Apeironomicon

{The Book of Infinity}

Grimoire of the mathematical dark arts

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Contents

Slavish subservience to the shibboleth: What is a space?

> In the classical teaching of Calculus, these ideas are immediately obscured by the accidental fact that, on a one-dimensional vector space, there is a one- to-one correspondence between linear forms and numbers, and therefore the derivative at a point is defined as a number instead of a linear form.
>
> This slavish subservience to the shibboleth of numerical interpretation at any cost becomes much worse...
>
> Jean Dieudonné

We swiftly recall background definitions and results we will be relying on, in order to establish notation and for general completeness. This section can be skipped or skimmed, and the reader can refer to it later at any point of confusion.

I.1 Incantations of Topological Spaces

Definition I.1.1 — A topological space $(X, \mathscr{O}(X))$ consists of a set X equipped with a *topology*:

$$\mathscr{O}(X) \subseteq \mathscr{P}(X) := \{S \mid S \subseteq X\},$$

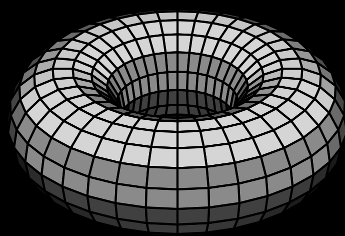
i.e. a collection of subsets of X such that:

$$\bigcup_{U \in I} U \in \mathscr{O}(X) \qquad \text{for} \quad I \subseteq \mathscr{O}(X) \tag{$\exists$}$$

$$U \cap V \in \mathscr{O}(X) \qquad \text{for} \quad U, V \in \mathscr{O}(X) \tag{$\wedge$}$$

$$X \in \mathscr{O}(X) \qquad \text{for} \quad \text{ever and ever.} \tag{$\top$}$$

Behold, a picture of a topological space:



Remark I.1.2. First, we note that taking $I = \varnothing$ in ($\exists$) yields $\varnothing \in \mathscr{O}(X)$, which the reader might have expected in ($\top$). Now we address the cryptic labeling we have employed for these conditions. Indeed, recall:

$$\bigcup_{U \in I} U \coloneqq \{x \in X \mid \exists U : x \in U\} \tag{$\exists$}$$

$$U \cup V \coloneqq \{x \in X \mid x \in U \wedge x \in V\} \tag{$\wedge$}$$

$$X \coloneqq \{x \in X \mid \top\} \tag{$\top$}$$

where we use $\top$ to denote True. Hence, we may interpret $\bigcup$, $\cap$, and $\top$ within the *context* of the topological space X as the analogues for $\exists$, $\wedge$, and $\top$ respectively. Fear not, we plan on make this analogy much more rigorous...

Exercise I.1.3. Find the analogue for *negation* $\neg$ within the context of a topological space X.

(☠) Interpret the *principle of excluded middle*[1] in such a context.

(ii) Need this principle always hold? If not (*wink wink*), when does it?

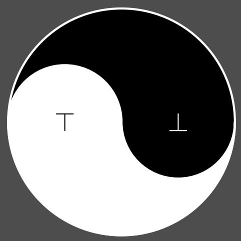
(iii) What can you say about a space where excluded middle holds?

(iv) Engage in deep contemplation about what this means.

DEFINITION I.1.4 (Sierpiński) — The *Sierpiński space* $(\Omega, \mathscr{O}(\Omega))$ is a topological space with two points

[1] *To be or not to be, that is the question!*

$\Omega := \{\top, \bot\}$ and only one non-trivial open set $\{\top\}$.



DEFINITION I.1.5 (Nice examples) —

(i) The real line $\mathbb{R}$ is a topological space with open sets being *actual open* sets;[a]

(ii) The path $I := [0,1]$ is a topological space with open sets coming from those in $\mathbb{R}$;[b]

(iii) In general, flat euclidean space $\mathbb{R}^n$ is again a topological space with *actual open* sets;[c]

(iv) The circle S^1 is a topological space with open sets coming from those in $\mathbb{R}^2$;

1. The torus $S_1 \times S_1$
2. The cylinder $S_1 \times \mathbb{R}$ or annulus $\mathbb{R} \times S_1$
3. The Möbius strip [[I got bored, finish box later]]

[a] That is, $U \subseteq \mathbb{R}$ is open when every point $x \in U$ admits an open interval $x \in (x-\varepsilon, x+\varepsilon) \subseteq U$
[b] More explicitly, $U \subseteq I$ is open when there is an open $\widetilde{U} \subseteq \mathbb{R}$ with $\widetilde{U} \cap I = U$.
[c] Just replace open intervals with open balls $B_\varepsilon(x) := \{y \in \mathbb{R}^n \mid \|x-y\| < \varepsilon\}$ of radius $\varepsilon > 0$

[[more examples]]

DEFINITION I.1.6 — A map $f\colon X \to Y$ induces three maps on power sets:

(*) The *preimage* map $f^{-1}\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ acts by $S_Y \mapsto f^{-1}(S_Y) := \{x \mid f(x) \in S_Y\}$;

(∃) The *direct image* map $f_\exists\colon \mathscr{P}(X) \to \mathscr{P}(Y)$ acts by $S_X \mapsto f_\exists(S_X)$ where

$$f_\exists(S_X) := \{y \in Y \mid \exists x \in f^{-1}(y) : x \in S_X\};$$

(∀) The *exclusive image* map $f_\forall\colon \mathscr{P}(X) \to \mathscr{P}(Y)$ acts by $S_X \mapsto f_\forall(S_X)$ where

$$f_\forall(S_X) := \{y \in Y \mid \forall x \in f^{-1}(y) : x \in S_X\}.$$

When X and Y are equipped with topologies, we say f is *continuous* when the preimage map f^* restricts to a map of topologies

$$f^*\colon \mathscr{O}(Y) \to \mathscr{O}(X).$$

and denote the set of all such continuous maps by $\mathrm{Hom}(X,Y)$.

Remark I.1.7. [[Why the preimage?]]

Exercise I.1.8. For $S_X \subseteq X$ and $S_Y \subseteq Y$, verify the following equivalent conditions:

$$\begin{array}{ccc} f_\exists(S_X) \subseteq S_Y & \Leftrightarrow & S_X \subseteq f^{-1}(S_Y) \\ S_Y \subseteq f_\forall(S_X) & \Leftrightarrow & f^{-1}(S_Y) \subseteq S_X \end{array}$$

Deduce that $f_\exists$ and f^* preserve unions, whereas f^* and $f_\forall$ preserve intersections.

Exercise I.1.9. Show that Ω *classifies* open sets/subspaces, that is, for every topological space X there is a bijection of sets:

$$\mathcal{O}(X) \xrightarrow[\sim]{|\cdot|} \mathrm{Hom}(X \to \Omega).$$

Moreover, show this *realization* map $|\cdot|$ is *natural* in X, i.e.

- For $f \in \mathrm{Hom}(X \to Y)$, there is a map $f^*\colon \mathrm{Hom}(Y \to \Omega) \to \mathrm{Hom}(X \to \Omega)$ given by *precomposition* $g \mapsto g \circ f$ such that the following diagram commutes[2]:

$$\begin{array}{ccc} \mathscr{O}(Y) & \xrightarrow{|\cdot|} & \mathrm{Hom}(Y \to \Omega) \\ {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathscr{O}(X) & \xrightarrow[|\cdot|]{} & \mathrm{Hom}(X \to \Omega) \end{array}$$

Note: For this reason, many people write f^* instead of f^{-1} to denote the preimage map.

Whereas points assemble into a topological space, topological spaces themselves assemble into a *higher* space. Before delving into the aspects of *higher* topologies, we first need to extend the notion of a set, obtaining a *higher* set or a *category*.

[[Gio: This definition I gave for a category kinda fucking sucks lmao. The last time I thought of the definition of a category was in the context of Lie/Étale groupoids, where this was the best way to look at it]]

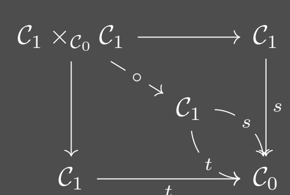
DEFINITION I.1.10 — A *category* $(\mathcal{C}_0, \mathcal{C}_1, \circ)$ consists of:

(i) Collections of *objects* $\mathcal{C}_0$ and *morphisms* $\mathcal{C}_1$ with *source*, *target*, and *identity* maps:

$$\mathcal{C}_0 \begin{array}{c} \xleftarrow{\;s\;} \\ \xrightarrow{\;i\;} \\ \xleftarrow[\;t\;]{} \end{array} \mathcal{C}_1$$

Note: We denote the subset $s^{-1}(a) \cap t^{-1}(b) \subseteq \mathcal{C}_1$ by $\mathrm{Hom}(a,b)$, and the morphisms $i(c)$ by id_c.

