Intro

# Higher Functional Analysis Program

## Based on recent work

[arXiv: 2410.05120 | Chen, F, Hungar, Penneys, Sanford]

[arXiv: 2404.05193 | F]

[arXiv: 2403.01651 | F + 10 wonderful human beings]

[arXiv: 2411.01678 | Henriques, Nivedita, Penneys]

[F+10] $\vdots$

[F] $\{\text{operator } 3\text{-algebra}\} \xrightarrow{\text{Rep}} \{\text{operator } 3\text{-categories}\}$

$B(\mathfrak{H})$, $\mathfrak{H}$; $4\mathrm{Hilb} \ni 3\mathrm{Hilb}$

[CFHPS] $\{\text{operator } 2\text{-algebra}\} \xrightarrow{\text{Rep}} \{\text{operator } 2\text{-categories}\}$

$B(\mathcal{X})$, $\mathcal{X}$; $3\mathrm{Hilb} \ni 2\mathrm{Hilb}$

$\{\text{operator algebras}\} \xrightarrow{\text{Rep}} \{\text{operator categories}\}$ [HNP]

$B(H)$, $H$; $2\mathrm{Hilb} \ni \mathrm{Hilb}$

$\mathbb{C} \xrightarrow{\text{Rep}} \{\text{Hilbert Spaces}\}$

$\mathrm{Hilb} \ni \mathbb{C}$

Giovanni Ferrer

THE Ohio State U.

ferrer.40@osu.edu

## Motivation

Facts: For an operator algebra $A$:

1. $A$ can always be viewed as acting on a Hilbert space $H$
   (i.e. $\exists$ faithful rep'n $A \curvearrowright H$)
2. The representations $\mathrm{Rep}(A)$ of $A$ contains interesting data:
   - Spectrum (irreps)
   - States (cyclic rep$^{\underline{n}}$s)
   - When $A$: factor, its type

   etc.

Hence, the representations of $A$ (which form an operator category) are worthy of study in their own right.

Inductively, this gives rise to higher Hilbert spaces and higher operator alg's.

**Def$^n$:** By an operator algebra, we mean:

$$\text{(f.d.) } H^*\text{-algebras} = \left\{ \begin{array}{l} \text{(f.d.) } C^*/W^*\text{-algebra } \mathcal{A} \\ + \text{ faithful trace } \mathrm{Tr} \end{array} \right\}$$

(not normalized)

**Rem$^k$:** We equip our operator alg's with traces in order to recover them from their rep$^n$s.
(+ quantum invariants for manifolds)

**Rem$^k$:** Equivalently, the data of a (f.d.) $H^*$-alg is:

(f.d.) $*$-algebra + Hilbert space structure on $\mathcal{A}$

s.t. $\langle ab, c\rangle = \langle b, a^*c\rangle = \langle a, cb^*\rangle$

(i.e. $L_a^* = L_{a^*}$ and $R_a^* = R_{a^*}$)

Rep(A) is a 1D math. structure: $\begin{cases} \text{objs: } H_A, K_A \quad \text{reps} \\ \text{arrows: } H \xrightarrow{f} K \quad A\text{-intertwiners} \end{cases}$

Fact: [completeness] For $H_A, K_A$ modules for $A$, we may form a module $(H \oplus K)_A = H_A \oplus K_A$

and $\{K \leq H \text{ is a submodule}\} \longleftrightarrow \{\text{projection } p_K \in B_A(H) \text{ A-module intertwiner}\}/\sim$

Fact: [finite basis] Every module $H_A$ can be decomposed as fin. $\oplus$ of distinct simple rep$^n$s

Fact: [higher $\langle \cdot, \cdot \rangle$] Given a module $H_A$ for $A$: $H^*$-alg, $\exists\, Tr_H$ on $B_A(H)$:

- For $\xi \in H$, consider: $L_\xi : A_A \to H_A$
  $a \mapsto \xi a$.
- For $\xi, \eta \in H$, set: $\langle \xi | \eta \rangle_A := L_\xi^* \circ L_\eta \in A$ ($\cong \mathrm{End}_A(A)$)

- Now set: $Tr_H(|\eta\rangle\langle\xi|) := Tr_A(\langle \xi | \eta \rangle_A)$
  (which extends to all $\mathrm{End}_A(H)$)

Rep(A) is a (f.d.) 2-Hilbert space (in the sense of [Baez])!

(Turns out all 2-Hilbert spaces arise in this way!)

