

## Lecture 7

### Some Advanced Topics using Propagation of Errors and Least Squares Fitting

#### Error on the mean (review from Lecture 4)

- Question: If we have a set of measurements of the same quantity:

$$x_1 \pm \sigma_1 \quad x_2 \pm \sigma_2 \dots x_n \pm \sigma_n$$

- ◆ What's the best way to combine these measurements?
- ◆ How to calculate the variance once we combine the measurements?
- ◆ Assuming Gaussian statistics, the Maximum Likelihood Methods combine the measurements as:

$$x = \frac{\sum_{i=1}^n x_i / \sigma_i^2}{\sum_{i=1}^n 1 / \sigma_i^2} \quad \text{weighted average}$$

- ◆ If all the variances ( $\sigma_1^2 = \sigma_2^2 = \dots \sigma_n^2$ ) are the same:

$$x = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{unweighted average}$$

- ◆ The variance of the weighted average can be calculated using propagation of errors:

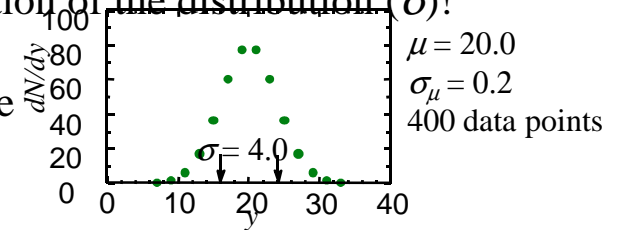
$$\sigma_x^2 = \sum_{i=1}^n \left[ \frac{\partial}{\partial x_i} x \right]^2 \sigma_i^2 = \sum_{i=1}^n \frac{1 / \sigma_i^4}{\left[ \sum_{i=1}^n 1 / \sigma_i^2 \right]^2} \sigma_i^2 = \frac{1}{\left[ \sum_{i=1}^n 1 / \sigma_i^2 \right]^2} \sum_{i=1}^n 1 / \sigma_i^2$$

$$\sigma_x^2 = \frac{1}{\sum_{i=1}^n 1 / \sigma_i^2} \quad \sigma_x \text{ is the error in the weighted mean}$$

- ◆ If all the variances are the same:

$$\sigma_x^2 = 1 / \sum_{i=1}^n 1 / \sigma_i^2 = 1 / [n / \sigma^2] = \frac{\sigma^2}{n}$$

- ☞ The error in the mean ( $\sigma_x$ ) gets smaller as the number of measurements ( $n$ ) increases.
- Don't confuse the error in the mean ( $\sigma_x$ ) with the standard deviation of the distribution ( $\sigma$ )!
- If we make more measurements
  - ☞ the standard deviation ( $\sigma$ ) of the distribution remains the same
  - ☞ the error in the mean ( $\sigma_x$ ) decreases



### More on Least Squares Fit (LSQF)

- In Lec 5, we discussed how we can fit our data points to a linear function (straight line) and get the "best" estimate of the slope and intercept. However, we did not discuss two important issues:
  - ◆ How to estimate the uncertainties on our slope and intercept obtained from a LSQF?
  - ◆ How to apply the LSQF when we have a non-linear function?
- Estimation of Errors from a LSQF
  - ◆ Assume we have data points that lie on a straight line:

$$y = \alpha + \beta x$$

- Assume we have  $n$  measurements of  $x$ 's and  $y$ 's.
- For simplicity, assume that each  $y$  measurement has the same error  $\sigma$ .
- Assume that  $x$  is known much more accurately than  $y$ .
  - ☞ ignore any uncertainty associated with  $x$ .
- Previously we showed that the solution for the intercept  $\alpha$  and slope  $\beta$  is:

$$\alpha = \frac{\sum_{i=1}^n y_i \sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i y_i \sum_{i=1}^n x_i}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \quad \text{and} \quad \beta = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}$$

- Since  $\alpha$  and  $\beta$  are functions of the measurements ( $y_i$ 's)
  - ▢ use the Propagation of Errors technique to estimate  $\sigma_\alpha$  and  $\sigma_\beta$ .