$(\circ)$ A *composition* $\circ$ rule for morphisms *in series*, i.e. a map:



Here $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ simply refers to the subset of $\mathcal{C}_1 \times \mathcal{C}_1$ with (f, g) such that $t(f) = s(g)$. Moreover, the commutativity of the top and bottom squares means that $\circ$ takes $(a \xrightarrow{f} b,\, b \xrightarrow{g} c)$ to some $a \xrightarrow{g \circ f} c$.

[2] By this, we mean that the two possible ways of getting from the top-left corner to the bottom-right corner agree. In this case, we simply mean that $|\cdot| \circ f^{-1} = f^* \circ |\cdot|$ holds.

Note: We require $\circ$ to be associative and unital, with identity id_c for every $c \in \mathcal{C}_0$.

Exercise I.1.11. Verify that topological spaces Top_0 and continuous maps Top_1 with their inherited function composition $\circ$ form a category, which we will denote by Top.

Exercise I.1.12. Verify that $\mathscr{O}(X)$ is a category with hom-spaces[3]

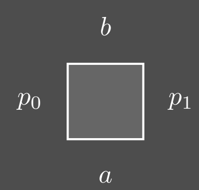
$$\mathrm{Hom}(U, V) := \begin{cases} \top & \text{if} \quad U \subseteq V \\ \bot & \text{else} \end{cases}$$

where we've used $\top$ (True) and $\bot$ (False) to denote the singleton set $* = \{\subseteq\}$ and the empty set $\varnothing = \{\,\}$ respectively in this context. In more concrete terms, there exists a morphism $U \xrightarrow{\subseteq} V$ if and only if $U \subseteq V$. [[find arbitrary coproducts, finite products, and terminal objects. In particular, what's the initial object]]

Definition I.1.13 — A *path* in a topological space X is a continuous map $p \colon I \to X$. We refer to $s(p) := p(0)$ as the *source* of p and to $t(p) := p(1)$ as its *target*.

Given two paths $p_0, p_1 \colon I \to X$ with the same source a and target b, a *homotopy* $H \colon p_0 \Rightarrow p_1$ is a continuous map $H \colon I \times I \to X$ such that:

$$H(0, \cdot\,) = p_0 \qquad H(1, \cdot\,) = p_1 \qquad H(\,\cdot\,, 0) = a \qquad H(\,\cdot\,, 1) = b$$



Exercise I.1.14. For a topological space X, verify there is a category $\Pi_1(X)$ of points in X and paths (up to homotopy) between them known as the homotopy *groupoid* of X. Indeed, show that every morphism in $\Pi_1(X)$ is invertible.
Note: A category such that every morphism is an isomorphism is known as a groupoid.
Note: We will later see that all groupoids arise in this way.

I.2 Incantations of Vector Spaces

I.3 Incantations of Smooth Spaces

Definition I.3.1 (Manifold) — We say that a topological space X is an *(n-dimensional) manifold* for $n \geq 0$ if it is:

- (locally trivial) For every $x \in X$, there is an open neighborhood $x \in U \subseteq X$ such that $U \cong \mathbb{R}^n$.

[3] More precisely, these are hom (-1)-spaces. We will later harness the power of negative categorical thinking; but for now, we simply make the reader aware of the fact that there are two (-1)-categories, corresponding to *truth* and *falsehood*. Is U contained in V?

Example I.3.2 — Flat euclidean space $\mathbb{R}^n$ is of course an n-dimensional manifold. Further examples include:

(i) $\mathbb{S}^n$

(ii) $\mathbb{T}^n$

(iii) $\mathbb{D}^n$

[[Gio: Add definition and pictures!]]

[[Gio: Schemes generalize manifolds: Manifold $M \leftrightarrow$ Sheaf $C\colon \mathscr{O}(M)^{\text{op}} \to$ Ring where locally $C(U) \cong C(\mathbb{R}^n)$ (by Ring I do kinda want C*-algebras). Notice $C\colon \mathscr{O}(\mathbb{R}^n)^{\text{op}} \to$ Ring is an affine scheme (i.e. the spectrum of the ring $C(\mathbb{R}^n)$ as a locally ringed space) A scheme is just a sheaf of rings which is locally equivalent to an affine scheme, that is, the spectrum of a chosen ring R]]

Remark I.3.3. Given an n-dimensional manifold M, such a trivialization $U \cong \mathbb{R}^n$ of an open $U \subseteq M$ can be thought of:

(i) As a *chart* $\varphi\colon U \xrightarrow{\sim} \mathbb{R}^n$ over U [[Gio: as in nautical maps (draw the analogy)]]

(ii) As a *parametrization* $\psi\colon \mathbb{R}^n \xrightarrow{\sim} U$ of U [[Gio: as in Calculus (draw the analogy)]]

Moreover, given two such trivializations $U \cong \mathbb{R}^n$ and $V \cong \mathbb{R}^n$ in M, there is a partially defined map $\mathbb{R}^n \to \mathbb{R}^n$ induced by their intersection $U \cap V$. Indeed, consider the map

$$t_{U,V}\colon \mathbb{R}^n \to U \cap V \to \mathbb{R}^n$$

where the first map is the parametrization of U co-restricted to $U \cap V$, and the second map is the chart over V restricted to $U \cap V$. Of course, we could have swapped the roles of U and V, yielding a partially-defined inverse $t_{U,V}^{-1} = t_{V,U}$.

Hence, in general, these so-called *transition maps* are homeomorphisms on their domains. However, $\mathbb{R}^n$ has more structure than just that of a topological space. Indeed, as the stage on which Calculus is defined, we may bootstrap the structure of $\mathbb{R}^n$ to manifolds in order to obtain a *differential calculus* or *geometry* on curved spaces.

Definition I.3.4 — An *(n-dimensional) smooth manifold* is a manifold M equipped with a choice of covering $M = \bigcup_i U_i$ by trivializations $U_i \cong \mathbb{R}^n$ such that:

- (transition maps) All transition maps are *smooth*, i.e. infinitely differentiable.

Such a choice of trivializing cover is known as a *smooth structure* or *smooth atlas.*[a]

[a] Technically, there is a way to compare atlases for which one should think of compatible atlases as encoding the same manifold. Therefore one should choose a maximal smooth atlas in order to make things work without hiccups. We will opt not to worry about such lame details.

Remark I.3.5. [[Gio: Equivalently, a smooth manifold is a covering of M by embeddings $\text{Hom}(\mathbb{R}^n \hookrightarrow M)$ such that blah. Can this be tied into covering sieves?]]

[[Gio: Riemannian Manifolds]]
[[Gio: Lorenzian Manifolds]]

I.4 Incantations of Homotopy Spaces

II

A Faustian bargain: What is an algebra?

> Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.
>
> — Michael Atiyah

[[Brett: I think there should be a chapter on "algebra" that emphasizes the role of algebras as dual to spaces - either here or after the chapter on sheaves - but beginning to build towards the Gelfand duality etc etc picture. and paralleling the space chapter, by giving a type of algebra for each type of space]]

[[Brett: also if spaces are Incantations then algebras could be Chants, maybe? not great at the theming. or go with the devil theme and call them Bargains or Contracts. i'm just loosely titling them for now to nail down what goes in each section]]

[[Gio: Oh this is fucking siiick. I love this! Hm what about conjurations?]]

Contrary to what Atiyah suggests, one really can have it all: algebra does not exist at the expense of geometry. Rather, algebra is the *dual* of geometry, and every type of space discussed in Chapter [[I]] has an algebraic doppelgänger.

II.1 boolean algebras, CABAs, lattices, locales - algebras of sets

II.2 vector spaces redux, this time focusing on algebra. probably make this a chapter on modules and basic homological algebra

II.3 c star algebras

II.4 lurie math, maybe?

III

Steps towards higher spaces: What is a site?

The wise man looks into space and does not regard the small as too little, nor the great as too big, for he knows that there is no limit to dimensions.

Zhuang Zhou

Recall that, for a category $\mathcal{X}$, a presheaf of a given flavor of mathematical structure on $\mathcal{X}$ is simply a contravariant functor:

$$\mathcal{X}^{\mathrm{op}} \to \mathsf{Structures}$$

where Structures can be taken to be category of Sets, Groups, Ab, Vec, Hilb, C*Alg, W*Alg, etc. depending on our interests. But where does the term "pre-sheaf" come from, and what would make a "pre"-sheaf not just "pre"?

