Braided 2 Categories and the Drinfel'd Center

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Contents

1 Let's recall some stuff first

Recall the periodic table of k-tuply monoidal n-categories:

	n=0	n=1	n=2	⋯
k=0	(pointed) set	(pointed) category	(pointed) 2-category	⋯
k=1	monoid	monoidal category	**monoidal 2-category**	⋯
k=2	abelian monoid	braided monoidal category	**braided monoidal 2-category**	⋯
k=3	”	symmetric monoidal category	sylleptic monoidal 2-category	⋯
k=4	”	”	symmetric monoidal 2-category	⋯
k=5	”	”	”	⋯
⋮	⋮	⋮	⋮	⋱

We see that a *monoidal 2-category* is just a 3-category with one object.

We will however work with the strictest version possible while still retaining full generality:

Notation 1.1 — By a *monoidal 2-category* we mean a Gray monoid, i.e.

(S1) a 2-category $\mathcal{A}$ together with

(S2) a 2-functor $\boxtimes : \mathcal{A} \boxtimes_G \mathcal{A} \to \mathcal{A}$ where $\boxtimes_G$ is the Gray product,

such that:

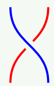
(A1) $\boxtimes$ is (strictly) associative with unit $\mathbf{1} \in \mathbb{C}$.

2 Braided Monoidal 2-Categories

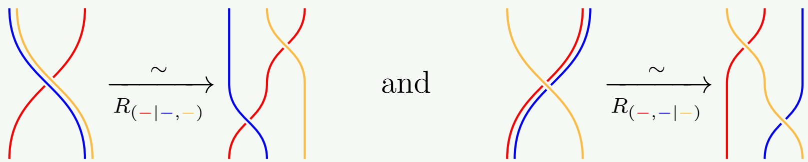
A braided monoidal 2-category is just a 4-category with a single object and a single 1-morphism. Unfortunately, 4-categories are *hard*. However, we can provide a semi-strict (Gray) definition for braided monoidal 2-categories. This definition is due to [3].

Definition 2.1 — An *braided monoidal 2-category* consists of:

(S1) A monoidal 2-category $(\mathcal{A}, \boxtimes, I)$

(S2) A 2-natural equivalence $\beta_{-,-} : - \boxtimes - \overset{\sim}{\Rrightarrow} - \boxtimes -$ represented by 

(S3) Invertible modifications $R_{(-|-,-)}$ and $R_{(-,-|-)}$ where:



satisfying:

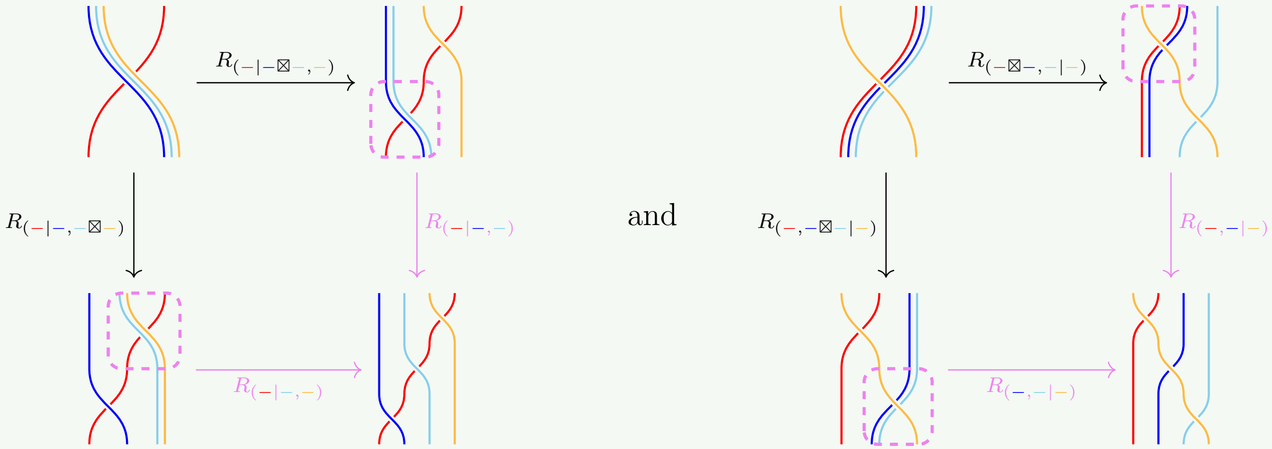
(A1) the (1,3)-crossing and (3,1)-crossing axioms,

(A2) the (2,2)-crossing axiom,

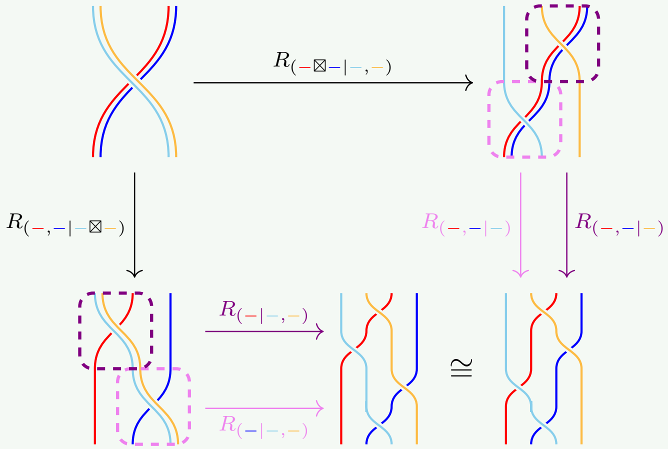
(A3) the Yang-Baxter axiom, and

(A4) unit axioms.

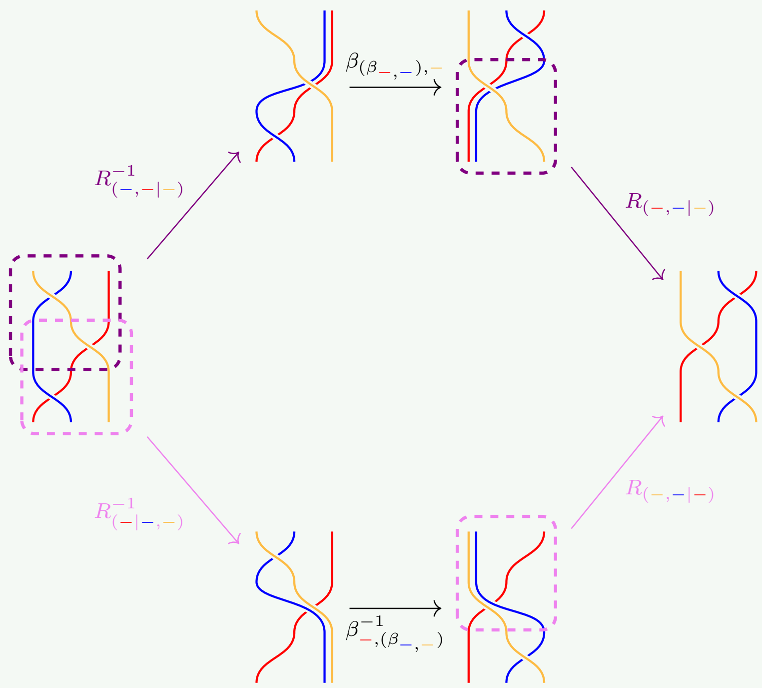
(A1) The (1,3)-crossing and (3,1)-crossing axioms:



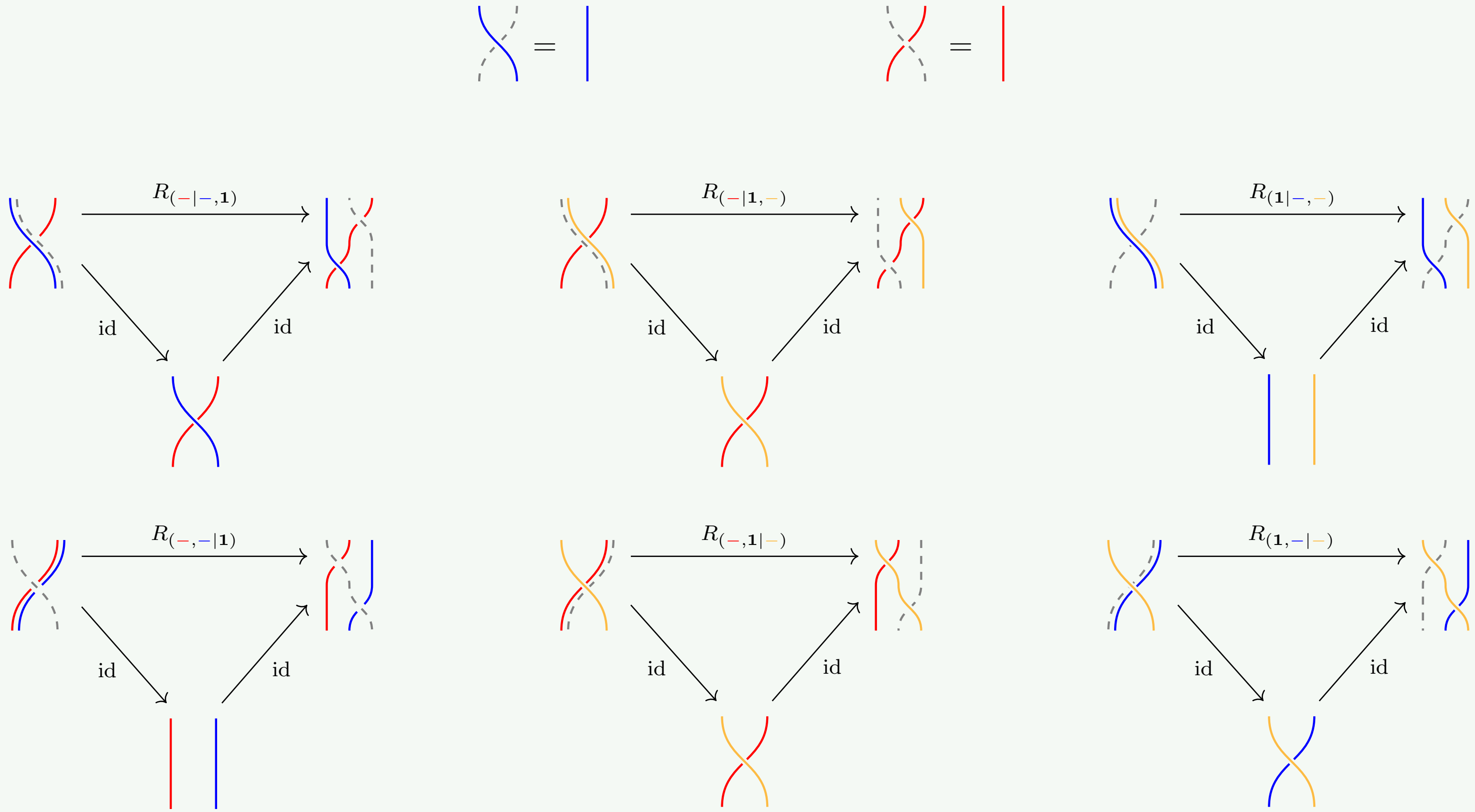
(A2) The (2,2) crossing axiom:



(A3) The Yang-Baxter axiom:



(A4) Unit axioms:



Example 2.2 — Examples of braided monoidal 2-categories

- The Drinfeld center $\mathcal{Z}(\mathcal{A})$ of a monoidal 2-category $\mathcal{A}$,
- The braided monoidal 2-category $\mathsf{BrMod}{-}\mathcal{B}$ of braided module categories over a braided fusion 1-category $\mathcal{B}$.
- The braided fusion 2-categories $\mathcal{S}$ and $\mathcal{T}$

Exercise 2.1. Come up with more examples of braided monoidal 2-categories.

3 The Drinfel'd Center of a Monoidal 2-Category

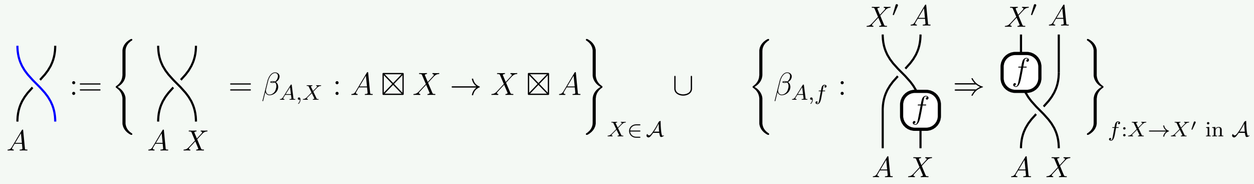
This definition is also from [3].

3.1 The Base 2-Category $\mathcal{Z}(\mathcal{A})$

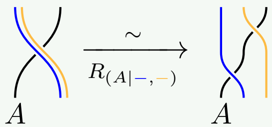
Definition 3.1 — Given a monoidal 2-category $\mathcal{A}$, we define $\mathcal{Z}(\mathcal{A})$ to be the 2-category consisting of:

(0) Objects are triples $\big(A, \beta_{A,-}, R_{(A|-,-)}\big)$ consisting of:

- an object $A \in \mathcal{A}$,
- a 2-natural equivalence $\beta_{A,-} : A \boxtimes - \overset{\sim}{\Rightarrow} - \boxtimes A$ represented by