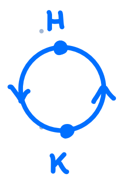
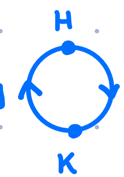
## Recap

**Data:** $\mathcal{H}$ : ($\mathbb{C}$-linear) 1-category + $\mathrm{Tr} = \{\mathrm{Tr}_H : \mathcal{E}nd(H) \to \mathbb{C}\}_{H \in \mathcal{H}}$

$*$-operation $\left( H \xrightarrow{f} K \right)^* = H \xleftarrow{f^*} K$

$\mathcal{H}\langle H, K \rangle := \Big\{$  $\Big\}$ (f.d.) Hilbert space

$\mathcal{H}\langle H, H \rangle$ w. composition is an $H^*$-algebra

- tracial: $\mathrm{Tr}^{\mathcal{H}}_H\left( H \xrightarrow{f} K \xrightarrow{g} H \right) = \mathrm{Tr}^{\mathcal{H}}_K\left( K \xrightarrow{g} H \xrightarrow{f} K \right)$ $\Big( \mathrm{Tr}^{\mathcal{H}}_H\big( f$  $g \big) = \mathrm{Tr}^{\mathcal{H}}_H\big( g$  $f \big) \Big)$
- pos. def: $\mathrm{Tr}^{\mathcal{H}}_H\left( H \xrightarrow{f} K \xrightarrow{f^*} H \right) \geq 0$ w. equality only when $H \xrightarrow{f} K = 0$.

# 2-Hilbert spaces

**Rem$^{k}$:** Equivalently, the data of a 2-Hilbert space is a:

(f.s.s.) $*$-category + Hilbert space structures on

$$\mathcal{H}\langle H, K\rangle := \{ H \Downarrow K \}$$

s.t. $\langle H \xrightarrow{f} L \xrightarrow{g} K,\ H \xrightarrow{h} K \rangle = \langle L \xrightarrow{g} K,\ L \xrightarrow{f^*} H \xrightarrow{g} K \rangle = \langle H \xrightarrow{f} L,\ H \xrightarrow{g} K \xrightarrow{h^*} L \rangle$

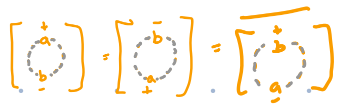
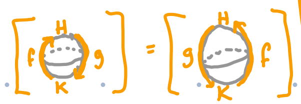
**Intuition:**

$\mathbb{C} : \mathrm{Hilb} :: \mathrm{Hilb} : 2\text{-}\mathrm{Hilb}$

| | |
|---|---|
| $\mathbb{C} \ni \langle \xi, \eta \rangle$ | $\mathrm{Hilb} \ni \mathcal{H}\langle H, K\rangle$ |
| $\xi + \eta$ | $H \oplus K$ |
| for $\lambda \in \mathbb{C}$: $\lambda \cdot \xi$ | for $\mathbb{C}^n \in \mathrm{Hilb}$: $\mathbb{C}^n \cdot H = H^{\oplus n}$ |
| $H^* = \mathrm{Fun}(H \to \mathbb{C})$ | $\mathcal{H}^* := \mathrm{Fun}(\mathcal{H} \to \mathrm{Hilb})$ |
| $\bar{H} \to H^*$<br>$\xi \mapsto \langle \xi \vert \cdot \rangle$ | $\bar{\mathcal{H}} \to \mathcal{H}^*$<br>$H \mapsto \mathcal{H}\langle H, \cdot\rangle$<br>$f \downarrow \ \mapsto\ \downarrow - \circ f^*$<br>$K \mapsto \mathcal{H}\langle H, \cdot \rangle$ |
| Riesz-Rep. Thm<br>$\bar{H} \cong H^*$ | Yoneda Embedding Thm<br>$\bar{\mathcal{H}} \cong \mathcal{H}^*$ |

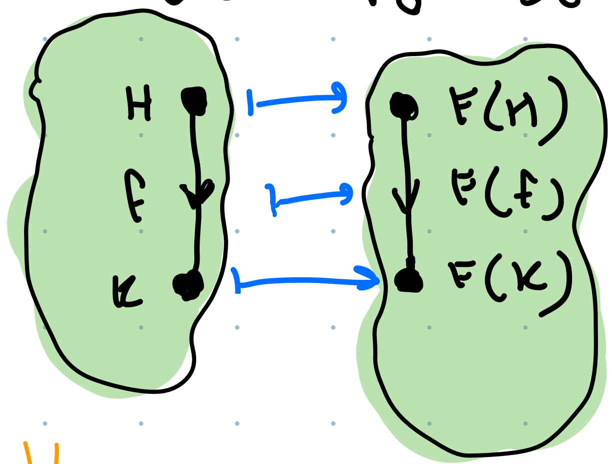
Functionals

(co) sheaves

| inner product | trace |
|---|---|
| evaluation of oriented 0-spheres in a 1-sphere | evaluation of oriented 1-spheres in a 2-sphere |
| <br>$\langle a, b \rangle = \langle \bar{b}, \bar{a} \rangle = \overline{\langle b, a \rangle}$ | <br>$\mathrm{Tr}_K(f \circ g) = \mathrm{Tr}_H(g f)$ |