Idea. *If presheaves are like functions on a space, sheaves are to be thought of as the* ***continuous*** *functions.*

But in order to talk about "continuity" of a functor on a category, we need the notion of a *topology* on a category: thus turning it into a *higher space* known as a *site*.

[[Gio: Question for Brett: Vec/Hilb should play the role of Set. Of course, this is not an actual topos but *morally* this should be the canonical example of a *linear topos*. In what way is this true? Wtf is up with subobjects? :( ]]

[[Gio: Cool Reference: Bohr Topos (nLab) is a topos theoretic approach to quantum mechanics involving C*-algebras. From the looks of it, everything seems to be done in terms of sheaves of sets...]]

Idea. *The category $\mathcal{O}(X)$ of open sets is the same data as a topological space X, and is hence a space.*[1]

Albeit unorthodox, we may describe the topology on X as follows:

For every $x \in X$, there is a collection of so-called open neighborhoods $T_x \subseteq \mathcal{O}(X)$ of x where each $U \in T_x$ contains $x \in U$. These are required to satisfy the following three axioms:

[1] In this note we will use this idea to motivate the notion of a *site*. In particular, we will focus on abstracting "open covers" to arbitrary categories. However, there is another, equally valid direction in formalizing the very same idea. Indeed, one can instead consider lattices which behave like the lattice $\mathcal{O}(X)$ by admitting:

- finite limits (meets, greatest lower bounds, intersections) $\wedge = \cap$,
- arbitrary colimits (joins, least upper bounds, unions) $\bigvee = \bigcup$
- initial (minimal, smallest, False) and terminal (maximal, greatest, True) objects $0 = \varnothing$ and $1 = X$.

These lattices are known as *frames/locales* in the field of *pointless geometry* (pun intended). These also play a role in intuitionistic logic, where they are known as *complete Heyting algebras*. Of course these structures will turn out to be closely related to sites. In fact, the representations of a site $\mathcal{X}$ will form a so-called *topos* $Sh(\mathcal{X})$, where a topos is the categorification (homotopification) of a locale.

Algebraic geometry cheat sheet

Category level 0	Category level 1
Set	Category
Topology	Grothendieck Topology
Space	Site
Function $X \to \mathbb{C}$	Presheaf $\mathcal{X}^{\mathrm{op}} \to \mathsf{Hilb}$
Continuous function $f\colon X \to \mathbb{C}$	Sheaf $F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Hilb}$
Abelian group $C(X)$	Topos $Sh(\mathcal{X})$
$(f+g)(x) := f(x) + g(x)$	$(F \oplus G)(U) := F(U) \oplus G(U)$
Vector space $C(X)$	2-vector space $Sh(\mathcal{X})$
Scalars $\lambda \in \mathbb{C}$	Hilbert spaces $\Lambda \in \mathsf{Hilb}$
$(\lambda \triangleright f)(x) := \lambda \cdot f(x)$	$(\Lambda \triangleright F)(U) := \Lambda \otimes F(U)$
Algebra $C(X)$	Monoidal category $Sh(\mathcal{X})$
$(f \cdot g)(x) := f(x) \cdot g(x)$	$(F \otimes G)(U) := F(U) \otimes G(U)$
Commutative C*-algebra $C(X)$	Symmetric *C*-2-algebra* $Sh(\mathcal{X})$
$\overline{f}(x) := \overline{f(x)}$	$\overline{F}(U) := \overline{F(U)}$

(T1) If $V \in T_y$ is an open neighborhood of $y \in X$, then $V \cap T_x \subseteq T_y$.[2]

(T2) For every collection of points $\{x_j\} \subset X$, and every collection of open neighborhoods $\{U_{ij}\} \subseteq T_{x_j}$, we have $\bigcup_{ij} U_{ij} \in T_{x_j}$ for every x_j.

(T3) $X \in T_x$ is an open neighborhood for every $x \in X$

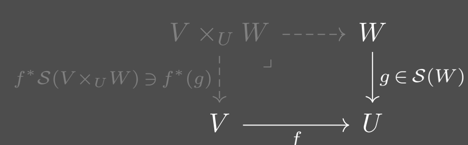
Notice (T1) captures the fact that the topology on X is closed under finite intersections, (T2) captures closure under arbitrary unions, and (T3) implies X is open.[3]

One thing to note is that the points $x \in X$ generally live "outside" of $\mathcal{O}(X)$, as the singletons $\{x\}$ are seldom open. Hence, one should *morally* modify these conditions in terms of coverings for open sets U, instead of neighborhoods for a point x.[4] This leads us to the following notion of a Grothendieck topology:

Definition III.0.1 — A *Grothendieck topology* τ on a category $\mathcal{X}$ consists of the following data:

For every $U \in \mathcal{X}$, there is a collection of so-called covering sieves τ_U where each $S \in \tau_U$ is a subfunctor of $\mathcal{X}(- \to U)$.[a] These are required to satisfy the following three axioms:

(T1) For $f \in \mathcal{X}(V \to U)$ and $\mathcal{S} \in \tau_U$, we have $f^*\mathcal{S} \in \tau_V$ where $f^*\mathcal{S}$ is the pullback[b] sieve:



(T2) For every covering sieve $\mathcal{S} \in \tau_U$, and every collection of covering sieves $\{\mathcal{S}_{U_j} \in \tau_{U_j}\}$, we have $\bigcup_j \mathcal{S}_{U_j} \circ \mathcal{S} \in \tau_U$ where

$$\left(\bigcup_j \mathcal{S}_{U_j} \circ \mathcal{S}\right)(V) = \bigcup_j \mathcal{S}_{U_j}(V) \circ \mathcal{S}(U_j) := \left\{ V \xrightarrow{g} U_j \xrightarrow{f} U \,\middle|\, g \in \mathcal{S}_{U_j}(V) \text{ and } f \in \mathcal{S}(U_j) \right\}$$

[2] Here we're not worrying about $\varnothing$

[3] Either include $\varnothing$ in each T_x (which is admittedly not great conceptually), or include it at the end

[4] We note that one should also be able to interpret this discussion in terms of filters and ultrafilters.

(T3) $\mathcal{X}(- \to U) \in \tau_U$ is a covering sieve for every $U \in \mathcal{X}$.

A site $(\mathcal{X}, \mathcal{T})$ is then a category $\mathcal{X}$ equipped with a Grothendieck topology τ.

[a] More concretely, a covering sieve $\mathcal{S}$ picks out a collection of morphisms $\mathcal{S}(V) \subseteq \mathcal{X}(V \to U)$ for every $V \in \mathcal{X}$, which we say *cover* U. In the case of $\mathcal{X} = \mathcal{O}(X)$, note that $\mathcal{X}(V \to U)$ is either $\varnothing$ or $\{\subseteq\}$. Hence, when defining a covering sieve $\mathcal{S}$ for U, we have a choice of whether or not to include V as part of our cover for U whenever $V \subset U$.

[b] In the case when $\mathcal{X} = \mathcal{O}(X)$, pulling back along an inclusion $V \subseteq U$ is the same as taking an intersection with V.

Remark III.0.2. As one can see, the notation for sieves gets a bit heavy and obfuscates this relatively easy concept: *A site is a category equipped with a notion of "covering", where*:

(T1) The preimage of a cover is a cover

(T2) Covering a cover is a cover

(T3) The whole space[5] covers everything

Alternatively, we can state these axioms in parallel to those of a topology as long as we're willing to squint at the meaning of "intersections", "unions", and "trivial":

(T1) Finite *intersections* of covers are covers

(T2) Arbitrary *unions* of covers are covers[6]

(T3) *Trivial* covers are covers

Of course, by construction, we recover our guiding example:

Example III.0.3 (Spaces are spaces) — The category $\mathcal{O}(X)$ of open sets with inclusions for a topological space X admits a natural Grothendieck topology τ, where $\mathcal{S} \in \tau_U$ is a covering sieve on an open set $U \subseteq X$ if and only if

$$U = \bigcup_{\mathcal{S}(V) \neq \varnothing} V$$

When working in categorification, one notices the following common motif:

Idea. *Structures of a certain mathematical flavor assemble into* ***higher*** *mathematical structures with a resembling taste.*

We see this for example as abelian groups themselves form an abelian category Ab, vector spaces form a 2-vector space Vec, Hilbert spaces form a 2-Hilbert space Hilb, and so on. Thus, in the spirit of (vertical) categorification, we should expect that topological spaces themselves form a higher space, i.e. a *site*.