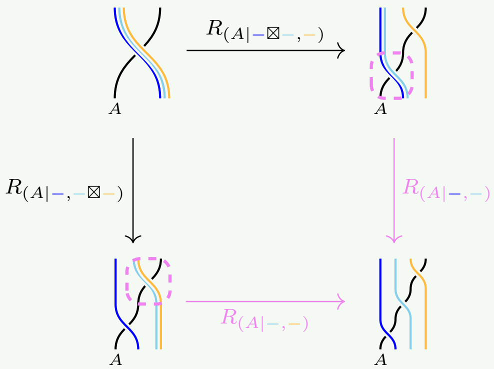


- an invertible modification $R_{(A|-,-)}$ where

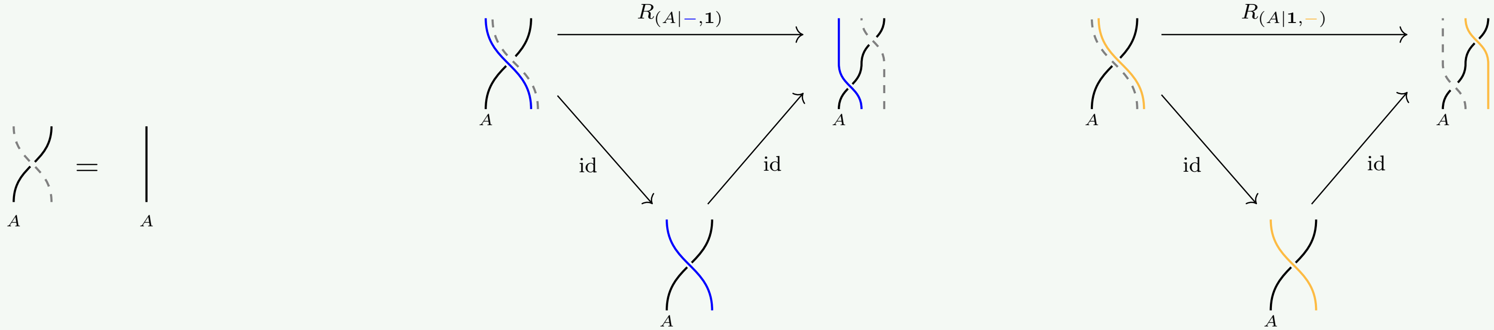


satistfying the (1,3)-crossing and unit axioms.

- The (1,3)-crossing axiom:

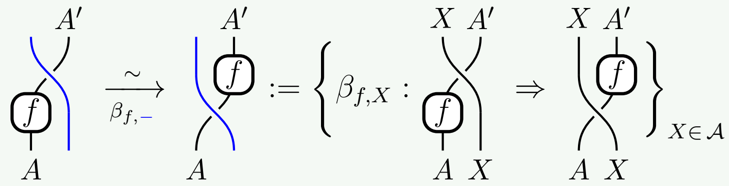


- Unit axioms:



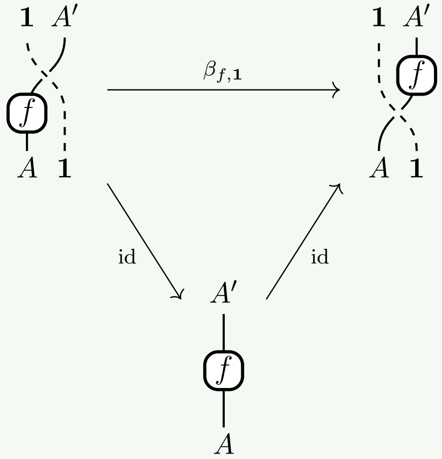
(1) A 1-morphism $(f, \beta_{f,-}) : \big(A, \beta_{A,-}, R_{(A|-,-)}\big) \to \big(A', \beta_{A',-}, R_{(A'|-,-)}\big)$ consists of:

- a 1-morphism $f : A \to A'$ in $\mathcal{A}$,
- an invertible modification represented by

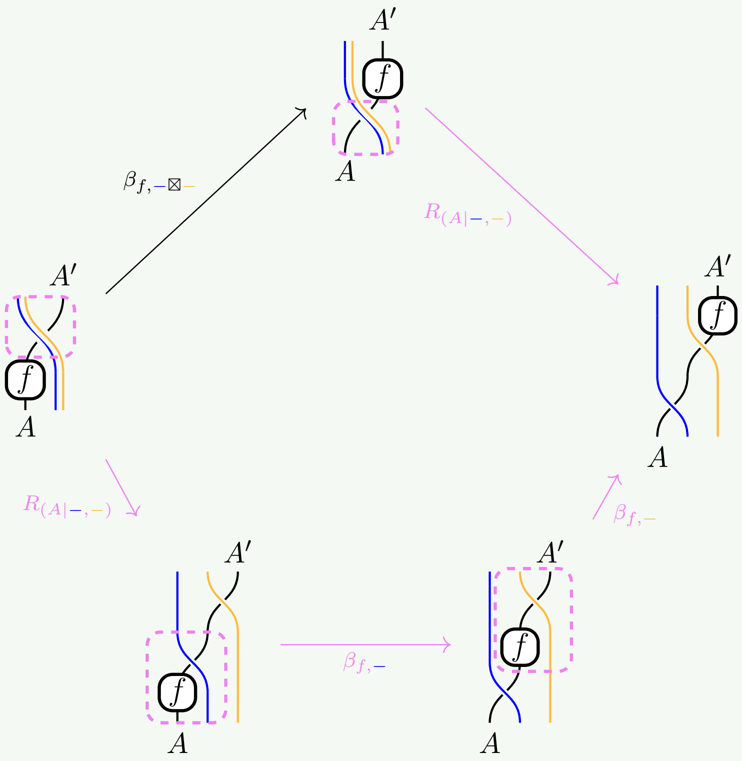


such that $\beta_{f,-}$ satisfies a unit axiom and $R_{(A|-,-)}$ becomes natural in $f : A \to A'$.

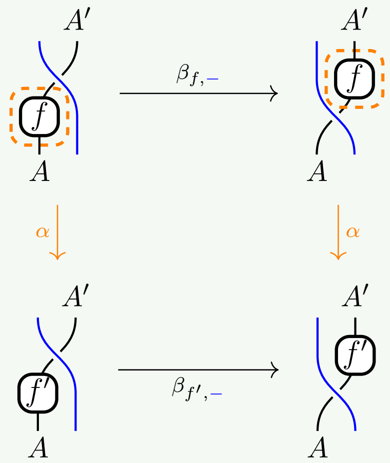
- Unit axiom:



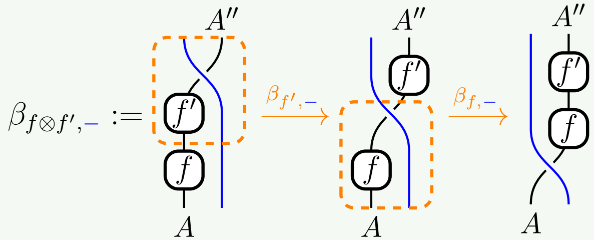
- Naturality of $R_{(A|-,-)}$:



(2) A 2-morphism $\alpha : (f, \beta_{f,-}) \Rightarrow (f', \beta_{f',-})$ is a 2-morphism $\alpha : f \Rightarrow f'$ in $\mathcal{A}$ such that $\beta_{f,-}$ becomes 2-natural in $\alpha : f \Rightarrow f'$, i.e.