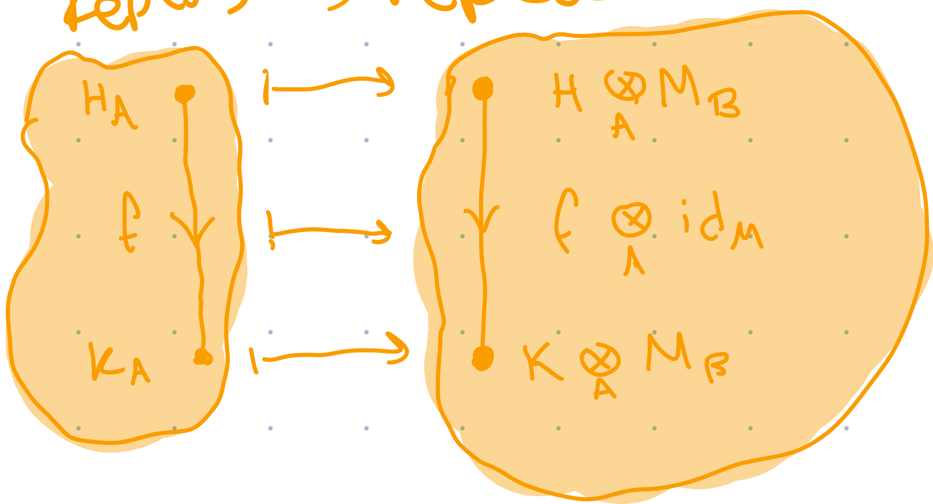
# Operators on 2-Hilbert spaces

<u>Def$^{\underline{n}}$</u>: An operator $\mathcal{H} \xrightarrow{F} \mathcal{K}$ between 2-Hilbert spaces is a †-functor




<u>Ex</u>: For ${}_AM_B$ an A-B bimodule,

There's an operator $\mathrm{Rep}(A) \xrightarrow{R_M} \mathrm{Rep}(B)$




(Turns out all operators arise in this way!)

## Adjoints

Prop$^{\underline{n}}$ [HNP]: For an operator $\mathcal{H} \xrightarrow{F} \mathcal{K}$, there's a (unitary) adjoint operator:

i.e. $\exists!$ $\mathcal{H} \xleftarrow[F^*]{} \mathcal{K}$

s.t. $\mathcal{H}\langle H, F^*(K)\rangle = \mathcal{K}\langle F(H), K\rangle$ (as Hilbert spaces)

(Fact: $(GF)^* = F^* G^*$ and $F^{**} = F$.)

Ex: For ${}_A M_B$ an A-B bimodule, consider ${}_B M^*_A := \{\varphi : M \to \mathbb{C} \mid (b \cdot \varphi \cdot a)(m) = \varphi(a \cdot m \cdot b)\}$

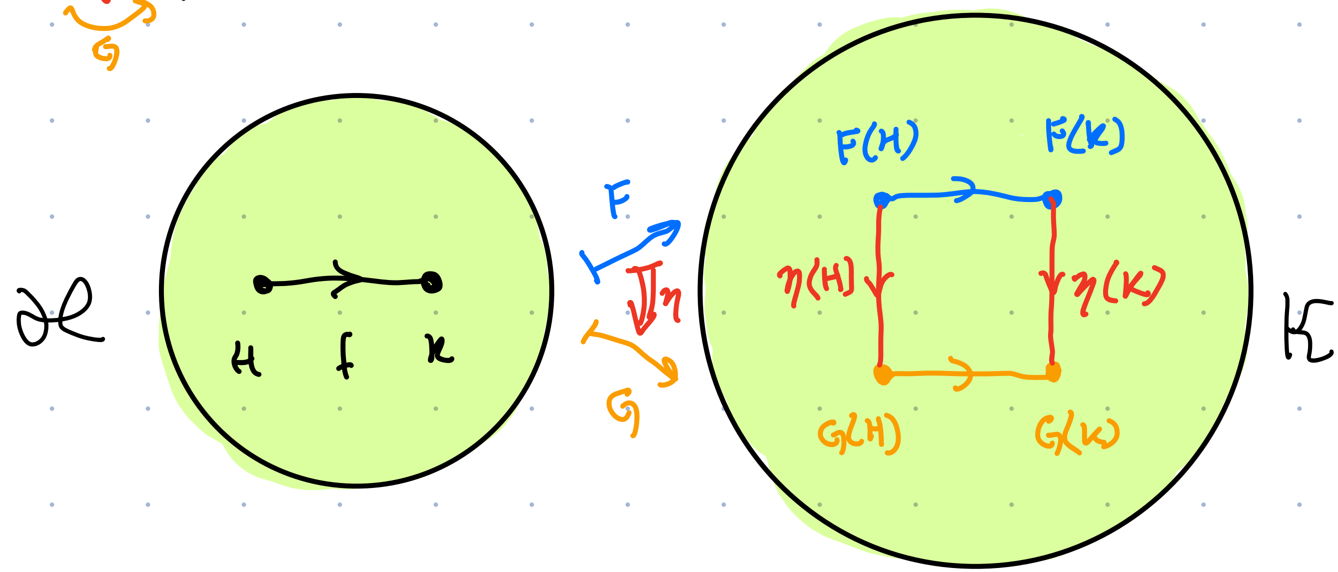
The operator $\operatorname{Rep}(A) \xrightarrow{R_M} \operatorname{Rep}(B)$