Vertical categorification

Category level 0	Category level 1
Abelian groups	Abelian category Ab
Vector spaces	2-vector space Vec
Hilbert spaces	2-Hilbert space Hilb
Topological spaces	Site CHaus

[5] Following the tradition of Yoneda, we identify a space $\mathcal{X}$ with its Yoneda embedding.

[6] The ambiguity of this statement is particularly egregious, as we will see later on.

Example III.0.4 (Spaces are a space) — The category CHaus of compact Hausdorff spaces forms a site, where each non-trivial covering sieve $\mathcal{S}$ on a compact Hausdorff space X corresponds to a finite[a] cover $\{U_i\}_{i=1}^n$ of X with compact Hausdorff spaces U_i, i.e., $\mathcal{S}$ is determined by a surjective map:

$$\coprod_{i=1}^{n} U_i \twoheadrightarrow X.$$

[a] Here one might be worried about satisfying (T2). However it is true that the arbitrary "union" of finite covers $\{\mathcal{S}_{X_j}\}$ is again finite... not great, but the reader was warned. Indeed, in the formal statement of (T2), the union $\bigcup_j \mathcal{S}_{U_j}$ gets post-composed with a chosen finite cover $\mathcal{S}$, and now $\bigcup_j \mathcal{S}_{U_j} \circ \mathcal{S}$ is again finite.

We note that these covering sieves in CHaus are quite tame. In general, Grothendieck topologies can be much more fine or coarse. Indeed, just as in point-set topology, there are maximal and minimal Grothendieck topologies:

Example III.0.5 — For a category $\mathcal{X}$,

- The discrete topology is obtained by declaring every subfunctor $\mathcal{X}(- \to U)$ to be a covering sieve for every $U \in \mathcal{X}$;
- The trivial topology is obtained by declaring that the only covering sieves are $\mathcal{X}(- \to U)$.
- The canonical topology is obtained by declaring that the only covering sieves are the representable presheaves, i.e. those isomorphic to $\mathcal{X}(- \to U)$ for some $U \in \mathcal{X}$.

IV

Being hungry, they carry the sheaves: What is a sheaf?

> Those who go out weeping, carrying seed to sow, will return with songs of joy, carrying sheaves with them.
>
> Psalms 126:6

Now that we have fleshed out the notion of a topology on a category $\mathcal{X}$, we may talk about presheaves of a certain flavor

$$\mathcal{X}^{\mathrm{op}} \to \mathsf{Structures}$$

which preserve the topology we've chosen on our site.

Idea. If we think of a covering sieve $\mathcal{S}$ for $U \in \mathcal{X}$ as an honest covering of U, "preserving" $\mathcal{S}$ corresponds to satisfying a certain "gluing condition"[1] with respect to this covering. In the case when $\mathcal{X} = \mathcal{O}(X)$ for a topological space X, we think of a presheaf F as assigning a whole structure's-worth of functions over each $U \subset X$. Indeed, consider the prototypical example where $\mathsf{Structures} = \mathsf{Groups}$ and $F = C(- \to G)$

$$C(- \to G) \colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Groups}$$

assigns to each open $U \subset X$ the group of G-valued continuous functions for a topological group G:[2]

$$C(U \to G) \coloneqq \{f \colon U \to G \text{ continuous}\}.$$

Now for an open covering $\{V_i\} \subseteq X$ of U, consider the commutative diagram

$$\begin{array}{ccc} C(U \to G) & \longrightarrow & C(V_j \to G) \\ \downarrow & & \downarrow \\ C(V_i \to G) & \longrightarrow & C(V_i \cap V_j \to G) \end{array} \qquad \begin{array}{ccc} (f \colon U \to G) & \longmapsto & f|_{V_j} \\ \downarrow & & \downarrow \\ f|_{V_i} & \longmapsto & f|_{V_i \cap V_j} \end{array}$$

Observe that each $f \colon U \to G$ is uniquely determined by the collection of functions $(f_i \coloneqq f|_{V_i})$ where $f_i|_{V_j} = f_j|_{V_i}$. Conversely, given such a collection $(f_i \colon V_i \to G)$ of continuous functions, we may uniquely *glue* these to obtain a continuous $f \colon U \to G$. The slick way to express this categorically is that the following diagram is a

[1] You'll also hear of algebraic geometers talking about "descent conditions", which are synonymous.
[2] This is actually closely related to how one thinks of a topological group as a *condensed* group in Condensed Mathematics.

pullback square[3]:

$$\begin{array}{ccc} C(U \to G) & \longrightarrow & \coprod_j C(V_j \to G) \\ \downarrow & \lrcorner & \downarrow \\ \coprod_i C(V_i \to G) & \longrightarrow & \coprod_{ij} C(V_i \cap V_j \to G) \end{array}$$

In any case, what the category theory is trying to express is that the space $C(U \to G)$ is built from each $C(V_i \to G)$ by gluing along their intersections $C(V_i \cap V_j \to G)$.

Definition IV.0.1 (Sheaf) — A sheaf of groups $F\colon \mathcal{X} \to \mathsf{Groups}$ on a site $(\mathcal{X}, \tau)$ is a presheaf satisfying:

- For every covering sieve $\mathcal{S}$ on $U \in \mathcal{X}$, we have a pullback square:

$$\begin{array}{ccc} F(U) & \longrightarrow & \coprod_j F(V_j) \\ \downarrow & \lrcorner & \downarrow \\ \coprod_i F(V_i) & \longrightarrow & \coprod_{ij} F(V_i \times_U V_j) \end{array}$$

Of course, we may equivalently describe this gluing condition more concretely:

- For every covering sieve $\mathcal{S}$ on $U \in \mathcal{X}$, we have that every collection $(f_i \in F(V_i))$ with $f_i|_{V_j} = f_j|_{V_i}$ can be uniquely glued into $f \in F(U)$ such that each $f|_{V_i} = f_i$.[a]

[a]Here one actually needs to consider collections $(f_{i,s} \in F(V_i))$ indexed not only by objects $V_i \in \mathcal{X}$, but also by morphisms $s \in \mathcal{S}(V_i) \subseteq \mathcal{X}(V_i \to U)$. One then needs to reinterpret our restriction notation $f|_{V_i}$ as $F(s)(f)$ where $F(s)\colon F(U) \to F(V_i)$. As the notation is already quite cumbersome, we will omit such ennui.

Again, by construction, we obtain our first example:

Example IV.0.2 (Continuous functions form sheaves) — For a topological group G, the sheaf $C(- \to G)$ on $\mathcal{O}(X)$ is known as the *sheaf of germs of continuous G-valued functions on X*.

In the following section, we will discuss the meaning of this curious term “germ”. However, prior to such a digression, we present more examples of sheaves:

Example IV.0.3 (Manifolds form sheaves) — For a smooth manifold X, its *structure sheaf*

$$C^\infty \colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Vec}$$

assigns to an open set $U \subset X$ the vector space

$$C^\infty(U) := \{f\colon U \to \mathbb{R} \text{ or } \mathbb{C} \text{ smooth}\},$$

where the target depends on which flavor of manifolds one desires.

Non-example IV.0.4 (C*-algebras) — With the previous example in mind, one might wish to construct a C*-algebraic analogue of C^∞ as follows: For a compact Hausdorff space, we define

$$C\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{C^*Alg}$$

[3]or equalizer diagram, pick your poison.

which assigns to an open set $U \subset X$ the C*-algebra

$$C(U) := \{f : U \to \mathbb{C} \text{ continuous}\}.$$

Of course there's a problem in that functions in $C(U)$ need not be bounded because U is seldom compact, i.e. this obviously doesn't form a C*-algebra. But we may try to modify this construction by considering:

$$C_0 : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{C^*Alg} \text{ (not necessarily unital)}$$

on a locally compact Hausdorff space X, which assigns to an open set $U \subset X$ the C*-algebra

$$C_0(U) := \{f : U \to \mathbb{C} \text{ continuous and vanishing at infinity}\}.$$

Albeit a more promising candidate, as this does form a presheaf, we note that C_0 does *not* satisfy the desired gluing condition for sheaves.

Exercise IV.0.5. Find a locally compact Hausdorff space X so that $C_0 : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{C^*Alg}$ is not a sheaf on X. In particular, build a family of continuous functions vanishing at infinity $(f_i : U_i \to \mathbb{C})$ on an open cover $\{U_i\}$ of X which does *not* glue to a global function $f : X \to \mathbb{C}$ which vanishes at infinity.
Hint: Consider $X = \mathbb{R}$ covered by $U_i = (i-1, i+1)$ for $i \in \mathbb{Z}$.