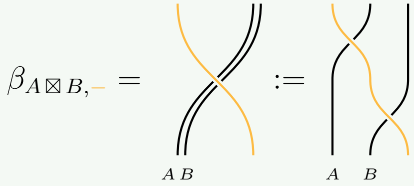
For $\big(A, \beta_{A,-}, R_{(A|-,-)}\big) \xrightarrow{(f,\beta_{f,-})} \big(A', \beta_{A',-}, R_{(A'|-,-)}\big) \xrightarrow{(f',\beta_{f',-})} \big(A'', \beta_{A'',-}, R_{(A''|-,-)}\big)$ in $\mathcal{Z}(\mathcal{A})$, their 1-composite $(f, \beta_{f,-}) \otimes (f', \beta_{f',-})$ is defined to be $(f \otimes f', \beta_{f \otimes f',-})$ where:

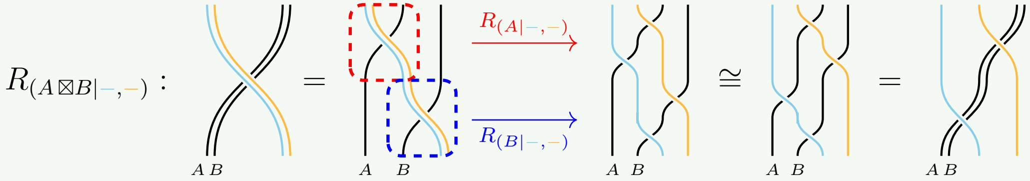


The compositions $\otimes$ and $\circ$ of 2-morphisms in $\mathcal{Z}(\mathcal{A})$ are the same as in $\mathcal{A}$.

3.2 The Monoidal Structure

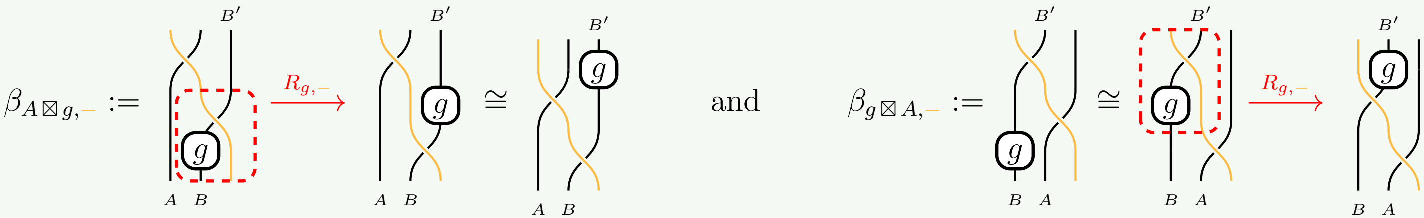
Definition 3.2 — For objects $(A, \beta_{A,-}, R_{(A|-,-)})$ and $(B, \beta_{B,-}, R_{(B|-,-)})$, we define $(A, \beta_{A,-}, R_{(A|-,-)}) \boxtimes (B, \beta_{B,-}, R_{(B|-,-)})$ to be $\big(A \boxtimes B, \beta_{A\boxtimes B,-}, R_{(A\boxtimes B|-,-)}\big)$ where:





For an object $\big(A, \beta_{A,-}, R_{(A|-,-)}\big)$ and a 1-morphism $(g : B \to B', \beta_{g,-})$, we define:

$$\big(A, \beta_{A,-}, R_{(A|-,-)}\big) \boxtimes (g, \beta_{g,-}) := (A \boxtimes g, \beta_{A\boxtimes g,-}) \quad \text{and} \quad (g, \beta_{g,-}) \boxtimes \big(A, \beta_{A,-}, R_{(A|-,-)}\big) := (g \boxtimes A, \beta_{g\boxtimes A,-}) \text{ where:}$$

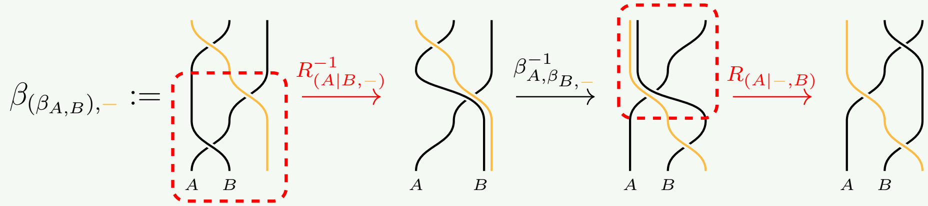


The other $\boxtimes$-products are defined as in $\mathcal{A}$.

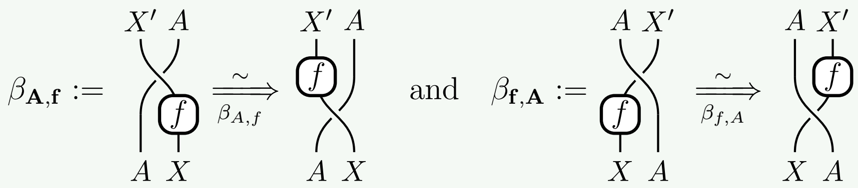
3.3 The Braiding

Definition 3.3 — We define the braiding $(\beta_{-,-}, R_{(-|-,-)}, R_{(-,-|-)})$ on $\mathcal{Z}(\mathcal{A})$ as folllows:

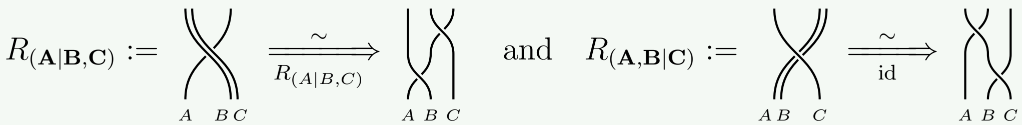
(β0) For objects $\mathbf{A} = \big(A, \beta_{A,-}, R_{(A|-,-)}\big)$ and $\mathbf{B} = \big(B, \beta_{B,-}, R_{(B|-,-)}\big)$, we define $\beta_{\mathbf{A},\mathbf{B}} := (\beta_{A,B}, \beta_{(\beta_{A,B}),-})$ where



(β1) For an object $\mathbf{A} = \big(A, \beta_{A,-}, R_{(A|-,-)}\big)$ and a 1-morphism $\mathbf{f} = (f : X \to X', \beta_{f,-})$, we define



(R) For objects $\mathbf{A} = \big(A, \beta_{A,-}, R_{(A|-,-)}\big)$, $\mathbf{B} = \big(B, \beta_{B,-}, R_{(B|-,-)}\big)$, and $\mathbf{C} = \big(C, \beta_{C,-}, R_{(C|-,-)}\big)$, we define



3.4 Some Facts

Theorem 3.4. Given any monoidal 2-category $(\mathcal{A}, \boxtimes, \mathbf{1})$, the Drinfeld center $\mathcal{Z}(\mathcal{A})$ is a braided monoidal 2-category.

