A duality formalism in the spirit of functional analysis: Problem Sheet

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1 Foundations of Hilbert spaces

1.1 Hilbert spaces

We will briefly recall basic definitions to establish our notation.

Definition 1.1. A Hilbert space consists of

- an underlying vector space H;
- an inner product $\langle -|-\rangle_H \colon \overline{H} \otimes H \to \mathbb{C}$.

This data is require to satisfy the following conditions:

- (Hermitian) $\langle \eta|\xi\rangle_H = \overline{\langle \xi|\eta\rangle}_H$ for every $\eta, \xi \in H$;
- (Positive) $\langle \eta|\eta\rangle_H \geq 0$ for every $\eta \in H$;
- (Definite) $\langle \eta|\eta\rangle_H = 0$ if and only if $\eta = 0$;
- (Complete) The inner product $\langle -|-\rangle_H$ gives rise to a norm $\| - \|_H$ which gives rise to a metric $d_H(-,-)$. We demand that H is Cauchy complete with respect to this metric.

Definition 1.2. A (closed) subspace $K \subseteq H$ of a Hilbert space H consists of a sub-vector space K which is closed with respect to the metric on H.[1]

Definition 1.3. Let H be a Hilbert space. The conjugate Hilbert space $\overline{H}$ is given by the vector space with symbols

$$\overline{H} \coloneqq \{\overline{\xi} \mid \xi \in H\},$$

sum $\overline{\xi} + \overline{\eta} \coloneqq \overline{\xi + \eta}$, and scalar action $\lambda\overline{\xi} \coloneqq \overline{\overline{\lambda}\xi}$. We define the inner product on $\overline{H}$ by

$$\langle \overline{\eta}|\overline{\xi}\rangle_{\overline{H}} \coloneqq \overline{\langle \eta|\xi\rangle}_H = \langle \xi \mid \eta\rangle_H.$$

For a linear transformation $f \colon H \to K$, we define $\overline{f} \colon \overline{H} \to \overline{K}$ by $\overline{f}\overline{\xi} \coloneqq \overline{f\xi}$.

[1] In this note, all subspaces of a Hilbert space will be required to be closed.

1.2 Operators

Definition 1.4. Let $f\colon H \to K$ be a linear map between Hilbert spaces. We define $\|f\| \in [0, \infty]$ to be the following quantity

$$\|f\| \coloneqq \sup_{\|\xi\|_H = 1} \|f\xi\|_K.$$

Definition 1.5. We say a linear map $f\colon H \to K$ between Hilbert spaces is bounded whenever $\|f\| < \infty$. We denote the collection of all such bounded maps by $\mathrm{Hom}(H \to K)$. We use the special notation $B(H) \coloneqq \mathrm{Hom}(H \to H)$ and $H^* \coloneqq \mathrm{Hom}(H \to \mathbb{C})$.[2]

Exercise 1.6. Let H be a Hilbert space and K be a vector space. Given an isomorphism of vector space $f\colon H \to K$, consider the following inner product on K

$$\langle k_1 | k_2 \rangle_k \coloneqq \langle f^{-1} k_1 | f^{-1} k_2 \rangle_H.$$

Verify that this indeed defines an inner product which equips K with the structure of a Hilbert space.

Remark 1.7. The inner product seen in Exercise 1.6 is known as the pull-back along f, and this process is sometimes known as *transfer of structure.*

1.3 Adjoints

Definition 1.8. Let $f\colon H \to K$ be a bounded map between Hilbert spaces H and K. We say that $f^\dagger\colon K \to H$ is the adjoint for f when

$$\langle fh | k \rangle_K = \langle h | f^\dagger k \rangle_H \quad \text{for every } h \in H,\ k \in K.$$

Exercise 1.9. Show that adjoints are unique whenever they exist.[3]

Exercise 1.10. Show that $(-)^\dagger\colon \mathsf{Hilb}^{\mathrm{op}} \to \mathsf{Hilb}$ is a contravariant functor.[4]

Hint: Show that $\mathrm{id}^\dagger = \mathrm{id}$ *and* $(fg)^\dagger = g^\dagger f^\dagger$.

1.4 Unitaries

Definition 1.11. We say that an operator $u \in B(H)$ is a unitary when $u^{-1} = u^\dagger$.