Example IV.0.6 (C*-algebras v2.0) — A way to fix the previous non-example is to instead consider

$$C : \mathcal{O}(X)^{\mathrm{op}} \to \dagger\mathsf{Alg}$$

as a sheaf of †-algebras on our chosen compact Hausdorff space X. It just so happens that $C(X)$ is a C*-algebra[a], which is a *property* and not a *structure* on a †-algebra.

[a] More generally $C(\widetilde{X})$ is a C*-algebra for every component $\widetilde{X} \subseteq X$ as closed sets in a Hausdorff space are compact.

Example IV.0.7 (C*-algebras v3.0) — A second way to fix the previous non-example is to instead consider C as a sheaf on the category of closed sets $\mathscr{C}(X)$ with inclusions:

$$C : \mathscr{C}(X)^{\mathrm{op}} \to \mathsf{C^*Alg}$$
$$Y \mapsto C(Y).$$

Here we note that $C(Y)$ is a C*-algebra for every $Y \subseteq X$ as closed sets in a compact Hausdorff space are again compact Hausdorff. Of course, we may view each $C(Y)$ as globally defined sections supported on Y:

$$C(Y) \cong \{f \in C(X) \mid \mathsf{supp}[f] \subseteq Y\}.$$

In any case, we will refer to this as the *structure sheaf* for $C(X)$.

Remark IV.0.8. We provide an even "fancier" description of the structure sheaf. Indeed, notice there is a sheaf

$$よ_{\mathbb{C}} : \mathsf{CHaus}^{\mathrm{op}} \to \mathsf{C^*Alg}$$
$$X \mapsto \mathsf{LCHaus}(X \to \mathbb{C})$$

where LCHaus is the site of locally compact Hausdorff spaces. Equivalently, we may write

$$\mathsf{LCHaus}(X \to \mathbb{C}) = \lim_{r\to\infty} \mathsf{CHaus}(X \to B^r(0)) \qquad \text{where} \qquad \mathbb{C} = \lim_{r\to\infty} B^r(0),$$

is viewed as the colimit of discs of radius $r > 0$ centered at the origin 0. Now, for a particular $X \in \mathsf{CHaus}$, we may construct a site over X, namely the slice category $\mathsf{CHaus}\,/X$ of compact Hausdorff spaces $Y \xrightarrow{f} X$. Then $よ_{\mathbb{C}}$ induces a sheaf on $\mathsf{CHaus}\,/X$ by

$$\begin{aligned} C\colon (\mathsf{CHaus}\,/X)^{\mathrm{op}} &\to \mathsf{C^*Alg} \\ (Y \xrightarrow{f} X) &\mapsto \{Y \xrightarrow{f} X \xrightarrow{g} \mathbb{C} \mid g \in C(X)\} \end{aligned}$$

In particular, when $Y \xrightarrow{f} X$ is an inclusion $Y \subseteq X$, then $C(Y \xrightarrow{f} X) = C(Y)$ by the Tietze extension theorem.

Perhaps even more abstractly, $\mathsf{CHaus}\,/X$ can be viewed as the category $e\ell(よ_X)$ of elements of the sheaf

$$\begin{aligned} よ_X\colon \mathsf{CHaus}^{\mathrm{op}} &\to \mathsf{CHaus} \\ Y &\mapsto \mathsf{CHaus}(Y \to X). \end{aligned}$$

There is an action of each $よ_X(Y)$ on $C(X)$ by composition:

$$\mathsf{CHaus}(Y \to X) \times \mathsf{CHaus}(X \to \mathbb{C}) \xrightarrow{\circ} \mathsf{CHaus}(Y \to \mathbb{C})$$

from which we induce a sheaf $よ_X \triangleright C(X)\colon \mathsf{CHaus}^{\mathrm{op}} \to \mathsf{C^*Alg}$, and hence our desired one on the category of elements

$$C\colon (\mathsf{CHaus}\,/X)^{\mathrm{op}} \to \mathsf{C^*Alg}\,.$$

[[Now just you pray we don't generalize this further.]]

In any case, the site $\mathsf{CHaus}\,/X$ and $\mathscr{C}(X)$ are morally the same... [[how can this be formalized though? Are they "Morita equivalent" in some sense, i.e. do they have the same sheaves?]]

Example IV.0.9 (C*-algebras v4.0) — Conversely, we define the *structure cosheaf* of $C(X)$ by

$$\begin{aligned} S\colon \mathscr{O}(X) &\to \mathsf{C^*Alg}_0 \\ U &\mapsto \{f \in C(X) \mid \mathsf{supp}[f] \subseteq U\} \end{aligned}$$

which takes values in (not necessarily unital) C*-algebras $\mathsf{C^*Alg}_0$. For $U \subseteq V$, we include $S(U) \subseteq S(V)$ by

$$\begin{aligned} S(U) &\to S(V) \\ f &\mapsto (f \cup 0)(v) := \begin{cases} f(v) & v \in U \\ 0 & v \notin U \end{cases} \end{aligned}$$

Notice these are sections to the usual restriction maps $C(V) \to C(U)$. Equivalently, we may dually view $\mathscr{O}(X) = \mathscr{C}(X)^{\mathrm{op}}$ for which

$$\begin{aligned} S\colon \mathscr{C}(X)^{\mathrm{op}} &\to \mathsf{C^*Alg}_0 \\ Y &\mapsto Y^{\perp} = \{f \in C(X) \mid f(Y) = 0\} \end{aligned}$$

takes values in ideals of $C(X)$. We note that S is *not* a sheaf on $\mathscr{C}(X)$. Moreover, singletons $\{x\} \in \mathscr{C}(X)$ are sent to maximal ideals $x^{\perp} \cong \mathbb{C}$.

In summary, one can interpret this dual construction as a sheaf with the ideal space Ideal$(C(X))$ as its target:

$$S\colon \mathscr{C}(X)^{\mathrm{op}} \to \mathsf{Ideal}(C(X))$$

where the points of X correspond to maximal ideals $\widehat{C(X)} = \mathrm{Spec}(C(X)) \cong \mathsf{MaxIdeals}(C(X))$.

Sheafs of smooth functions? Continuous functions? Vanishing at certain sets? Imaginably one can cook up many more sheaves of this form. Indeed, one may consider sheaves of holomorphic functions on compact Riemann surfaces, meromorphic functions on such surfaces equipped with divisors, you name it. In fact, one might be willing to naively conjecture: *the nicer the flavor of functions, the easier it is to form a sheaf out of them.* However, there is a certain element of robustness these families of functions must satisfy. We provide the following counterexample as a warning.

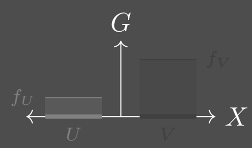
NON-EXAMPLE IV.0.10 (Constant function presheaf) — Let G be a group. We define to *constant function* presheaf on a space X, denoted by:

$$C_c(- \to G)\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Groups},$$

by assigning to each open $U \subseteq X$, the group of constant functions on U:

$$C_c(U \to G) := \{f\colon U \to G \text{ constant}\}.$$

This presheaf in general fails to satisfy the require gluing condition. Indeed, as long as G is non-trivial and there exist disjoint open sets $U, V \subset X$, we can always construct constant functions f_U and f_V which do not glue to a constant function:



There is a way to fix this example, which is to consider a slightly larger, more robust class of functions.

EXAMPLE IV.0.11 (Locally constant function sheaf) — Let G be a group. We define to *locally constant function* sheaf on a space X, denoted by:

$$C_{lc}(- \to G)\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Groups},$$

by assigning to each open $U \subseteq X$, the group of locally[a] constant functions on U:

$$C_{lc}(U \to G) := \{f\colon U \to G \text{ locally constant}\}.$$

[a] We say that a function f is locally *flavored* when every $x \in X$ admits an open neighborhood $x \in U \subseteq X$ such that f has said *flavor* on U.

Exercise IV.0.12. Let G be a group equipped with its discrete topology. Show that

$$C(U \to G) = C_{lc}(U \to G) \text{ for any space } U.$$

We will see that there is a more systematic way of enhancing a presheaf into a sheaf, i.e. a free construction:

$$\mathsf{Sheaves} \underset{\text{forget}}{\overset{\text{free}}{\rightleftarrows}} \mathsf{Presheaves} \quad (\text{free} \dashv \text{forget})$$

For this we will introduce the "stalk picture" and finally discuss "germs" of functions. Before this, we include a brief digression on spaces of sheaves.

[[What about sheaves of solutions of a PDE on a manifold? Talk with Gabe!]]