Proof. An incomplete proof of this theorem appears in [Baez + Neuchl], which is completed and corrected by [Crans]. □

Theorem 3.5. Given any braided monoidal 2-category $\big(\mathcal{A}, \boxtimes, \mathbf{1}, \beta_{-,-}, R_{(-|-,-)}, R_{(-,-|-)}\big)$, there exists an embedding $\zeta : \mathcal{A} \hookrightarrow \mathcal{Z}(\mathcal{A})$ given by:

$$\begin{aligned}(A \in \mathcal{A}) &\mapsto \big(A, \beta_{A,-}, R_{(A|-,-)}\big)\\ (f : A \to A') &\mapsto (f, \beta_{f,-})\\ (\alpha : f \Rightarrow f') &\mapsto \alpha.\end{aligned}$$

Note that this implies that every braided monoidal 2-category is equivalent to one for which $R_{(-,-|-)}$ is trivial.

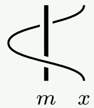
4 Braided Module Categories

This section is based on chapter 3 of [4]. From here on out, $\mathcal{B}$ will always be braided fusion 1-category.

4.1 Definitions

Definition 4.1 — A braided (right) module category of $\mathcal{B}$ is:

(S1) a finite semisimple (right) $\mathcal{B}$-module category $(\mathcal{M}, \triangleleft : \mathcal{M} \boxtimes \mathcal{B} \to \mathcal{M}, \ldots)$

(S2) a natural isomorphism $\sigma_{-,-} : \triangleleft \Rightarrow \triangleleft$ represented by:  $= \Big\{$ 

$:= \sigma_{m,x} : m \triangleleft x \to m \triangleleft x \Big\}_{m \in \mathcal{M}, x \in \mathcal{B}}$

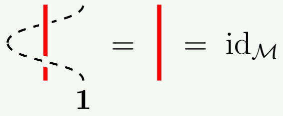
satisfying:

(A1) a unit axiom,

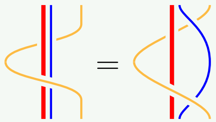
(A2) compatibility with $\triangleleft$ and braiding, and

(A3) compatibility with the $\otimes$-product on $\mathcal{B}$.

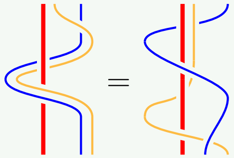
(A1) Unit axiom:



(A2) Compatibility with $\triangleleft$ and braiding:



(A3) Compatibility with the $\otimes$-product on $\mathcal{B}$:



Remark 4.1. The term *braided* in the previous definition is justified as follows:

- Recall that the Artin braid group of type B is the group B_n generated by $\sigma_0, \dots, \sigma_{n-1}$ subject to the relations:

$$\begin{aligned} \sigma_1\sigma_0\sigma_1\sigma_0 &= \sigma_0\sigma_1\sigma_0\sigma_1, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{whenever } |i-j|>1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i=1,\dots,n-1. \end{aligned}$$

- Given $X_1, \dots, X_{n-1} \in \mathcal{B}$ and $M \in \mathcal{M}$, there are isomorphisms

$$M \triangleleft X_1 \triangleleft \cdots \triangleleft X_{n-1} \to M \triangleleft X_{\sigma(1)} \triangleleft \cdots \triangleleft X_{\sigma(n-1)}, \quad \text{for } \sigma \in B_n,$$

  compatible with the composition of braids.

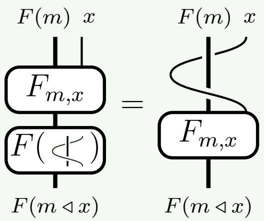
Definition 4.2 — A braided module functor $(\mathcal{M}, \triangleleft, \sigma_{-,-}) \xrightarrow{(F, F_{-,-})} (\mathcal{M}', \triangleleft', \sigma'_{-,-})$ is:

(S1) a linear functor $F : \mathcal{M} \to \mathcal{M}'$, and

(S2) a natural isomorphism $F_{-,-} = \{F_{m,x} : F(m \triangleleft x) \to F(m) \triangleleft' x\}_{m \in M, x \in \mathcal{B}}$

such that:

(A1) $F_{m,x} \circ F(\sigma_{m,x}) = \sigma'_{f(m),x} \circ F_{m,x}$.

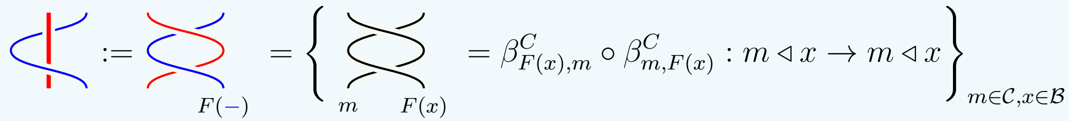


Remark 4.2. Note that being a *braided* module functor is a property of a module functor, *not* extra structure.

Definition 4.3 — A transformation $\alpha : (F, F_{-,-}) \Rightarrow (F', F'_{-,-})$ of braided module functors is simply a natural transformation $\alpha : F \Rightarrow F'$ of the underlying $\mathcal{B}$-module functors.

Definition 4.4 — We define $\mathsf{BrMod}{-}\mathcal{B}$ to be the 2-category of braided modules over $\mathcal{B}$, braided module functors, and natural transformations.

Example 4.5 — Any braided monoidal functor $F : \mathcal{B} \to \mathcal{C}$ of braided fusion 1-categories equips $\mathcal{C}$ with the structure of a braided $\mathcal{B}$-module category with $c \triangleleft b := c \otimes_C F(b)$ and module braiding:



- In particular, when $\mathcal{C}$ is a braided tensor category containing $\mathcal{B}$, we can equip $\mathcal{C}$ with this braided $\mathcal{B}$-module category structure. In this case, the category of braided module endofunctors is braided equivalent to:

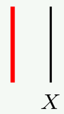
$$\mathcal{Z}_{(2)}(\mathcal{B} \subset \mathcal{C}) := \{c \in \mathcal{C} \mid \beta_{c,b} \circ \beta_{b,c} = \mathrm{id}_{b \otimes c} \text{ for all } b \in \mathcal{B}\}$$

- A special case of this is when $\mathcal{C} = \mathcal{B}$, where we see $\mathcal{B}$ as the rank one free braided $\mathcal{B}$-module category. Then, the category of braided module endofunctors of $\mathcal{B}$ is braided equivalent to the Müger center $\mathcal{Z}_{(2)}(\mathcal{B}) := \mathcal{Z}_{(2)}(\mathcal{B} \subset \mathcal{B})$.