Exercise 1.12. Show that $\mathrm{id}_H \in B(H)$ is unitary.

Exercise 1.13. If $u, v \in B(H)$ are unitary, show that uv is again unitary.

Exercise 1.14. Let $u \in B(H)$ be unitary and $\lambda \in \mathbb{C}$. Show that λu is unitary if and only if λ is a unit complex number.

1.5 Projections

Definition 1.15. We say that an operator $p \in B(H)$ is a projection when $p = p^\dagger = p^2$.

Exercise 1.16. If $p \in B(H)$ is a projection, show $p = \mathrm{id}$ when restricted to the subspace $\mathrm{Im}\, p$.

Exercise 1.17. If $p \in B(H)$ is a projection, show that $\mathrm{id}_H - p$ is again a projection.

Exercise 1.18. Let $p \in B(H)$ be a projection and $\lambda \in \mathbb{C}$. Show that λp is a projection only when $\lambda = 1$.

[2] In this note, we'll call elements of $B(H)$ operators and elements of H^* functionals. In the literature conventions may vary.
[3] Later on we'll show they always do for a bounded operator.
[4] Here, $(-)^\dagger$ is the identity on objects.

Exercise 1.19. If $p, q \in B(H)$ are commuting projections[5], show that pq is again a projection.

Definition 1.20. We say that two projections $p, q \in B(H)$ are orthogonal whenever $pq = qp = 0$.

Exercise 1.21. Let $p, q \in B(H)$ be projections. Show that $pq = 0$ if and only if $qp = 0$.

Exercise 1.22. Let $p, q \in B(H)$ be orthogonal projections. Show that $p + q$ is again a projection.

1.6 Direct sums

In this section, we will compare different approaches to define the direct sum of Hilbert spaces.

Definition 1.23 (External direct sum)**.** For Hilbert spaces H and K, we define their external direct sum $H \oplus_{ext} K \coloneqq \{(h, k) \mid h \in H, k \in K\}$ as the Hilbert space with inner product

$$\langle (h_1, k_1) | (h_2, k_2) \rangle \coloneqq \langle h_1 | h_2 \rangle + \langle k_1 | k_2 \rangle$$

Remark 1.24. Notice that the constructed Hilbert space $H \oplus_{ext} K$ makes sense regardless of what H and K are.

Definition 1.25 (Internal direct sum)**.** For a Hilbert spaces S with subspaces $H \subseteq S$ and $K \subseteq S$, we say that $S = H \oplus_{int} K$ is the internal direct sum of H and K when

(I1) For every $h \in H$ and $k \in K$, $\langle h | k \rangle = 0$. We denote this by $H \perp K$.

(I2) For every $s \in S$, there exists $h \in H$ and $k \in K$ such that $s = h + k$. We denote this by $S = H + K$.

Remark 1.26. Notice that the construction $H \oplus_{int} K$ only makes sense when H and K are subspaces of a larger Hilbert space S.

Definition 1.27 (Universal direct sum)**.** For Hilbert space H and K, their universal direct sum is any Hilbert space U equipped with maps $i_H : H \to U$ and $i_K : K \to U$ such that

(U1) $i_H^\dagger i_H = \mathrm{id}_H$ and $i_K^\dagger i_K = \mathrm{id}_K$;

(U2) $i_K^\dagger i_H = 0$ and $i_H^\dagger i_K = 0$;

(U3) $i_H i_H^\dagger + i_K i_K^\dagger = \mathrm{id}_U$.

When a Hilbert space U admits such maps, we will denote this by $U = H \oplus_{uni} K$.

Remark 1.28. Notice that this definition makes sense for any H and K, yet is not a construction.

Exercise 1.29. Let H, K be Hilbert spaces with universal direct sum $U = H \oplus_{uni} K$. Deduce that $p_H \coloneqq i_H i_H^\dagger$ and $p_K \coloneqq i_K i_K^\dagger$ are orthogonal projections.

Exercise 1.30. In this exercise, we will show that the three notions of direct sums agree.

(i) (The exterior direct sum is a universal direct sum.) For Hilbert spaces H and K, consider the maps $i_H : H \to H \oplus_{ext} K$ and $i_K : K \to H \oplus_{ext} K$ defined by $h \mapsto (h, 0)$ and $k \mapsto (0, k)$ respectively. Show that $H \oplus_{ext} K = H \oplus_{uni} K$ by verifying identities (U1), (U2), and (U3).