has adjoint $\operatorname{Rep}(A) \xleftarrow[R_{M^*}]{} \operatorname{Rep}(B)$

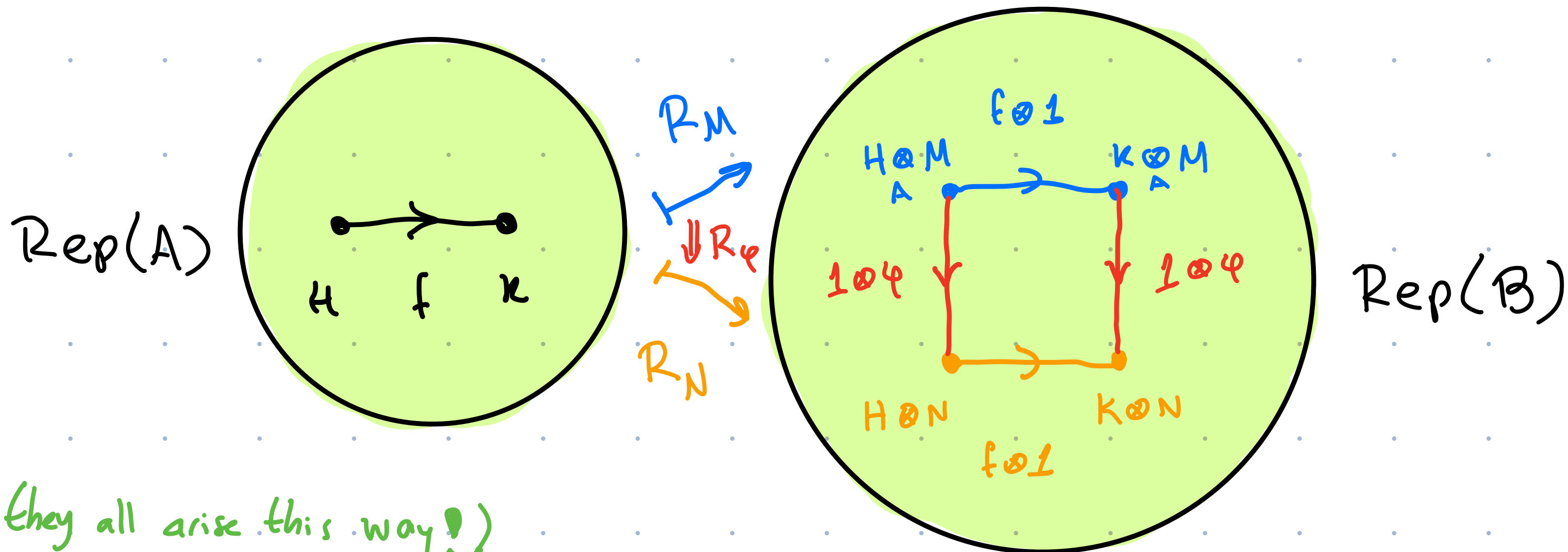
(i.e. $R_M^* = R_{M^*}$) (also, $M^* \cong \overline{M}$, the contragradient bimod.)

# Operators between operators

Def$^{\underline{n}}$: Operators $\mathcal{H} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{K}$ with $\Downarrow\eta$ allow operators (nat. trans.) between them!




Ex: For an (adjointable) A-B bimodule intertwiner $\varphi : {}_A M_B \to {}_A N_B$ :




(Turns out they all arise this way!)

# More adjoints

One can define another adjoint $\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow \eta^*"] & \mathcal{K} \end{tikzcd}$ for such higher operators so that:

$$B(\mathcal{H},\mathcal{K}) := \left\{ \begin{array}{ll} \underline{\text{objects}}: & \begin{tikzcd} \mathcal{H} \arrow[r, "F"] & \mathcal{K} \end{tikzcd} \\ \underline{\text{arrows}}: & \begin{tikzcd} \mathcal{H} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{K} \end{tikzcd} \end{array} \right\}$$

forms a 2-Hilbert space!