IV.1 Sheaves on smooth spaces

[[Sheaf of differential forms]]

[[Sheaf of functionals of simplices]]

Where the sea advances insensibly in silence: What is a topos?

> It is better to have a good category with bad objects than a bad category with good objects.
>
> — Attributed to A. Grothendieck

Consider the following example of a sheaf, which is the "dirac-delta" function's higher analogue. Of course this can be stated in terms of any sort of mathematical structure. But, as we are not interested in *centipede mathematics*[1], we will state it here for sheaves of groups.

Example V.0.1 (Skyscraper sheaves) — Let G be a group and $p \in X$ a point in a space X. The *Skyscraper sheaf* over a point $p \in X$, denoted by

$$G\delta_p \colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Groups}$$

is given on open sets by:

$$G\delta_p(U) = \begin{cases} G & p \in U \\ 0 & \text{else} \end{cases}$$

and has "restriction" maps for $U \subseteq V$:

$$G\delta_p(V) \to G\delta_p(U) = \begin{cases} \mathrm{id}_G & p \in U \subseteq V \\ 0 & \text{else} \end{cases}$$

One can more generally extend this construction to $G\chi_P$ for characteristic functions χ_P where $P \subset X$, "add" these $G\chi_P \oplus_X H\chi_Q$ to assign different groups G, H to different points in $P, Q \subseteq X$, etc.

Exercise V.0.2. What conditions on $P \subset X$, if any, does one need to impose so that one can define a sheaf $G\chi_P$ on X?

[1] This is the tradition of removing as many hypothesis from a theorem as possible while retaining its form. *How many legs can you remove from a centipede until it is no longer a centipede?* One? Fifty? Fifty-one? Ninety-nine? A hundred? Moreover, how much discussion can one include in a footnote until it is no longer a footnote? We leave this as an exercise to the reader.

The idea of sheaves being "continuous[2] functions" on a site leads us to the following insight:

Idea. *The space of sheaves $Sh(\mathcal{X})$ on a site $\mathcal{X}$ plays the role of $C(X)$ on a space X.*

From practice, we know that the C*-algebra $C(X)$ is quite a nice mathematical object, due to the fact that $\mathbb{C}$ is quite rich. More generally, the tradition of Yoneda teaches us that algebraic structures on an object T correspond to structures on $\mathrm{Hom}(- \to T)$, which in turn descend onto structure for each $\mathrm{Hom}(X \to T)$. So the fact that $\mathbb{C}$ is a C*-algebra is what endows $C(X)$ is said structure.

Now the same can be said for higher structures: When $\mathcal{T}$ is, for example, an abelian category, it follows that the category $Sh(\mathcal{X} \to \mathcal{T})$ of sheaves on a site $\mathcal{X}$ valued in $\mathcal{T}$ will also form an abelian category. This is the guiding principle of *condensed mathematics*[3], where topological abelian groups are replaced by sheaves of abelian groups on a suitable site CHaus of topological spaces.

Indeed, albeit the category of topological abelian groups is not an abelian category, we may identify a topological abelian group G with its associated sheaf:

$$C(- \to G) \colon \mathsf{CHaus}^{op} \to \underbrace{\mathsf{Ab}}_{\text{abelian category}}$$

which does live in an abelian category of condensed abelian groups $\mathsf{cAb} := Sh(\mathsf{CHaus} \to \mathsf{Ab})$.[4]

> Definition V.0.3 — A category of the form $Sh(\mathcal{X})$ for a site $\mathcal{X}$ is known as a *(Grothendieck) topos.*

Idea. *Condensed structures are like structures that need not have enough points.*

For a condensed structure $F \colon \mathsf{CHaus}^{\mathrm{op}} \to \mathsf{Structure}$, we may consider the *underlying structure*

$$|F| := F(*) = F(\{p\}),$$

which are to be thought of as the space of "points" of F.[5] Notice how $|F|$ could be trivial yet $\Omega_F := F(S^1)$, the loop space of F, could be non-trivial. For example, consider the condensed abelian group

$$H^1(-;\mathbb{Z}) := \mathsf{CHaus}^{\mathrm{op}} \to \mathsf{Ab}$$

which only has one point $H^1(*;\mathbb{Z}) = \{0\}$ and an infinite loop space $H^1(S^1;\mathbb{Z}) = \mathbb{Z}$. One way to think about this is that H^1 is the condensed abelian group representing the *infinitely small circle.*

Moreover, since we may view finite sets Δ as discrete compact Hausdorff spaces in CHaus, there is also an *underlying simplicial structure* $F|_{\Delta^{op}} \colon \Delta^{op} \to \mathsf{Structure}$.

Indeed, for condensed sets, we obtain some nice adjunctions:

$$\mathsf{sSet} \underset{\mathsf{Kan}}{\overset{\text{よ}|_{\Delta^{op}}}{\rightleftarrows}} \mathsf{cSet} \underset{|\cdot|}{\overset{\text{よ}}{\rightleftarrows}} \mathsf{Top}$$

Note that here, instead of $|F|$ landing in Set, we equip $F(*)$ with an organic topology.

Exercise V.0.4. What topology do we need to equip $F(*)$ with in order to obtain the desired adjunction?

[2] This last example might be a bit counterintuitive, since characteristic functions are generally not continuous. But what we normally think of as continuity will arise as a local-triviality condition later on.

[3] One will find different formalisms, all based on this principle, which have their own ways of dealing with size issues:

- *Condensed mathematics* only considers spaces smaller than an uncountable inaccessible cardinal κ, taking a (large) colimit on κ whenever needed. In fact, they tend to restrict themselves to so-called *pro-finite sets*, which form a site with more-or-less the same sheaves.
- *Pyknotic mathematics* only considers spaces smaller than the first strongly inaccessible cardinal κ.
- *Quasi-mathematics* completely disregards size issues. A *quasi-topological space* in the sense of Spanier is precisely a sheaf $\mathsf{CHaus}^{\mathrm{op}} \to \mathsf{Ab}$ on the large category CHaus. This is the philosophy we will follow, noting that "quasi-mathematics" is not a standard term.

[4] Here it is curious that $\mathsf{CHaus}^{\mathrm{op}} \cong \mathsf{C}^*\mathsf{Alg}_{\text{comm.}}$ appears. Is this just a coincidence?

[5] Again practicing the tradition of Yoneda, the points of F are $\mathrm{Hom}(* \Rightarrow F) = F(*)$ where $* := \text{よ}_* = \mathsf{Top}(- \to *)$.

VI

Love in the Time of Cholera: What is a germ?

> Today, when I saw you, I realized that what is between us is nothing more than an illusion.
>
> — Gabriel García Márquez

We have seen how sheaves on a space X serve to encode classes of partially defined functions on X together with the way they glue together. In this section, our aim is to present the *"Stalk picture"* for sheaves, which is motivated by *fiber bundles.*

Definition VI.0.1 (Bundles) — A (locally trivial) fiber bundle $E \xrightarrow{p} B$ with fiber F consists of:

(E) A space E called the *total space*;

(B) A space B called the *base space*

(F) A space F called the *fiber*, which might be equipped with Structure.

(p) A continuous map $p\colon E \to B$ satisfying two conditions:

- (locally trivial) Each $b \in B$ admits an open neighborhood $U \subseteq B$ such that:

$$\begin{array}{ccccc} U \times F & = & E_U & \hookrightarrow & E \\ \downarrow{\scriptstyle p_U} & & \downarrow{\scriptstyle p|^U} & & \downarrow{\scriptstyle p} \\ U & = & U & \hookrightarrow & B \end{array}$$

  where p_U is the projection $(u, f) \mapsto u$ and $p|^U\colon E_U := p^{-1}(U) \to U$ is the co-restriction of p.

- (transition maps) For two such $U, V \subseteq X$, there is a transition map $t_{U,V} : U \cap V \to \mathrm{Aut}(F)$. Indeed, consider a basepoint $b \in U \cap V$. For each $e \in E_b := p^{-1}(b)$, we have two equivalent expressions (or coordinates): $(b, f) \in U \times F$ and $(b, f') \in V \times F$. We then define $t_{U,V}(b)$ by $f \mapsto f'$. When the fiber F has Structure, we require these transition maps to be structure preserving[a].

[a] That is, we compile $\mathrm{Aut}(F)$ in the category Structure.

