i=1}^N y_i$$

- ◆ define: $Q_i \equiv f(x_i, y_i)$

$$Q \equiv f(\mu_x, \mu_y) \text{ evaluated at the average values}$$

- ◆ expand Q_i about the average values:

$$Q_i = f(\mu_x, \mu_y) + (x_i - \mu_x) \left(\frac{\partial Q}{\partial x} \right)_{\mu_x} + (y_i - \mu_y) \left(\frac{\partial Q}{\partial y} \right)_{\mu_y} + \text{higher order terms}$$

- ◆ assume the measured values are close to the average values

- ☞ neglect the higher order terms:

$$Q_i - Q = (x_i - \mu_x) \left(\frac{\partial Q}{\partial x} \right)_{\mu_x} + (y_i - \mu_y) \left(\frac{\partial Q}{\partial y} \right)_{\mu_y}$$

$$\sigma_Q^2 = \frac{1}{N} \sum_{i=1}^N (Q_i - Q)^2$$

$$= \frac{1}{N} \sum_{i=1}^N (x_i - \mu_x)^2 \left(\frac{\partial Q}{\partial x} \right)_{\mu_x}^2 + \frac{1}{N} \sum_{i=1}^N (y_i - \mu_y)^2 \left(\frac{\partial Q}{\partial y} \right)_{\mu_y}^2 + \frac{2}{N} \sum_{i=1}^N (x_i - \mu_x)(y_i - \mu_y) \left(\frac{\partial Q}{\partial x} \right)_{\mu_x} \left(\frac{\partial Q}{\partial y} \right)_{\mu_y}$$

$$= \sigma_x^2 \left(\frac{\partial Q}{\partial x} \right)_{\mu_x}^2 + \sigma_y^2 \left(\frac{\partial Q}{\partial y} \right)_{\mu_y}^2 + 2 \left(\frac{\partial Q}{\partial x} \right)_{\mu_x} \left(\frac{\partial Q}{\partial y} \right)_{\mu_y} \frac{1}{N} \sum_{i=1}^N (x_i - \mu_x)(y_i - \mu_y)$$

- ◆ If the measurements are uncorrelated

- ☞ the summation in the above equation is zero

$$\sigma_Q^2 = \sigma_x^2 \left(\frac{\partial Q}{\partial x} \right)_{\mu_x}^2 + \sigma_y^2 \left(\frac{\partial Q}{\partial y} \right)_{\mu_y}^2$$

uncorrelated errors

- ◆ If x and y are correlated, define σ_{xy} as:

$$\sigma_{xy} = \frac{1}{N} \sum_{i=1}^N (x_i - \mu_x)(y_i - \mu_y)$$

$$\sigma_Q^2 = \sigma_x^2 \left(\frac{\partial Q}{\partial x} \right)_{\mu_x}^2 + \sigma_y^2 \left(\frac{\partial Q}{\partial y} \right)_{\mu_y}^2 + 2 \left(\frac{\partial Q}{\partial x} \right)_{\mu_x} \left(\frac{\partial Q}{\partial y} \right)_{\mu_y} \sigma_{xy}$$

correlated errors

- Example: Power in an electric circuit.

$$P = I^2 R$$

- ◆ Let $I = 1.0 \pm 0.1$ amp and $R = 10 \pm 1 \Omega$

☞ $P = 10$ watts

- ◆ calculate the variance in the power using propagation of errors

$$\sigma_P^2 = \sigma_I^2 \left(\frac{\partial P}{\partial I} \right)_{I=1}^2 + \sigma_R^2 \left(\frac{\partial P}{\partial R} \right)_{R=10}^2 = \sigma_I^2 (2IR)^2 + \sigma_R^2 (I^2)^2 = (0.1)^2 (2 \cdot 1 \cdot 10)^2 + (1)^2 (1^2)^2 = 5 \text{ watts}^2$$

☞ $P = 10 \pm 2$ watts

- If the true value of the power was 10 W and we measured it many times with an uncertainty (σ) of ± 2 W and Gaussian statistics apply