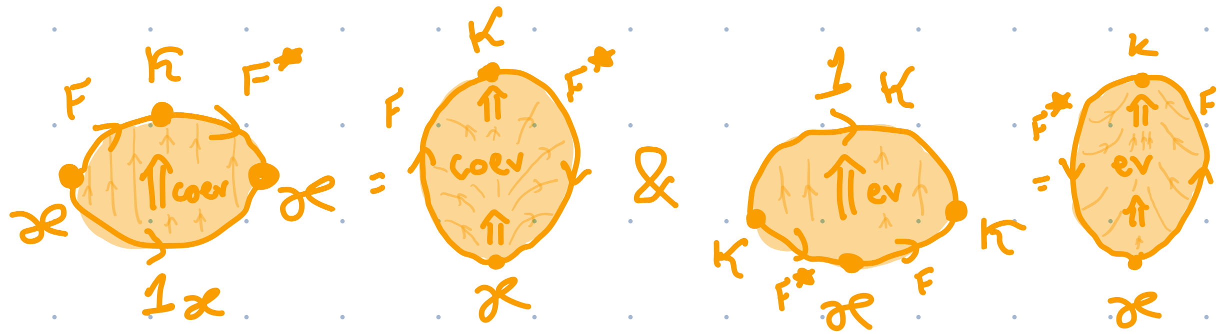
with unitary trace:

$$\mathrm{Tr}_F\left( \begin{tikzcd} \mathcal{H} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{K} \end{tikzcd} \right) = \sum_{\substack{\{H_i\} \subseteq \mathcal{H} \\ \text{ONB}}} \mathrm{Tr}^{\mathcal{K}}_{F(H_i)}\big(\eta(H_i)\big) \, \overbrace{\mathrm{Tr}^{\mathcal{H}}_{H_i}(id_{H_i})}^{d_{H_i}}$$

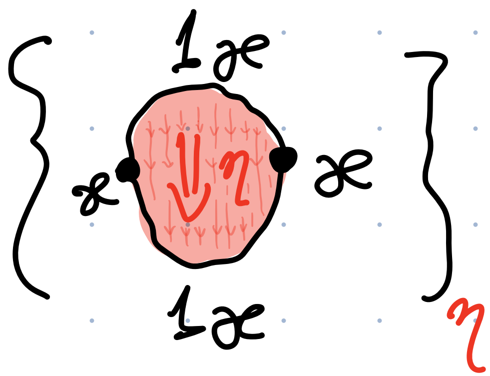
$$\left( = \text{“}\int_{H \in \mathcal{H}} \eta(H)\, dH\text{”} \right)$$

(Operators between (f.d.) 2-Hilbert spaces form a 2-Hilbert space)

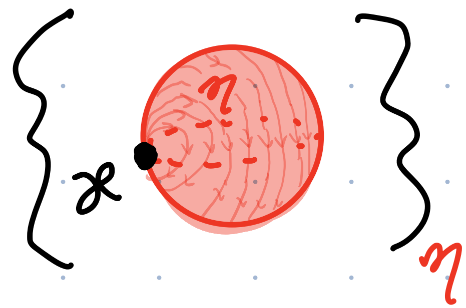
Fact: $F$ and $F^*$ admit:




Fact: $\mathcal{B}(1_{\mathcal{X}}) :=$




$=$

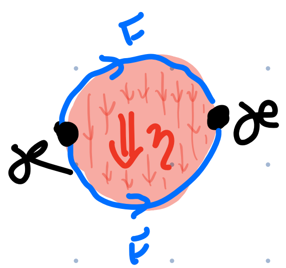



$(\pi_2(\mathcal{X})!)$

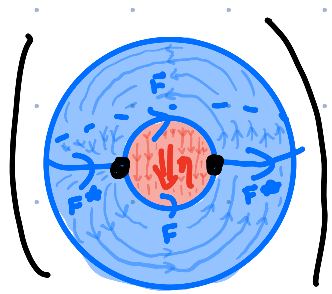
is a commutative $C^*/W^*$-algebra with a spherical weight