(ii) (The internal direct sum is a universal direct sum.) Let S be a Hilbert space with subspaces $H \subseteq S$ and $K \subseteq S$. Consider the inclusions $\iota_H : H \hookrightarrow S$ and $\iota_K : K \hookrightarrow S$ given by $h \mapsto h$ and $k \mapsto k$ respectively. Show that $H \oplus_{int} K \cong H \oplus_{uni} K$ by verifying identities (U1), (U2), and (U3).

[5] i.e. $pq = qp$

(iii) (Universal direct sums are unique) Suppose U and U' admit maps i_H, i_K and i'_H, i'_K that witness U and U' as universal direct sums of H and K respectively. Show that the map $i'_H i_H^\dagger + i'_K i_K^\dagger \colon U \to U'$ is a unitary isomorphism, i.e. that

$$(i'_H i_H^\dagger + i'_K i_K^\dagger)^\dagger = i_H (i')_H^\dagger + i_K (i')_K^\dagger$$

is the inverse of $i'_H i_H^\dagger + i'_K i_K^\dagger$.

Therefore, any universal direct sum is equivalent to the exterior direct sum and the internal direct sum.

Remark 1.31. Because of Exercise 1.33, we will generally write $H \oplus K$ for the external, internal, and universal direct sums. In this note however, we might include their subscripts for pedagogical reasons.

Exercise 1.32. Let H, K be Hilbert spaces and consider the external direct sum $H \oplus_{ext} K$ with its universal maps i_H and i_K. Verify the projections p_H and p_K from Exercise 1.33 are given by $(h, k) \mapsto (h, 0)$ and $(h, k) \mapsto (0, k)$ respectively.

Exercise 1.33. Let H, K be Hilbert spaces and consider the internal direct sum $H \oplus_{int} K$ with its universal maps i_H and i_K. Verify the projections p_H and p_K from Exercise 1.33 are given by $h + k \mapsto h$ and $h + k \mapsto k$ respectively.

1.7 Orthogonal complements

Definition 1.34. Let $K \subseteq H$ be a subspace of a Hilbert space H. We define the orthogonal complement $K^\perp$ of K in H to be the subspace

$$K^\perp := \{h \in H \mid \langle h|k \rangle \text{ for all } k \in K\}.$$

Exercise 1.35. Verify $K^\perp$ is indeed a subspace of H.

Exercise 1.36. Show $K^{\perp\perp} = K$.

Remark 1.37. Although absent in the notation $K^\perp$, this does depend on both K and H. Indeed, if $K \subseteq H'$ for some other Hilbert space H', the orthogonal complements $K^\perp \subseteq H$ and $K^\perp \subseteq H'$ may differ.

Exercise 1.38. Let H be a Hilbert space. Show that subspaces of H are complemented, i.e. if $K \subseteq H$ is a subspace then $H = K \oplus_{int} K^\perp$.

Exercise 1.39. Conversely, let S be a Hilbert space with subspaces $H \subseteq S$ and $K \subseteq S$. Show that $S = H \oplus_{int} K$ if and only if $H = K^\perp$.

Definition 1.40. Let $K \subseteq H$ be a subspace of a Hilbert space H. By Exercise 1.38, $H = K \oplus_{int} K^\perp = K \oplus_{uni} K^\perp$ with inclusions $i_K \colon K \to H$ and $i_{K^\perp} \colon K^\perp \to H$. By Exercise 1.33, there are projections $p_K := i_K i_K^\dagger$ and $p_{K^\perp} = i_{K^\perp} i_{K^\perp}^\dagger = \mathrm{id}_H - p_K$. We call p_K the projection corresponding to the subspace K.

Exercise 1.41. Let $K \subseteq H$ be a subspace of a Hilbert space H with corresponding projection p_K. Show that $\operatorname{Ker} p_K = K^\perp$. Deduce $(\operatorname{Ker} p_K)^\perp = K$.

Exercise 1.42. Let $p \in B(H)$ be a projection. Show that $\operatorname{Im} p = (\operatorname{Ker} p)^\perp$.