We will quickly talk only of bundles $E \xrightarrow{p} B$, suppressing the fibers from our notation when possible. In order to clarify this talk of Structure and structure preserving maps, let us instantiate our cases of interest:

- In the case when $\mathsf{Structure} = \mathsf{Vec}$, such a fiber bundle is known as a *vector bundle* and we require the transition maps $t_{U\cap V} : U \cap V \to \mathrm{Aut}(V)$ to have their image in *linear* automorphisms $\mathsf{Vec}(V \to V)$.
- When $\mathsf{Structure} = \mathsf{Hilb}$, these are known as *Hilbert bundles* and we require the transition maps to land in bounded maps (really, the "correct" choice is unitary maps). These are closely related to so-called *Riemannian* manifolds.
- One can also consider $\mathsf{Structure} = \mathsf{sHilb}$, which are then related to the *semi-Riemannian* manifolds appearing in general relativity.
- Finally, for *C*-bundles* with $\mathsf{Structure} = \mathsf{C^*Alg}$, we require that $t_{U\cap V}$ land in $*$-algebra automorphisms.

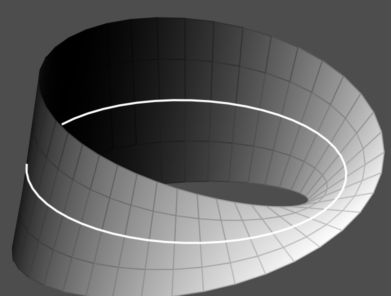
More generally, one speaks of so-called *structure groups*:

Definition VI.0.2 — We say that a bundle has structure group $G \leq \mathrm{Aut}(F)$ when every $\mathrm{Im}\, t_{U\cap V} \subseteq G$.

These structure groups are more refined than just equipping the fibers F of a bundle $E \xrightarrow{p} B$ with Structure[1]. For example, we may talk of n-dimensional vector bundles with structure group $O(n)$. Similarly, we can consider Hilbert bundles with fiber H having structure group $U(H)$.

Before we move on to relating these bundles to our story about sheaves, we present some examples.

Example VI.0.3 (Möbius) — Aside from cylinder/annulus, the trivial line bundle $S^1 \times \mathbb{R}$ on S^1, there is a once-twisted line bundle $\ddot{\mathrm{M}} \to S^1$ known as the Möbius strip:



Example VI.0.4 — For an n-dimensional (smooth) manifold M, its *tangent bundle* TM has fibers $\mathbb{R}^n$ where, more concretely, the fiber over a basepoint $b \in M$ is its *tangent space* T_bM. Viewing this as the space of derivations at b,

$$T_bM := \{\partial \colon C^\infty(M) \to \mathbb{R} \text{ or } \mathbb{C} \mid \partial(fg) = f(b)\partial(g) + \partial(f)g(b) \text{ for all } f, g \in C^\infty(M)\}.$$

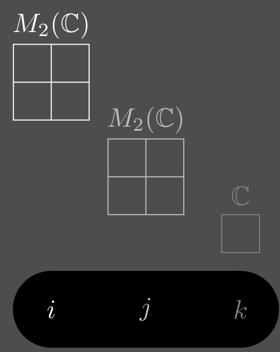
Exercise VI.0.5. Figure out how to equip TM with a topology so that $TM \to M$ is a vector bundle.

[1] Equivalently, one could restrict the morphisms in the category Structure, so that $\mathrm{Aut}(F)$ is our desired group in this subcategory. For example, one could pass from Hilb to the subcategory $\mathsf{Hilb}_{\mathsf{isom}}$ of Hilbert spaces with isometric maps in order to obtain bundles with structure group $U(H)$. This, however, would be a notational nightmare.

Non-example VI.0.6 (Unitary algebras) — Let A be a unitary algebra, id est, a finite dimensional C*/W*-algebra. We may write $A = \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$ where $Z(A) = \bigoplus_{i=1}^N \mathbb{C}$ with $\operatorname{Spec} A = \operatorname{Spec} Z(A) = \{1, \ldots, N\}$. We may view A as a (co)sheaf $\mathcal{A}$ over $\operatorname{Spec} A$ by:

$$\mathcal{A}(U) := \bigoplus_{i \in U} M_{n_i}(\mathbb{C}) \quad \text{for } U \subseteq \operatorname{Spec}(A).$$

In the stalk picture, $\mathcal{A}$ can be viewed as:



Note that $\mathcal{A}$ is in general not a bundle as the fibers $M_{n_i}(\mathbb{C})$ of $\mathcal{A}$ not isomorphic. For $U \subseteq V$ note there are *both* restriction and and inclusion maps $\mathcal{A}(V) \leftrightarrows \mathcal{A}(U)$ which witnesss $\mathcal{A}$ as *both* a sheaf and cosheaf over $\operatorname{Spec} A$. These are non-commutative analogues of the structure sheaves and cosheaves seen in [[ref:examples]].

We now discuss how to view bundles as sheaves.

Example VI.0.7 (Bundles as sheaves) — Let $E \xrightarrow{p} B$ be a fiber bundle with fiber F equipped with some Structure. We define its *sheaf of sections*, denoted by:

$$\Gamma(- \to E) \colon \mathcal{O}(B)^{\mathrm{op}} \to \mathsf{Structure},$$

by assigning to an open set $U \subseteq B$ the space of *sections*[a] on U:

$$\Gamma(U \to E) := \{s \colon U \to E \text{ continuous or smooth} \mid U \xrightarrow{s} E \xrightarrow{p} U = \mathrm{id}_U\}.$$

[a] In general, the sections of a morphism $f \colon A \to B$ are its right-inverses, i.e. the $g \colon B \to A$ such that $fg = \mathrm{id}_B$.

A particular instance of this example to keep in mind is the sheaf of vector fields on a manifold:

$$\mathfrak{X}_M := \Gamma(- \to TM) \colon \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{Vec}.$$

Idea. *Sheafs are like bundles where we allow the "fibers", called* stalks*, to be different.*

In order to recover the fiber over a basepoint $b \in B$ from the section sheaf $\Gamma(- \to E)$ of a bundle E, we somehow need to "shrink" the $\Gamma(U \to E)$ by taking smaller and smaller open neighborhoods of $b \in U \in \mathcal{O}(B)$. Using the language of ultrafilters on $\mathcal{O}(X)$, or more generally, of directed limits in $\mathcal{X}$, we obtain the notion of a *stalk*:

DEFINITION VI.0.8 — For a sheaf $F\colon \mathcal{O}(X)^{\mathrm{op}}$, its *stalk* over the point $x \in X$ is given by the colimit

$$F_x := \lim_{\mathcal{O}(X)\ni U \ni x} F(U).$$

This stalk is determined by a universal property where each $F(U)$ admits a map F_x compatible with restrictions.

In the case where F is a sheaf of functions, for example when $F = C(- \to G)$ for a topological group G, each $f \in C(U \to G)$ determines a *germ* at $x \in U \in \mathcal{O}(X)$ i.e. its image under the map $C(U \to G) \to F_x$. Thus, the stalk F_x is known as the space of germs at x, and $F = C(- \to G)$ the sheaf of germs of G-valued functions.
To summarize our discussion so far:

$$\begin{array}{ccccc} \text{Presheaves} & \longleftrightarrow & \text{Sheaves} & \longleftrightarrow & \text{Fiber bundles} \\ \rightsquigarrow & & \rightsquigarrow & & \rightsquigarrow \\ \text{Functions} & \longleftrightarrow & \text{Continuous functions} & \longleftrightarrow & \text{Constant functions} \end{array}$$

Let us now view operator algebras as sheaves through this stalk picture:

DEFINITION VI.0.9 — Recall that the spectrum $\mathrm{Spec}(A)$ of a C*-algebra A is the set of unitary isomorphism classes of irreducible representations $\mathsf{Irr}(A)$ with the topology induced by the hull-kernel topology of $\mathsf{Prim}(\mathcal{A})$. [[We should do this in more detail somewhere]] For each $\pi \in \mathrm{Spec}(A)$, we assign the C*-algebra $\mathcal{A}_\pi = \pi(A) \subseteq B(H_\pi)$ to the fiber $\mathcal{A}_\pi$ over π. This assembles into both a sheaf and a cosheaf over $\mathrm{Spec}(A)$ with total space of sections A.