$$\sigma_Q^2 = \sigma_x^2 \left( \frac{\partial Q}{\partial x} \right)^2 + \sigma_y^2 \left( \frac{\partial Q}{\partial y} \right)^2 + 2\sigma_{xy} \left( \frac{\partial Q}{\partial x} \right) \left( \frac{\partial Q}{\partial y} \right)$$

- ▣ Assumed that each measurement is independent of each other:

$$\sigma_Q^2 = \sigma_x^2 \left( \frac{\partial Q}{\partial x} \right)^2 + \sigma_y^2 \left( \frac{\partial Q}{\partial y} \right)^2$$

$$\sigma_\alpha^2 = \sum_{i=1}^n \sigma_{y_i}^2 \left( \frac{\partial \alpha}{\partial y_i} \right)^2 = \sigma^2 \sum_{i=1}^n \left( \frac{\partial \alpha}{\partial y_i} \right)^2$$

$$\frac{\partial \alpha}{\partial y_i} = \frac{\partial}{\partial y_i} \frac{\sum_{i=1}^n y_i \sum_{j=1}^n x_j^2 - \sum_{i=1}^n x_i y_i \sum_{j=1}^n x_j}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2} = \frac{\sum_{j=1}^n x_j^2 - x_i \sum_{j=1}^n x_j}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2}$$

$$\sigma_\alpha^2 = \sigma^2 \sum_{i=1}^n \left( \frac{\sum_{j=1}^n x_j^2 - x_i \sum_{j=1}^n x_j}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2} \right)^2 = \sigma^2 \sum_{i=1}^n \left( \frac{\left( \sum_{j=1}^n x_j^2 \right)^2 + x_i^2 \left( \sum_{j=1}^n x_j \right)^2 - 2x_i \sum_{j=1}^n x_j \sum_{j=1}^n x_j^2}{\left( n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2 \right)^2} \right)$$

$$\begin{aligned} \sigma_{\alpha}^2 &= \sigma^2 \frac{n(\sum_{j=1}^n x_j^2)^2 + \sum_{i=1}^n x_i^2 (\sum_{j=1}^n x_j)^2 - 2(\sum_{j=1}^n x_j)^2 \sum_{j=1}^n x_j^2}{(n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2)^2} = \sigma^2 \frac{n(\sum_{j=1}^n x_j^2)^2 - \sum_{i=1}^n x_i^2 (\sum_{j=1}^n x_j)^2}{(n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2)^2} \\ &= \sigma^2 \sum_{j=1}^n x_j^2 \frac{n \sum_{j=1}^n x_j^2 - (\sum_{j=1}^n x_j)^2}{(n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2)^2} \\ \sigma_{\alpha}^2 &= \sigma^2 \frac{\sum_{j=1}^n x_j^2}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \quad \text{variance in the intercept} \end{aligned}$$

👉 We can find the variance in the slope ( $\beta$ ) using exactly the same procedure:

$$\begin{aligned} \sigma_{\beta}^2 &= \sum_{i=1}^n \sigma_{y_i}^2 \left( \frac{\partial \beta}{\partial y_i} \right)^2 = \sigma^2 \sum_{i=1}^n \left( \frac{\partial \beta}{\partial y_i} \right)^2 = \sigma^2 \sum_{i=1}^n \left( \frac{\partial}{\partial y_i} \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \right)^2 = \sigma^2 \sum_{i=1}^n \left( \frac{nx_i - \sum_{j=1}^n x_j}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} \right)^2 \\ &= \sigma^2 \frac{n^2 \sum_{j=1}^n x_j^2 + n(\sum_{j=1}^n x_j)^2 - 2n \sum_{i=1}^n x_i \sum_{j=1}^n x_j}{(n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2)^2} = \sigma^2 \frac{n^2 \sum_{j=1}^n x_j^2 - n(\sum_{j=1}^n x_j)^2}{(n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2)^2} \end{aligned}$$

$$\sigma_{\beta}^2 = \frac{n\sigma^2}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}$$

variance in the slope

- If we don't know the true value of  $\sigma$ ,

- ▢ estimate variance using the spread between the measurements ( $y_i$ 's) and the fitted values of  $y$ :

$$\sigma^2 \approx \frac{1}{n-2} \sum_{i=1}^n (y_i - y_i^{fit})^2 = \frac{1}{n-2} \sum_{i=1}^n (y_i - \alpha - \beta x_i)^2$$