4.2 α-Inductions and the Intermediate Category $\mathbf{A}(\mathcal{B})$

Definition 4.6 — The α-inductions [2] for a right $\mathcal{B}$-module category $\mathcal{M}$ are tensor functors:

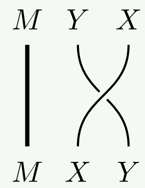
$$\alpha^{\pm}_{\mathcal{M}} : \mathcal{B} \to \mathrm{End}_{\mathcal{B}}(\mathcal{M}), \qquad \alpha^{\pm}_{\mathcal{M}}(X) := - \triangleleft X =$$



, for every $X \in \mathcal{B}$.

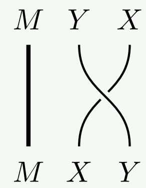
where $\mathsf{End}^{\mathsf{r.e.}}_{\mathcal{B}}(\mathcal{M})$ is the category of right exact $\mathcal{B}$-module endofunctors of $\mathcal{M}$. The $\mathcal{B}$-module functor structures on $\alpha^{\pm}_{\mathcal{M}}(X) : \mathcal{M} \to \mathcal{M}$ *do* differ and are given by:

$$\begin{array}{c} \alpha^{+}_{\mathcal{M}}(X)(M \triangleleft Y) \\ {\scriptstyle \alpha^{+}_{\mathcal{M}}(X)_{M,Y}} \uparrow \\ \alpha^{+}_{\mathcal{M}}(X)(M) \triangleleft Y \end{array} \quad \begin{array}{c} = \\ := \\ = \end{array}$$



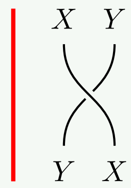
and

$$\begin{array}{c} \alpha^{-}_{\mathcal{M}}(X)(M \triangleleft Y) \\ {\scriptstyle \alpha^{-}_{\mathcal{M}}(X)_{M,Y}} \uparrow \\ \alpha^{-}_{\mathcal{M}}(X)(M) \triangleleft Y \end{array} \quad \begin{array}{c} = \\ := \\ = \end{array}$$



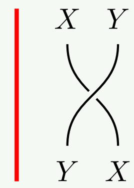
The monoidal functor structures of $\alpha^{\pm}_{\mathcal{M}}$ *also* differ and are given by:

$$\begin{array}{c} \alpha^{+}_{\mathcal{M}}(X \otimes Y) \\ {\scriptstyle (\alpha^{+}_{\mathcal{M}})_{X,Y}} \Uparrow \\ \alpha^{+}_{\mathcal{M}}(X) \circ \alpha^{+}_{\mathcal{M}}(Y) \end{array} \quad \begin{array}{c} = \\ := \\ = \end{array}$$

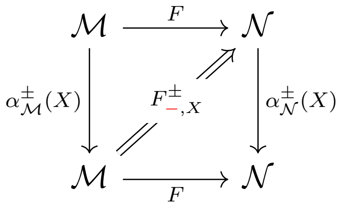


and

$$\begin{array}{c} \alpha^{-}_{\mathcal{M}}(X \otimes Y) \\ {\scriptstyle (\alpha^{-}_{\mathcal{M}})_{X,Y}} \Uparrow \\ \alpha^{-}_{\mathcal{M}}(X) \circ \alpha^{-}_{\mathcal{M}}(Y) \end{array} \quad \begin{array}{c} = \\ := \\ = \end{array}$$



Remark 4.3. Notice that, by definition, every $\mathcal{B}$-module functor $F : \mathcal{M} \to \mathcal{N}$ gives rise to natural transformations $F^{\pm}_{-,X}$ of $\mathcal{B}$-module functors:

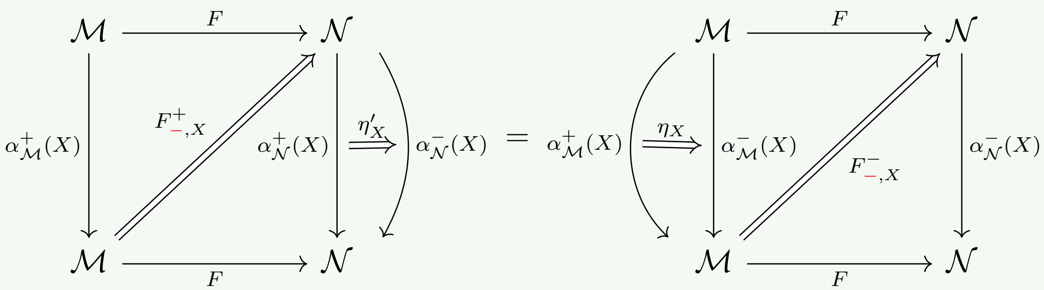


for all $X \in \mathcal{B}$.

Definition 4.7 — Let $\mathbf{A}(\mathcal{B})$ be the 2-category consisting of:

(0) objects are pairs $(\mathcal{M}, \eta)$ where $\mathcal{M}$ is a $\mathcal{B}$-module category and $\eta : \alpha^{+}_{\mathcal{M}} \xRightarrow{\sim} \alpha^{-}_{\mathcal{M}}$ is an isomorphism of tensor functors,

(1) a 1-morphism $F : (\mathcal{M}, \eta) \to (\mathcal{N}, \eta')$ is a $\mathcal{B}$-module functor $F : \mathcal{M} \to \mathcal{N}$ such that



(2) A 2-morphism $\alpha : F \Rightarrow F'$ is just a $\mathcal{B}$-module natural transformation.

5 Explicit Description of $\mathcal{Z}(\Sigma\mathcal{B})$

This section is based on sections 4.1 and 4.2 of [4]. First recall that $\Sigma\mathcal{B} \cong \mathsf{Mod}{-}\mathcal{B}$, the category of finite semisimple (right) $\mathcal{B}$-module categories, pointed by the rank one free module $\mathcal{B}$ ([5], 1.3.13). The goal of this section is to prove the following:

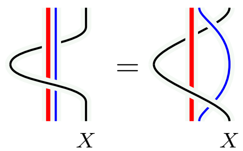
Theorem 5.1. $\mathcal{Z}(\Sigma\mathcal{B}) \cong \mathsf{BrMod}{-}\mathcal{B}$

We will proceed by showing $\mathcal{Z}(\mathsf{Mod}{-}\mathcal{B}) \cong \mathbf{A}(\mathcal{B}) \cong \mathsf{BrMod}{-}\mathcal{B}$.

Lemma 5.2 — There is a canonical 2-equivalence $\mathbf{A}(\mathcal{B}) \cong \mathsf{BrMod}{-}\mathcal{B}$.