Theorem VI.0.10 (Dauns-Hofmann). Every C*-algebra A can be organically realized as a sheaf

$$\mathcal{A}\colon \mathcal{O}(\mathrm{Spec}\, A)^{\mathrm{op}} \to \dagger\mathsf{Alg}$$

of [[just †-algebras]] on its spectrum $\mathrm{Spec}(A)$ with total section space $\mathcal{A}(\mathrm{Spec}(A)) = A$ [[actually 0?]] and fibers $A_\pi \cong \pi(A)$. [[Are these simple?]] Moreover, $\mathcal{A}$ is a $C\colon \mathcal{O}(\mathrm{Spec}\, A)^{\mathrm{op}} \to \dagger\mathsf{Alg}$ module. [[with $Z(A) \cong C(\mathrm{Spec}(A))$]]

Proof. Let A be a C*-algebra and let $U_I = \{\pi\colon A \to B(H) \text{ irrep.} \mid I \not\subseteq \ker \pi\}$ be a basis open set in $\mathrm{Spec}(A)$ corresponding to the ideal $I \subseteq A$. We define

$$\mathcal{A}(U_I) := A / \bigcap_{\pi \in U_I} \ker \pi.$$

For $I \subseteq J$, the map corresponding to $U_J \subseteq U_I$ is given by the canonical projection map induced by

$$\bigcap_{\pi \in U_I} \ker \pi \subseteq \bigcap_{\pi \in U_J} \ker \pi$$
$$\mathcal{A}(U_I) \to \mathcal{A}(U_J)$$

Using universal properties of quotients, one readily verifies the sheaf condition for $\mathcal{A}$. Moreover, the basis open set $U_A = \mathrm{Spec}\, A$ and $\bigcap_{\pi \in \mathrm{Spec}\, A} \ker \pi = 0$ as every $a \in A$ admits an irrep π of A with $\|a\| = \|\pi(a)\|$. Hence $\mathcal{A}(\mathrm{Spec}\, A) = A/0 = A$. Finally, $\lim_{\pi \in U_I} \mathcal{A}(U_I) = A/\ker \pi \cong \pi(A)$. □

Remark VI.0.11. In the case when $A = C(X)$, we note that $\mathrm{Spec}(A) = X$ and $\mathcal{A}$ recovers the usual sheaf $C\colon \mathcal{O}(X)^{\mathrm{op}} \to \dagger\mathsf{Alg}$ with total section space $C(X)$ and 1-dimensional fibers $C(X)_x = \mathbb{C}$. In the stalk picture, $\mathcal{A}$ is in fact just $\Gamma(X \times \mathbb{C} \to X)$.

Theorem VI.0.12 (Factor decomposition). [[To do]]

We now provide a method of *sheafifying* any presheaf F, by first viewing it as a "generalized bundle" and then taking its sheaf of sections.

DEFINITION VI.0.13 (Sheafification) —

[[Figure out where to move this]]

EXAMPLE VI.0.14 (2-functionals) — For a finite dimensional 2-Hilbert space $\mathcal{X}$, recall that the Yoneda embedding $\mathcal{X} \to \mathsf{Hilb}^{\mathcal{X}^{\mathrm{op}}}$ is a unitary equivalence. Hence, all Grothendieck topologies on $\mathcal{X}$ agree with the trivial topology and every presheaf on $\mathcal{X}$ is a sheaf.

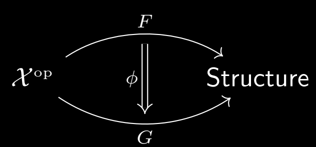
Remark VI.0.15. The analogous statement one category level down is that every functional on a Hilbert space is bounded/continuous.

VII

Failure is an option: What is sheaf cohomology?

> We don't make mistakes, just happy little accidents.
>
> — Bob Ross

Consider a morphism[1] $\phi\colon F \Rightarrow G$ of sheaves $F, G\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Structure}$:



Recall the following idea:

Idea. *Maps into an algebraic structure tend to absorb this structure and reflect its properties.*

In particular, when $\mathsf{Structure}$ forms an abelian category, we expect $Sh(\mathcal{X})$ to form a category which is also abelian.

So, for $\phi\colon F \Rightarrow G$, we should be able to construct sheaves $\operatorname{Ker}\phi$ and $\operatorname{Im}\phi$ that fit into a short exact sequence:

$$0 \to \operatorname{Ker}\phi \to F \xrightarrow{\phi} \operatorname{Im}\phi \to 0$$

The first construction one would guess is to define

$$\operatorname{Ker}\phi\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Structure} \qquad \text{and} \qquad \operatorname{Im}\phi\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Structure}$$

pointwise, i.e. on $U \in \mathcal{X}$ by

$$(\operatorname{Ker}\phi)(U) := \operatorname{Ker}(\phi_U\colon F(U) \to G(U)) \qquad \text{and} \qquad (\operatorname{Im}\phi)(U) := \operatorname{Im}(\phi_U\colon F(U) \to G(U))$$

Unfortunately, while $\operatorname{Ker}\phi$ is indeed a sheaf, this naive construction for $\operatorname{Im}\phi$ *fails* to be more than a presheaf.

[1] By this, we just mean a natural transformation as functors $\mathcal{X}^{\mathrm{op}} \to \mathsf{Structure}$

Non-example VII.0.1 (exp) — The continuous map $e^{i\pi(-)}\colon \mathbb{R} \to \mathbb{T}$ induces a sheaf map

$$C(- \to \mathbb{R}) \xRightarrow{\phi} C(- \to \mathbb{T})$$

given by post-composition $\phi \coloneqq よ_{e^{i\pi(-)}}$, i.e. for $U \in \mathcal{O}(X)$ we define the corresponding component by:

$$C(U \to \mathbb{R}) \xrightarrow{\phi_U} C(U \to \mathbb{T})$$
$$f(x) \mapsto e^{i\pi f(x)}$$

Consider the open cover of $\mathbb{T} = U \cup V$ where $U = \mathbb{T} - \{1\}$ and $V = \mathbb{T} - \{-1\}$ are contractible. We claim that $\mathrm{id}_U \in (\mathrm{Im}\,\phi)(U)$ and $\mathrm{id}_V \in (\mathrm{Im}\,\phi)(V)$ yet

$$\mathrm{id}_U \cup \mathrm{id}_V = \mathrm{id}_{\mathbb{T}} \notin (\mathrm{Im}\,\phi)(\mathbb{T}).$$

Exercise VII.0.2. Show the previous claim by proving:

- There exists a continuous section of $e^{i\pi(-)}$ on U, suggestively named $\frac{1}{i\pi}\ln$. Convince yourself that this is equivalent to $\mathrm{id}_U \in (\mathrm{Im}\,\phi)(U)$.

- Convince yourself the same holds true for V.

- Show there exists no continuous split monomorphism of $\mathbb{T} \to \mathbb{R}$ by homotopical[2] considerations:

$$\underbrace{\pi_1(\mathbb{R})}_{0} \leftarrow \underbrace{\pi_1(\mathbb{T})}_{\mathbb{Z}}$$

  Convince yourself this means that $e^{i\pi(-)}$ has no continuous section on $\mathbb{T}$, and hence $\mathrm{id}_{\mathbb{T}} \notin (\mathrm{Im}\,\phi)(\mathbb{T})$.

Okay, so just sheafify this construction to obtain the desired $\mathrm{Im}\,\phi$ sheaf, big whoop. *Well actually...*

Idea. *The failure of our naive construction is a **feature**, not a **bug**.*

Indeed, notice the obstruction we constructed was homotopical in nature: $\mathbb{T}$ has nontrivial holes in dimension 1 whereas $\mathbb{R}$ does not. [[Brett: somewhere i want to discuss more generally “failures of abelian-ness” or “failures to have kernels” - this is another one of those things that shows up everywhere. this is the pt behind stabilization, triangulated categories, condensed sets in the first place, derived functors, etc. do you want maybe a whole chapter on cohomology generally?]]

The idea behind *sheaf* cohomology is to exploit this failure in order to detect holes.

To recap, given a short exact sequence of sheaves on $\mathcal{O}(X)^{\mathrm{op}}$:

$$0 \to K \to F \to I \to 0$$

we only have exact sequences:

$$0 \to K(U) \to F(U) \to I(U)$$

[2] Recall that $\pi_1(Y) \coloneqq C(\mathbb{T} \to Y)/\sim$ up to homotopy for a (pointed) space Y.

which we will extend into long exact sequences:

$$
\begin{array}{l}
0 \longrightarrow K(U) \longrightarrow F(U) \longrightarrow I(U) \xrightarrow{\alpha^0} \\
H^1(U;K) \longrightarrow H^1(U;F) \longrightarrow H^1(U;I) \xrightarrow{\alpha^1} \\
H^2(U;K) \longrightarrow H^2(U;F) \longrightarrow H^2(U;I) \dashrightarrow \\
\cdots \\
H^n(U;K) \longrightarrow H^n(U;F) \longrightarrow H^n(U;I)
\end{array}
$$

VII.1 de Rham's Theorem