- 👉  $n - 2 =$  number of degree of freedom

- $=$  number of data points – number of parameters ( $\alpha, \beta$ ) extracted from the data

- If each  $y_i$  measurement has a different error  $\sigma_i$ :

$$\sigma_{\alpha}^2 = \frac{1}{D} \sum_{i=1}^n \frac{x_i^2}{\sigma_i^2}$$

$$\sigma_{\beta}^2 = \frac{1}{D} \sum_{i=1}^n \frac{1}{\sigma_i^2}$$

weighted slope and intercept

$$D = \sum_{i=1}^n \frac{1}{\sigma_i^2} \sum_{i=1}^n \frac{x_i^2}{\sigma_i^2} - \left( \sum_{i=1}^n \frac{x_i}{\sigma_i^2} \right)^2$$

- 👉 The above expressions simplify to the “equal variance” case.

- ▢ Don't forget to keep track of the “ $n$ 's” when factoring out  $\sigma$ . For example:

$$\sum_{i=1}^n \frac{1}{\sigma_i^2} = \frac{n}{\sigma^2} \quad \text{not} \quad \frac{1}{\sigma^2}$$



- LSQF with non-linear functions:

- ◆ For our purposes, a non-linear function is a function where one or more of the parameters that we are trying to determine (e.g.  $\alpha$ ,  $\beta$  from the straight line fit) is raised to a power other than 1.

- Example: functions that are non-linear in the parameter  $\tau$ .

$$y = A + x / \tau$$

$$y = A + x\tau^2$$

$$y = Ae^{-x/\tau}$$

- ☞ These functions are linear in the parameters  $A$ .

- ◆ The problem with most non-linear functions is that we cannot write down a solution for the parameters in a closed form using, for example, the techniques of linear algebra (i.e. matrices).

- Usually non-linear problems are solved numerically using a computer.

- Sometimes by a change of variable(s) we can turn a non-linear problem into a linear one.

- ☞ Example: take the natural log of both sides of the above exponential equation:

$$\ln y = \ln A - x / \tau = C - Dx$$

- A linear problem in the parameters  $C$  and  $D$ !

- In fact its just a straight line!

- ☞ To measure the lifetime  $\tau$  (Lab 6) we first fit for  $D$  and then transform  $D$  into  $\tau$ .

- ◆ Example: Decay of a radioactive substance. Fit the following data to find  $N_0$  and  $\tau$ .

$$N = N_0 e^{-t/\tau}$$

- $N$  represents the amount of the substance present at time  $t$ .

- $N_0$  is the amount of the substance at the beginning of the experiment ( $t = 0$ ).

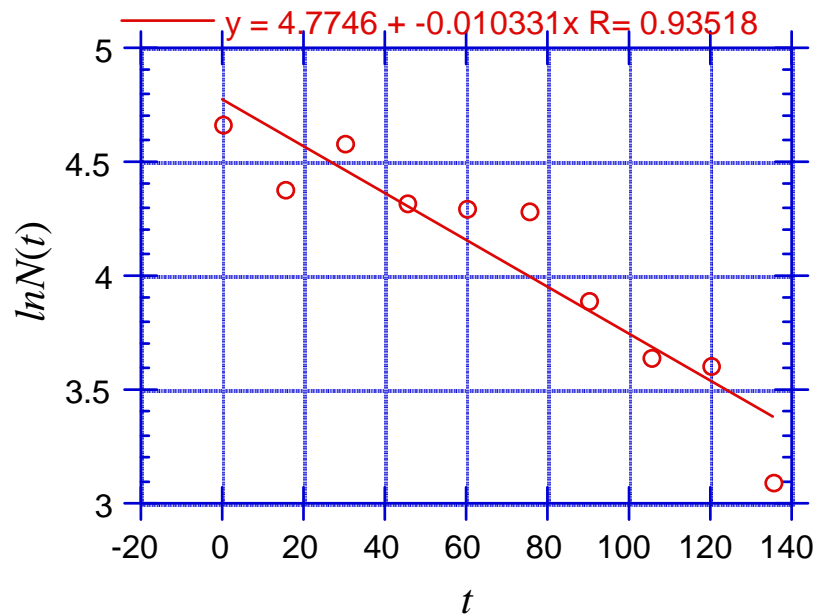
- $\tau$  is the lifetime of the substance.