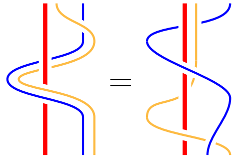
Proof of Lemma. Let $\mathcal{M}$ be a $\mathcal{B}$-module category. Note that a module braiding $\sigma_{-,-}$ on $\mathcal{M}$ is actually the same thing as a natural isomorphism $\eta : \alpha^+_{\mathcal{M}} \overset{\sim}{\Rightarrow} \alpha^-_{\mathcal{M}}$:

$$\begin{array}{ccc} M \triangleleft X & \xrightarrow{\sigma_{M,X}} & M \triangleleft X \\ \| & & \| \\ \alpha^+_{\mathcal{M}}(X)(M) & \xrightarrow[\eta_X(M)]{} & \alpha^-_{\mathcal{M}}(X)(M) \end{array} \qquad \text{for } X \in \mathcal{B} \text{ and } M \in \mathcal{M}.$$

- 

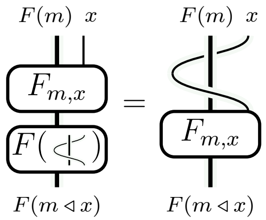
is equivalent to $\eta_X : \alpha^+_{\mathcal{M}}(X) \overset{\sim}{\Rightarrow} \alpha^-_{\mathcal{M}}(X)$ being an isomorphism of left $\mathcal{B}$-module functors, i.e.:

$$\begin{array}{ccc} \alpha^+_{\mathcal{M}}(X)(M) \triangleleft Y & \xrightarrow{(\alpha^+_{\mathcal{M}}(X))_{M,Y}} & \alpha^+_{\mathcal{M}}(X)(M \triangleleft Y) \\ {\scriptstyle \eta_X(M)\triangleleft \mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle \eta_X(M\triangleleft Y)} \\ \alpha^-_{\mathcal{M}}(X)(M) \triangleleft Y & \xrightarrow[(\alpha^-_{\mathcal{M}}(X))_{M,Y}]{} & \alpha^-_{\mathcal{M}}(X)(M \triangleleft Y) \end{array}$$

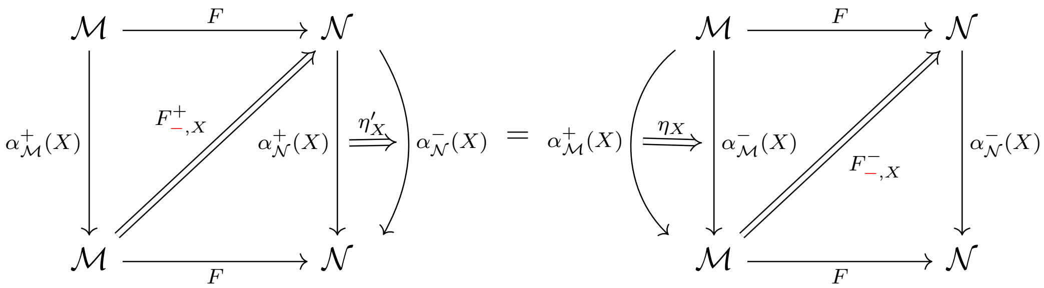
-  is equivalent to the monoidality of the natural isomorphism η, i.e.:

$$\begin{array}{ccc} \alpha^+_{\mathcal{N}}(A_1) \otimes \alpha^+_{\mathcal{N}}(A_2) & \xrightarrow{\eta_{A_1} \otimes \eta_{A_2}} & \alpha^-_{\mathcal{N}}(A_1) \otimes \alpha^-_{\mathcal{N}}(A_2) \\ {\scriptstyle (\alpha^+_{\mathcal{N}})_{A_1,A_2}} \downarrow & & \downarrow {\scriptstyle (\alpha^-_{\mathcal{N}})_{A_1,A_2}} \\ \alpha^+_{\mathcal{N}}(A_1 \otimes A_2) & \xrightarrow[\eta_{A_1 \otimes A_2}]{} & \alpha^-_{\mathcal{N}}(A_1 \otimes A_2) \end{array}$$

Furthermore, for a $\mathcal{B}$-module functor $F : \mathcal{M} \to \mathcal{N}$:

- F being braided, that is 

, is equivalent to:



Lastly, the 2-morphisms in each category just all $\mathcal{B}$-module natural transformations. ■

Sketch of proof of Theorem 5.1. In light of lemma 5.2, we need only show $\mathcal{Z}(\mathsf{Mod}\text{–}\mathcal{B}) \cong \mathbf{A}(\mathcal{B})$.

($\Leftarrow$) We construct a 2-functor $\mathbf{A}(\mathcal{B}) \to \mathcal{Z}(\mathsf{Mod}\text{–}\mathcal{B})$ as follows:

- Let $(\mathcal{N}, \eta : \alpha^+_{\mathcal{N}} \overset{\sim}{\Rightarrow} \alpha^-_{\mathcal{N}})$ be an object of $\mathbf{A}(\mathcal{B})$, so $\mathcal{N} \in \mathsf{Mod}\text{–}\mathcal{B}$.
- Recall that for any $\mathcal{M} \in \mathsf{Mod}\text{–}\mathcal{B}$, there exists an algebra A in $\mathcal{B}$ such that $\mathcal{M} \cong A - \mathsf{Mod}_{\mathcal{B}}$, the category of (left) $\mathcal{A}$-modules in $\mathcal{B}$ ([6], Cor. 7.10.5). Then,

  $$\mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M} \cong \mathcal{N} \boxtimes_{\mathcal{B}} (A - \mathsf{Mod}_{\mathcal{B}}) \cong \mathsf{Mod}_{\mathcal{N}}\text{–}\alpha^+_{\mathcal{N}}(A),$$

  where $\alpha^+_{\mathcal{N}}(A)$ is an algebra in $\mathsf{End}^{\mathsf{r.e.}}_{\mathcal{B}}(\mathcal{N})$ and its modules in $\mathcal{N}$ are objects in $N \in \mathcal{N}$ together with an action $\alpha^+_{\mathcal{N}}(A)(N) \to N$ satisfying the usual axioms.

  Similarly, $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \cong \mathsf{Mod}_{\mathcal{N}}\text{–}\alpha^-_{\mathcal{N}}(A)$.

  Hence, the isomorphism $\eta_A : \alpha^+_{\mathcal{N}}(A) \overset{\sim}{\Rightarrow} \alpha^-_{\mathcal{N}}(A)$ of algebras in $\mathsf{End}^{\mathsf{r.e.}}_{\mathcal{B}}(\mathcal{N})$ yields a 2-natural $\mathcal{B}$-module equivalence

  $$\beta_{\mathcal{M},\mathcal{N}} : \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M} \to \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}.$$

- Let $\mathcal{L} \cong A_1 - \mathsf{Mod}_{\mathcal{B}}$ and $\mathcal{M} \cong A_2 - \mathsf{Mod}_{\mathcal{B}}$. The invertible modification $R_{(\mathcal{N}|\mathcal{L},\mathcal{M})}$ arises from the following commutative diagram of algebra isomorphisms:

  $$\begin{array}{ccc} \alpha^+_{\mathcal{N}}(A_1) \otimes \alpha^+_{\mathcal{N}}(A_2) & \xrightarrow{\eta_{A_1} \otimes \eta_{A_2}} & \alpha^-_{\mathcal{N}}(A_1) \otimes \alpha^-_{\mathcal{N}}(A_2) \\ {\scriptstyle (\alpha^+_{\mathcal{N}})_{A_1,A_2}} \downarrow & & \downarrow {\scriptstyle (\alpha^-_{\mathcal{N}})_{A_1,A_2}} \\ \alpha^+_{\mathcal{N}}(A_1 \otimes A_2) & \xrightarrow[\eta_{A_1 \otimes A_2}]{} & \alpha^-_{\mathcal{N}}(A_1 \otimes A_2) \end{array}$$

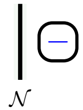
  Indeed, since $\alpha^{\pm}_{\mathcal{N}}$ is a central functor, $\alpha^{\pm}_{\mathcal{N}}(A_1) \otimes \alpha^{\pm}_{\mathcal{N}}(A_2)$ are algebras in $\mathsf{End}^{\mathsf{r.e.}}_{\mathcal{B}}(\mathcal{N})$ and $\eta_{A_1} \otimes \eta_{A_2}$ is an algebra isomorphism.