☞ 68% of the measurements would lie in the range [8,12] W

- ◆ Sometimes its convenient to put the above calculation in terms of relative errors:

$$\frac{\sigma_P^2}{P^2} = \frac{\sigma_I^2}{P^2} \left(\frac{\partial P}{\partial I} \right)^2 + \frac{\sigma_R^2}{P^2} \left(\frac{\partial P}{\partial R} \right)^2 = \frac{4\sigma_I^2}{I^2} + \frac{\sigma_R^2}{R^2} = 4 \left(\frac{0.1}{1} \right)^2 + \left(\frac{1}{10} \right)^2 = 0.1^2 (4+1)$$

- the uncertainty in the *current* dominates the uncertainty in the power

☞ current must be measured more precisely to greatly reduce the uncertainty in the power

- Example: The error in the average.

- ◆ The average of several measurements each with the same uncertainty (σ) is given by:

$$\mu = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$$

$$\sigma_\mu^2 = \sigma_{x_1}^2 \left(\frac{\partial \mu}{\partial x_1} \right)^2 + \sigma_{x_2}^2 \left(\frac{\partial \mu}{\partial x_2} \right)^2 + \dots + \sigma_{x_n}^2 \left(\frac{\partial \mu}{\partial x_n} \right)^2 = \sigma^2 \left(\frac{1}{n} \right)^2 + \sigma^2 \left(\frac{1}{n} \right)^2 + \dots + \sigma^2 \left(\frac{1}{n} \right)^2 = n \sigma^2 \left(\frac{1}{n} \right)^2$$

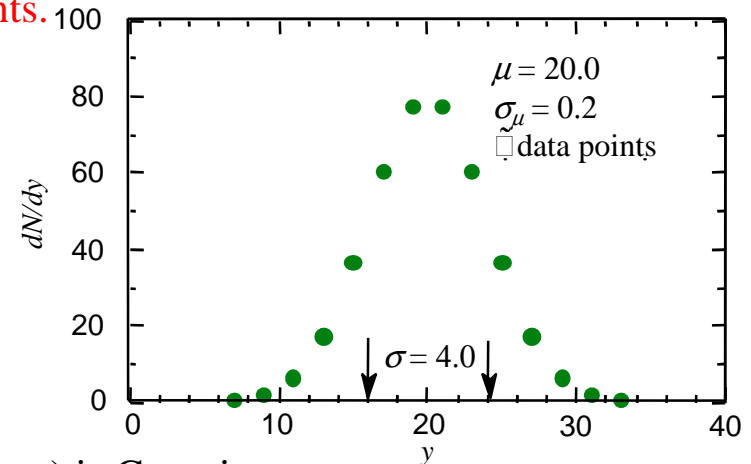
$$\sigma_\mu = \frac{\sigma}{\sqrt{n}}$$

“error in the mean”

- ☞ We can determine the mean better by combining measurements.
- ☞ The precision only increases as the square root of the number of measurements.
- 🕒 ■ Do not confuse σ_μ with σ !
- 🕒 ■ σ is related to the width of the *pdf* (e.g. Gaussian) that the measurements come from.
- 🕒 ■ σ does not get smaller as we combine measurements.

Problem in the Propagation of Errors

- In calculating the variance using propagation of errors
 - ◆ we usually assume the error in measured variable (e.g. x) is Gaussian
- If x is described by a Gaussian distribution
 - ◆ $f(x)$ may not be described by a Gaussian distribution!



- What does the standard deviation that we calculate from propagation of errors mean?

- ◆ Example: The new distribution is Gaussian.

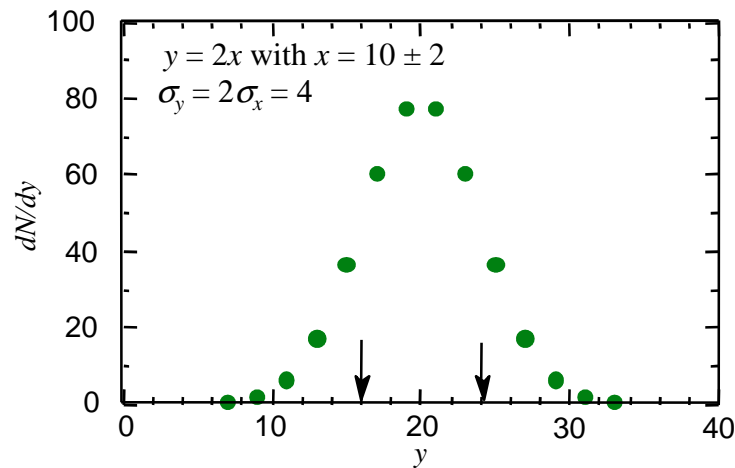
- Let $y = Ax$, with $A =$ a constant and x a Gaussian variable.