i.e. a weight $\Psi_{\mathcal{X}} : \mathcal{B}(1_{\mathcal{X}}) \to \mathbb{C}$

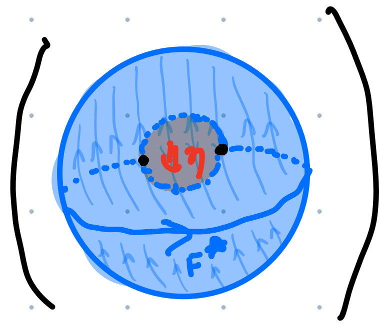
s.t. $\forall$




, $\Psi_{\mathcal{X}}\Big($




$\Big) = \Psi_{\mathcal{X}}\Big($



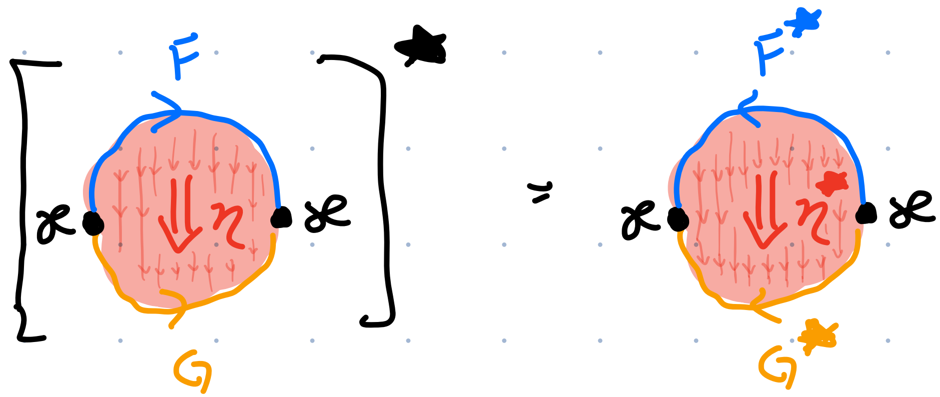

$\Big)$

# Higher Operator algebras

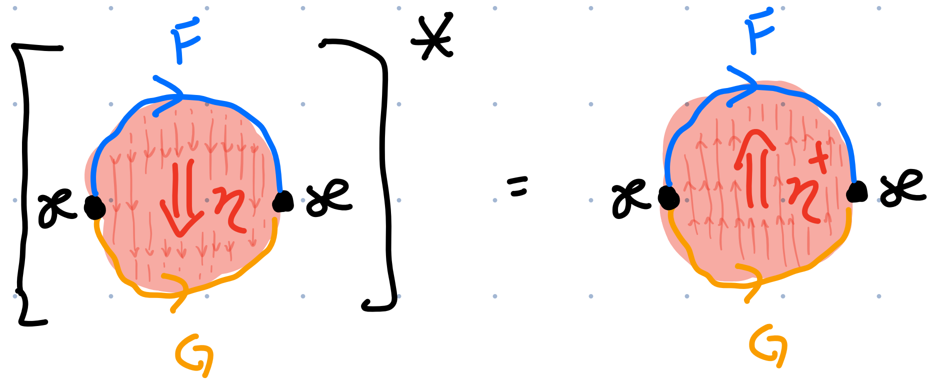
In particular, $\mathcal{B}(\mathcal{H})$ is a $\underbrace{\text{2-Hilbert space with } \otimes}_{\text{(prototypical example of O2A)}}$

<u>Fact</u>: $\mathcal{B}(\mathcal{H})$ has 2 involutions $\bigstar$ and $*$:

(horizontal)




(vertical)




+ spherical weight $\Psi_{\mathcal{H}}$ on $\mathcal{B}(1_{\mathcal{H}})$

# Higher operator algebras

<u>Def</u>$^{\underline{n}}$ [CFHPS] By an operator 2-algebra, we mean:

$$H^*\text{-multifusion cat.} := \Big\{ (\underset{\text{mFC}}{\mathcal{A}}, \overset{\text{involutions}}{*, \bigstar}, \underset{\text{spherical weight}}{\Psi}) \Big\}$$

<u>Rem</u>$^{\underline{k}}$: Equivalently, an $H^*$-mFC is:

$\mathcal{A}$ : UmFC + 2-Hilbert space structure

s.t. $\mathcal{A}\langle FG, H\rangle \cong \mathcal{A}\langle F, HG^{\bigstar}\rangle \cong \mathcal{A}\langle G, F^{\bigstar}H\rangle$ as Hilbert spaces.