  The fact that $R_{(\mathcal{N}|-,-)}$ satisfies the $(1,3)$-crossing and unit axioms follows from the monoidality of η.

This gives rise to a 2-functor $(\mathcal{N}, \eta) \mapsto \big(\mathcal{N}, \beta_{\mathcal{N},-}, R_{(\mathcal{N}|-,-)}\big)$.

($\Rightarrow$) We construct a 2-functor $\mathcal{Z}(\mathsf{Mod}\!-\!\mathcal{B}) \to \mathbf{A}(\mathcal{B})$ as follows:

- Note that for any $X \in \mathsf{End}_{\mathcal{B}}^{\mathsf{r.e.}}(\mathcal{B}) \cong \mathcal{B}$ and $\mathcal{N} \in \mathsf{Mod}\!-\!\mathcal{B}$



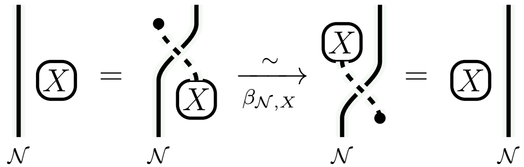
$$: \mathcal{B} \to \mathsf{End}_{\mathcal{B}}^{\mathsf{r.e.}}(\mathcal{N}) \quad \text{given by} \quad X \mapsto \rho_{\mathcal{N}} \circ (\mathrm{id}_{\mathcal{N}} \boxtimes_{\mathcal{B}} X) \circ \rho_{\mathcal{N}}^{-1}$$

$$: \mathcal{B} \to \mathsf{End}_{\mathcal{B}}^{\mathsf{r.e.}}(\mathcal{N}) \quad \text{given by} \quad X \mapsto \lambda_{\mathcal{N}} \circ (X \boxtimes_{\mathcal{B}} \mathrm{id}_{\mathcal{N}}) \circ \lambda_{\mathcal{N}}^{-1}$$

  are isomorphic to $\alpha_{\mathcal{N}}^{+}$ and $\alpha_{\mathcal{N}}^{-}$ respectively.

- For an object $\big(\mathcal{N}, \beta_{\mathcal{N},-}, R_{(\mathcal{N}|-,-)}\big) \in \mathcal{Z}(\mathsf{Mod}\!-\!\mathcal{B})$, consider:

$$\eta_X : \alpha_{\mathcal{N}}^{+}(X) \;\cong$$



$$\cong\; \alpha_{\mathcal{N}}^{-}(X)$$

  Since $\beta_{\mathcal{N},-}$ is a 2-natural transformation, $\beta_{\mathcal{N},X\circ Y} = \beta_{\mathcal{N},X} \circ \beta_{\mathcal{N},Y}$, implying that η is an isomorphism of tensor functors.

This gives rise to a 2-functor $\big(\mathcal{N}, \beta_{\mathcal{N},-}, R_{(\mathcal{N}|-,-)}\big) \mapsto (\mathcal{N}, \eta)$ which is a quasi-inverse to $\mathbf{A}(\mathcal{B}) \to \mathcal{Z}(\mathsf{Mod}\!-\!\mathcal{B})$. $\square$

Corollary 5.3 — $\mathsf{BrMod}{-}\mathcal{B}$ may be equipped with the structure of a braided monoidal 2-category due to the equivalence $\mathcal{Z}(\Sigma\mathcal{B}) \cong \mathsf{BrMod}{-}\mathcal{B}$ of monoidal 2-categories. This structure can be described explicitly (see [3], Remark 4.13).

Remark 5.1. The forgetful functor $\mathcal{Z}(\Sigma\mathcal{B}) \cong \mathsf{BrMod}{-}\mathcal{B} \xrightarrow{\text{forget}} \mathsf{Mod}{-}\mathcal{B} \cong \Sigma\mathcal{B}$ is fully faithful on 2-morphisms since every module natural transformations between two braided module functors is allowed.

Remark 5.2. Any $\mathcal{B}$-module summand of a braided $\mathcal{B}$-module category can be equipped with the structure of a braided $\mathcal{B}$-module category.

Hence, a braided $\mathcal{B}$-module category is indecomposable if and only if it is indecomposable as a $\mathcal{B}$-module category.

Warning 5.4 — For a general non-connected fusion 2-category $\mathcal{A}$, the canonical map $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is faithful on 2-morphisms but *not necessarily* full.

Exercise 5.1. Find such a fusion 2-category $\mathcal{A}$ such that the map $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is *not* full.

Corollary 5.5 — $\Omega\mathcal{Z}(\Sigma\mathcal{B}) = \mathcal{Z}_{(2)}(\mathcal{B})$.

Proof of Corollary. The unit object in $\mathcal{Z}(\Sigma\mathcal{B})$ corresponds to the "rank-1 free module" $\mathcal{B} \in \mathsf{BrMod}{-}\mathcal{B}$. Then recall that in example 4.5, we saw that the endomorphism category of $\mathcal{B}$ is $\mathcal{Z}_{(2)}(\mathcal{B})$. ■

References

[1] J. Baez and M. Neuchl, Higher-dimensional algebra I: braided monoidal 2-categories, Adv. Math. **121** (1996), 196-244.

[2] J. Böckenhauer, D.E. Evans, and Y. Kawahigashi, On α-induction, chiral generators and modular invariants for subfactors, Publ. Res. Inst. Math. Sci., **37** (2001), no. 1, 1–35.

[3] S. Crans, Generalized centers of braided and sylleptic monoidal 2-categories, Adv. Math. **136** (1998), 183-223.

[4] A. Davydov and D. Nikshych. Braided Picard groups and graded extensions of braided tensor categories. 2021. arXiv:2006.08022

[5] C. L. Douglas and D. J. Reutter. Fusion 2-categories and a state-sum invariant for 4-manifolds. 2018. arXiv:1812.11933.

[6] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. Tensor categories, vol. 205 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015.

[7] N. Johnson and D. Yau. 2-Dimensional Categories. 2020. arXiv:2002.06055

[8] D. Penneys. MATH 8800 Course Notes, Spring 2021.