- ☞ $\mu_y = A\mu_x$ and $\sigma_y = A\sigma_x$

- Let the probability distribution for x be Gaussian:

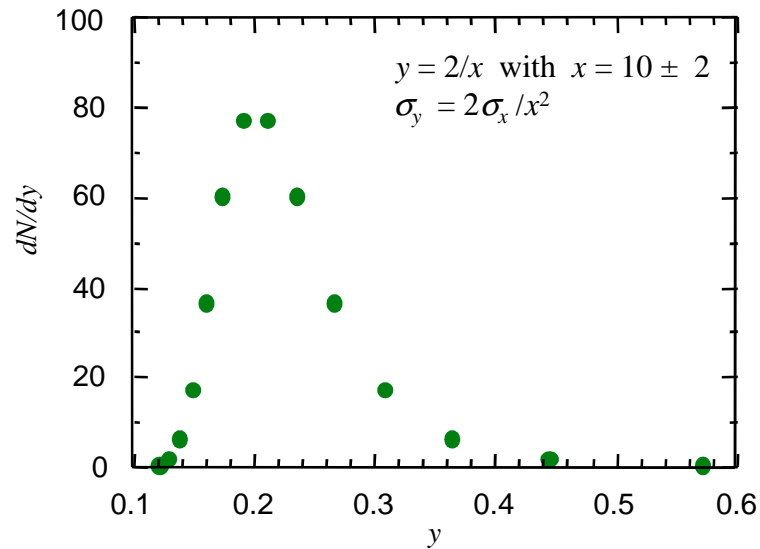
$$p(x, \mu_x, \sigma_x) dx = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx = \frac{1}{\frac{\sigma_y}{A} \sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu_y}{A}\right)^2}{2\left(\frac{\sigma_y}{A}\right)^2}} \frac{1}{A} dy = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} dy = p(y, \mu_y, \sigma_y) dy$$

- ☞ The new probability distribution for y , $p(y, \mu_y, \sigma_y)$, is also described by a Gaussian.



Start with a Gaussian with $\mu = 10, \sigma = 2$
 Get another Gaussian with $\mu = 20, \sigma = 4$

- ◆ Example: When the new distribution is non-Gaussian: $y = 2/x$.
 - The transformed probability distribution function for y does not have the form of a Gaussian.



Start with a Gaussian with $\mu = 10$, $\sigma = 2$.
DO NOT get another Gaussian!
 Get a *pdf* with $\mu = 0.2$, $\sigma = 0.04$.
 This new *pdf* has longer tails than a Gaussian *pdf*:
 $\text{Prob}(y > \mu_y + 5\sigma_y) = 5 \times 10^{-3}$, for Gaussian $\approx 3 \times 10^{-7}$

- *Unphysical situations can arise if we use the propagation of errors results blindly!*
 - ◆ Example: Suppose we measure the volume of a cylinder: $V = \pi R^2 L$.
 - Let $R = 1$ cm exact, and $L = 1.0 \pm 0.5$ cm.
 - Using propagation of errors:

$$\sigma_V = \pi R^2 \sigma_L = \pi/2 \text{ cm}^3$$

$$V = \pi \pm \pi/2 \text{ cm}^3$$
 - If the error on V (σ_V) is to be interpreted in the Gaussian sense
 - ☞ finite probability ($\approx 3\%$) that the volume (V) is < 0 since V is only 2σ away from than 0!
 - ☞ **Clearly this is unphysical!**
 - ☞ **Care must be taken in interpreting the meaning of σ_V .**