$i$	1	2	3	4	5	6	7	8	9	10
$t_i$	0	15	30	45	60	75	90	105	120	135
$N_i$	106	80	98	75	74	73	49	38	37	22
$y_i = \ln N_i$	4.663	4.382	4.585	4.317	4.304	4.290	3.892	3.638	3.611	3.091

$$D = -\beta = -\frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n y_i \sum_{i=1}^n x_i}{n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2} = -\frac{10 \times 2560.41 - 40.773 \times 675}{10 \times 64125 - (675)^2} = 0.01033$$

$$\tau = 1/D = 96.80 \text{ sec}$$

- The intercept is given by:  $C = 4.77 = \ln A$  or  $A = 117.9$



- ◆ Example: Find the values  $A$  and  $\tau$  taking into account the uncertainties in the data points.
  - The uncertainty in the number of radioactive decays is governed by Poisson statistics.
  - The number of counts  $N_i$  in a bin is assumed to be the average ( $\mu$ ) of a Poisson distribution:

$$\mu = N_i = \text{Variance}$$

- The variance of  $y_i (= \ln N_i)$  can be calculated using propagation of errors:

$$\sigma_y^2 = \sigma_N^2 (\partial y / \partial N)^2 = (N)(\partial \ln N / \partial N)^2 = (N)(1/N)^2 = 1/N$$

- The slope and intercept from a straight line fit that includes uncertainties in the data points:

$$\alpha = \frac{\sum_{i=1}^n \frac{y_i}{\sigma_i^2} \sum_{i=1}^n \frac{x_i^2}{\sigma_i^2} - \sum_{i=1}^n \frac{x_i y_i}{\sigma_i^2} \sum_{i=1}^n \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2} \sum_{i=1}^n \frac{x_i^2}{\sigma_i^2} - \left(\sum_{i=1}^n \frac{x_i}{\sigma_i^2}\right)^2} \quad \text{and} \quad \beta = \frac{\sum_{i=1}^n \frac{1}{\sigma_i^2} \sum_{i=1}^n \frac{x_i y_i}{\sigma_i^2} - \sum_{i=1}^n \frac{x_i}{\sigma_i^2} \sum_{i=1}^n \frac{y_i}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2} \sum_{i=1}^n \frac{x_i^2}{\sigma_i^2} - \left(\sum_{i=1}^n \frac{x_i}{\sigma_i^2}\right)^2}$$

Taylor P. 198  
and Problem 8.9

☞ If all the  $\sigma$ 's are the same then the above expressions are identical to the unweighted case.

$$\alpha = 4.725 \quad \text{and} \quad \beta = -0.00903$$

$$\tau = -1/\beta = 1/0.00903 = 110.7 \text{ sec}$$

- To calculate the error on the lifetime, we first must calculate the error on  $\beta$ :

$$\sigma_\beta^2 = \frac{\sum_{i=1}^n \frac{1}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2} \sum_{i=1}^n \frac{x_i^2}{\sigma_i^2} - \left(\sum_{i=1}^n \frac{x_i}{\sigma_i^2}\right)^2} = \frac{652}{652 \times 2684700 - (33240)^2} = 1.01 \times 10^{-6}$$

$$\sigma_\tau^2 = \sigma_\beta^2 (\partial \tau / \partial \beta)^2 \Rightarrow \sigma_\tau = \sigma_\beta \left(1/\beta^2\right) = \frac{1.005 \times 10^{-3}}{(9.03 \times 10^{-3})^2} = 12.3$$

- ☞ The experimentally determined lifetime is

$$\tau = 110.7 \pm 12.3 \text{ sec.}$$