Exercise 1.43 (Fundamental theorem of projections)**.** Show there is a one-to-one correspondence given by

$$\begin{aligned} \{\text{subspaces } K \subseteq H\} &\longleftrightarrow \{\text{projections } p \in B(H)\} \\ K &\longmapsto p_K \\ \operatorname{Im} p &\longleftarrow p \end{aligned}$$

Hint: Use previous exercises to deduce that $\operatorname{Im} p_K = K$ *and* $p_{\operatorname{Im} p} = p$.

1.8 Quotients

Definition 1.44. Let $K \subseteq H$ be a subspace of a Hilbert space H. Consider the vector space

$$H/K := \{\xi + K \mid \xi \in H\} \quad \text{for} \quad \xi + K := \{\xi + k \mid k \in K\},$$

with sum $(\xi_1 + K) + (\xi_2 + K) := (\xi_1 + \xi_2) + K$ and scalar action $\lambda(\xi + K) := (\lambda\xi) + K$.

Exercise 1.45. Verify that the quotient map $K^\perp \to H/K$ given by $h \mapsto h + K$ is an isomorphism of vector spaces.

By Exercise 1.6, the isomorphism from Exercise 1.45 allows us to transfer the inner product on $K^\perp$ to endow H/K with the structure of a Hilbert space.

Exercise 1.46 (First isomorphism theorem)**.** Let $f\colon H \to K$ be a bounded operator between Hilbert spaces. In this exercise, we will show that $H/\operatorname{Ker} f \cong \operatorname{Im} f$.

1. Show that the restriction $f|_{(\operatorname{Ker} f)^\perp}\colon (\operatorname{Ker} f)^\perp \to K$ is injective.
2. Deduce that $(\operatorname{Ker} f)^\perp \cong \operatorname{Im} f$.
3. Conclude that $H/\operatorname{Ker} f \cong \operatorname{Im} f$.

Exercise 1.47 (Rank-Nullity)**.** Let $f\colon H \to K$ be a bounded map between Hilbert spaces. Deduce that $H \cong \operatorname{Ker} f \oplus \operatorname{Im} f$.

Exercise 1.48. Let $\varphi \in H^*$ be a functional. Show that $\varphi \neq 0$ if and only if $(\ker\varphi)^\perp \cong \mathbb{C}$.

1.9 Riesz Representation Theorem

Theorem 1.49 (Riesz)**.** *If $\varphi \in H^*$ is a functional on a Hilbert space H, then there exists a unique $\eta \in H$ such that*

$$\varphi(\xi) \underset{(\star)}{=} \langle \eta|\xi\rangle \quad \textit{for every } \xi \in H.$$

Exercise 1.50. The goal of this exercise is to prove the Riesz representation theorem for Hilbert spaces. The statement is trivial for $\varphi = 0$, so let $\varphi \in H^*$ be a non-zero functional on a Hilbert space H.

1. First suppose there already exists $\eta \in H$ such that $\varphi(\xi) \underset{(\star)}{=} \langle \eta|\xi\rangle$ for every $\xi \in H$.
   - Show that η is unique, i.e. if $\eta' \in H$ satisfies $(\star)$ then $\eta = \eta'$.
   - Prove that $\eta \in (\ker\varphi)^\perp$.
   - For any non-zero $\eta_0 \in (\ker\varphi)^\perp$, show there exists some $\lambda \in \mathbb{C}$ such that $\eta = \lambda\eta_0$. Then compute that $\lambda = \frac{\overline{\varphi(\eta_0)}}{\langle \eta_0|\eta_0\rangle}$ by plugging in $\xi = \eta$ in $(\star)$.
2. In full generality, show there must exist an $\eta \in H$ satisfying $\varphi(\xi) \underset{(\star)}{=} \langle \eta|\xi\rangle$ for every $\xi \in H$.

   Hint:
   - Choose a non-zero $\eta_0 \in (\ker\varphi)^\perp$ and set $\eta := \frac{\overline{\varphi(\eta_0)}}{\langle \eta_0|\eta_0\rangle}\eta_0$.
   - Deduce $\varphi(\eta) = \langle \eta|\eta\rangle$.
   - Let $\xi \in H$. Show that $\varphi(\xi)\eta - \varphi(\eta)\xi \in \ker\varphi$.
   - Use the fact that $\eta \in (\ker\varphi)^\perp$ to conclude that $\varphi(\xi) \underset{(\star)}{=} \langle \eta|\xi\rangle$ for every $\xi \in H$.