We may now study modules $\quad \mathcal{A} \longrightarrow B(\mathcal{H})$

$H^*$-mFC $\quad$ ⊗-functor preserving $*$ and $\bigstar$ $\quad$ 2-Hilbert space

# Even higher Hilbert spaces

Rep(A) is a 2D math. struct.:

$$\left\{\begin{array}{ll} \text{objs:} & \mathcal{H}_A,\ \mathcal{K}_A,\ \ldots \ \text{rep}^{\text{n}}\text{s} \\ \text{arrows:} & \mathcal{H} \xrightarrow{F} \mathcal{K} \\ \text{surfaces:} & \mathcal{H} \underset{G}{\overset{F}{\Downarrow \eta}} \mathcal{K} \end{array}\right\}$$

with 2 involutions $*$, $*$

Fact: Direct sums and submodules play well with $A$-actions

So do operators that intertwine $A$-actions

Fact: Every rep$^{\text{n}}$ $\mathcal{H}_A$ can be decomposed $\overset{\text{finite}}{\boxplus}$ of irreducible rep$^{\text{n}}$s

Fact: Given a module $\mathcal{H}_A$ for $A$: $\exists$ spherical weight $\Psi_{\mathcal{H}} : \mathcal{B}_A(1_{\mathcal{H}}) \to \mathbb{C}$

$$\Psi_{\mathcal{H}}\left(\mathcal{H}\bullet\ \eta\right) = \sum_{\substack{\{H_i\} \subseteq \mathcal{H} \\ \text{ONB}}} \mathrm{Tr}^{\mathcal{H}}_{H_i}\left(H_i \bullet \circlearrowright^{\eta(H_i)}\right) \mathrm{Tr}^{\mathcal{H}}_{H_i}\left(1_{H_i}\right)$$

$$\left(= \text{``}\int_{\mathcal{H}} \eta \, dH\text{''}\right)$$

Rep(A) is a (f.d.) 3-Hilbert space (in the sense of [CFHPS])!

(Turns out all 3-Hilbert spaces arise in this way!)

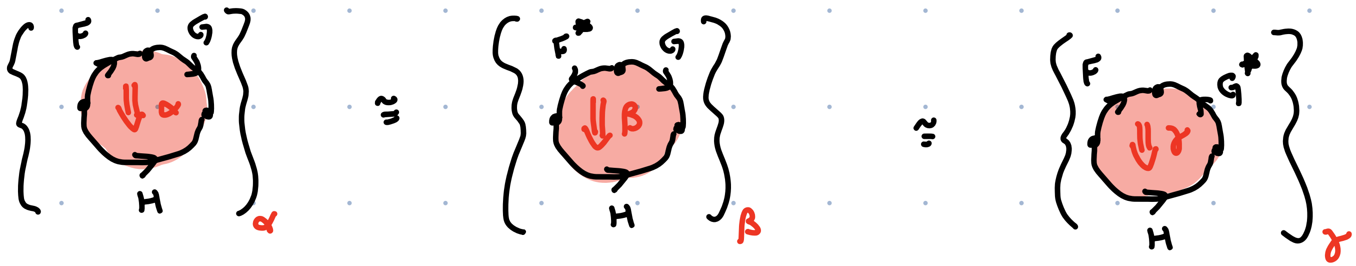
## Even higher Hilbert spaces

Rem^k: Equivalently, the data of a 3-Hilbert space is a:

(f.s.s.) $*$-2-category + 2-Hilbert space structures

on each space $\mathfrak{H}\langle \mathcal{X}, \mathcal{K}\rangle := \left\{ \begin{tikzcd} \mathcal{X} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathcal{K} \end{tikzcd} \right\}$ s.t.

$$\mathfrak{H}\langle \mathcal{X} \xrightarrow{F} \mathcal{K} \xrightarrow{G} \mathcal{L}, \mathcal{X} \xrightarrow{H} \mathcal{L}\rangle = \mathfrak{H}\langle \mathcal{L} \xrightarrow{G} \mathcal{K}, \mathcal{L} \xrightarrow{F^*} \mathcal{X} \xrightarrow{G} \mathcal{K}\rangle = \mathfrak{H}\langle \mathcal{X} \xrightarrow{F} \mathcal{L}, \mathcal{X} \xrightarrow{G} \mathcal{K} \xrightarrow{H^*} \mathcal{L}\rangle$$




One repeats the same story to obtain:

(1) operators between 3-Hilbert spaces
(2) operators between operator
(3) operators between operators between operators

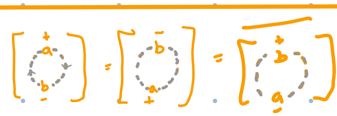
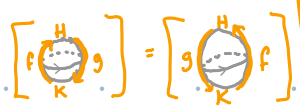
Intuition:

$\mathbb{C} : \mathrm{Hilb} :: \mathrm{Hilb} : 2\text{-}\mathrm{Hilb} :: 2\text{-}\mathrm{Hilb} : 3\text{-}\mathrm{Hilb}$

functionals

| | | |
|---|---|---|
| $\mathbb{C} \ni \langle \xi, \eta \rangle$ | $\mathrm{Hilb} \ni \mathcal{H}\langle H, K \rangle$ | $2\mathrm{Hilb} \ni \mathfrak{H}\langle \mathcal{H}, \mathcal{K} \rangle$ |
| $\xi + \eta$ | $H \oplus K$ | $\mathcal{H} \boxplus \mathcal{K}$ |
| for $\lambda \in \mathbb{C}$: $\lambda \cdot \xi$ | for $\mathbb{C}^n \in \mathrm{Hilb}$: $\mathbb{C}^n \cdot H = H^{\oplus n}$ | for $\mathrm{Hilb}^n \in 2\mathrm{Hilb}$, $\mathrm{Hilb}^n \cdot \mathcal{H} = \mathcal{H}^{\oplus n}$ |
| $H^* = \mathrm{Fun}(H \to \mathbb{C})$ | $\mathcal{H}^* := \mathrm{Fun}(\mathcal{H} \to \mathrm{Hilb})$ sheaves | $\mathfrak{H}^+ := \mathrm{Fun}(\mathfrak{H} \to 2\mathrm{Hilb})$ |
| $\bar{H} \to H^*$<br>$\xi \mapsto \langle \cdot, \xi \rangle$ | $\bar{\mathcal{H}} \to \mathcal{H}^*$<br>$H \mapsto e\langle H, \cdot \rangle$<br>$f \downarrow \;\mapsto\; \downarrow - \circ f^+$<br>$K \mapsto e\langle H, \cdot \rangle$ | $\bar{\mathfrak{H}} \to \mathfrak{H}^+$<br>similar |
| Riesz-Rep. Thm<br>$\bar{H} \cong H^*$ | Yoneda Embedding Thm<br>$\bar{\mathcal{H}} \cong \mathcal{H}^*$ | Future (easy) thm.<br>$\bar{\mathfrak{H}} \cong \mathfrak{H}^+$ |

stacks

| | | |
|---|---|---|
| inner product | trace | spherical weight |
| evaluation of oriented 0-spheres in a 1-sphere | evaluation of oriented 1-spheres in a 2-sphere | evaluation of oriented 2-spheres in a 3-sphere |
| <br>$\langle a, b \rangle = \langle \bar{b}, \bar{a} \rangle = \overline{\langle b, a \rangle}$ | <br>$\mathrm{Tr}_K(f \circ g) = \mathrm{Tr}_H(g \circ f)$ |  |

Outro

# Higher Functional Analysis Program

**Based on:**

[arXiv: 2410.05120 | Chen, F, Hungar, Penneys, Sanford]

[arXiv: 2404.05193 | F]

[arXiv: 2403.01651 | F + 10 wonderful human beings]

[arXiv: 2411.01678 | Henriques, Nivedita, Penneys]

[F+10] ⋮

[F] {operator 3-algebras} $\xrightarrow{\mathrm{Mod}^\dagger}$ {operator 3-categories}

$\mathrm{End}(\mathfrak{H}) \circlearrowright \mathfrak{H}$

$4\mathrm{Hilb} \ni 3\mathrm{Hilb}$

[CFHPS] {operator 2-algebras} $\xrightarrow[\mathrm{Mod}^\dagger]{\sim}$ {operator 2-categories} ①

$\mathrm{End}(\mathcal{H}) \circlearrowright \mathcal{H}$

$\Sigma^\dagger$

$3\mathrm{Hilb} \ni 2\mathrm{Hilb}$

We define these

{operator algebras} $\xrightarrow[\mathrm{Mod}^\dagger]{\sim}$ {operator categories}

$B(H) \circlearrowright H$

$\Sigma^\dagger$

$2\mathrm{Hilb} \ni \mathrm{Hilb}$

$\mathbb{C} \xrightarrow[\mathrm{Mod}^\dagger]{}$ {Hilbert Spaces}

$\mathbb{C} \circlearrowright \mathbb{C}$

$\Sigma^\dagger$

$\mathrm{Hilb} \ni \mathbb{C}$

② We show $\mathrm{Mod}^\dagger$ are equivalences

③ We provide a procedure called unitary condensation $\Rightarrow$ unitary suspension

$\Sigma^\dagger : n\mathrm{Hilb} \to (n+1)\mathrm{Hilb}$

④ Weird fact:

| 1D | 2D | 3D |
|---|---|---|
| ⓪ Every (f.d.) vector space admits a Hilbert space structure | ⓪ Every (f.s.s.) 2-vector space admits a 2-Hilbert space structure | ⓪ Every (f.s.s.) 3-vector space admits a 3-Hilbert space structure |
| ① Every algebraic map is "bounded" | ① Every algebraic map is "bounded" | ① Every algebraic map is "bounded" |
| | ② Every map between them is "bounded" | ② Every map between them is "bounded" |
| | | ③ Every map between those is "bounded" |

nope!

But!
[Reutter]
[CFHPS]

thanks!