Unitary Fusion Categories and the Levin-Wen Model

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Introduction

The mathematical theory of tensor categories first emerged in the context of representation theory, topology, and operator algebras.

In particular, these objects of study together with the manner in which they fuse assemble into a tensor category.

Recent developments in physics have led to the discovery of exotic phases of matter, now known as topological phases.

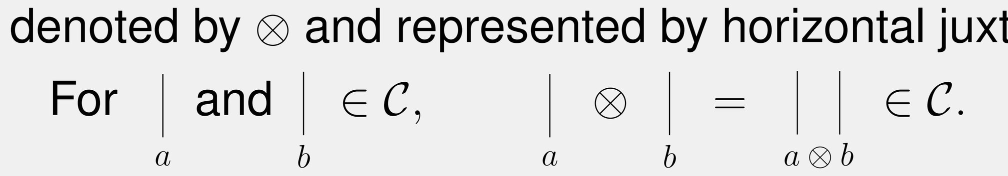
Tensor category theory provides a mathematical framework for characterizing and classifying these phases.

This is made explicit by the Levin-Wen construction.

Unitary monoidal categories

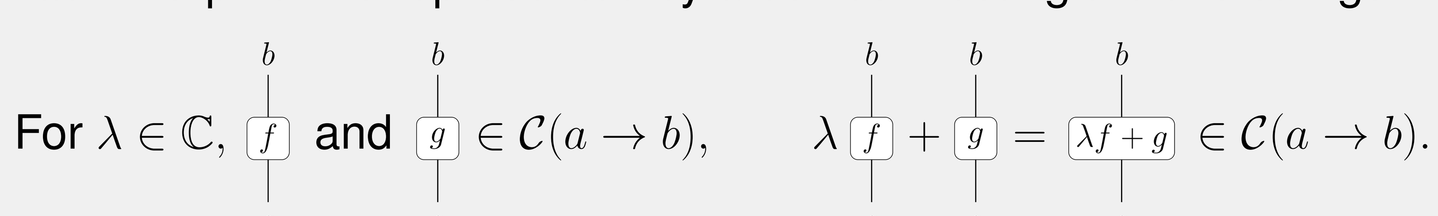
A unitary monoidal category $\mathcal{C}$ consists of:

- *Objects*, represented by labelled vertical strings, together with an operation, denoted by $\otimes$ and represented by horizontal juxtaposition:

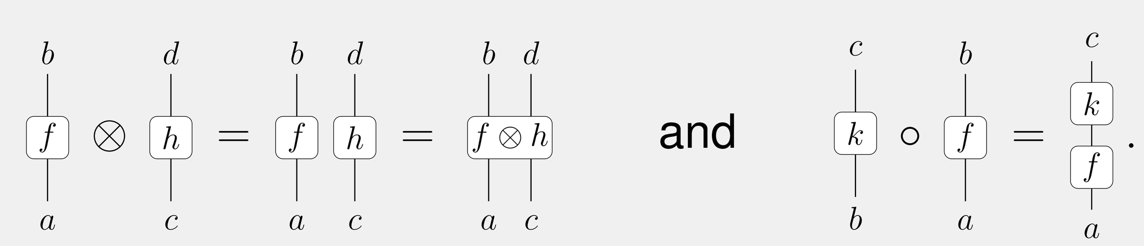


- A vector space $\mathcal{C}(a \to b)$ of *morphisms* between each pair of objects in $\mathcal{C}$.

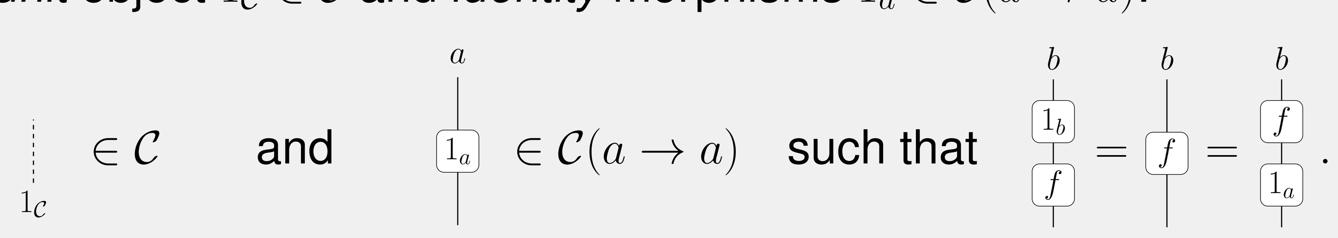
  Each morphism is represented by a box connecting the two strings:



  We also have bilinear horizontal and vertical *compositions*:

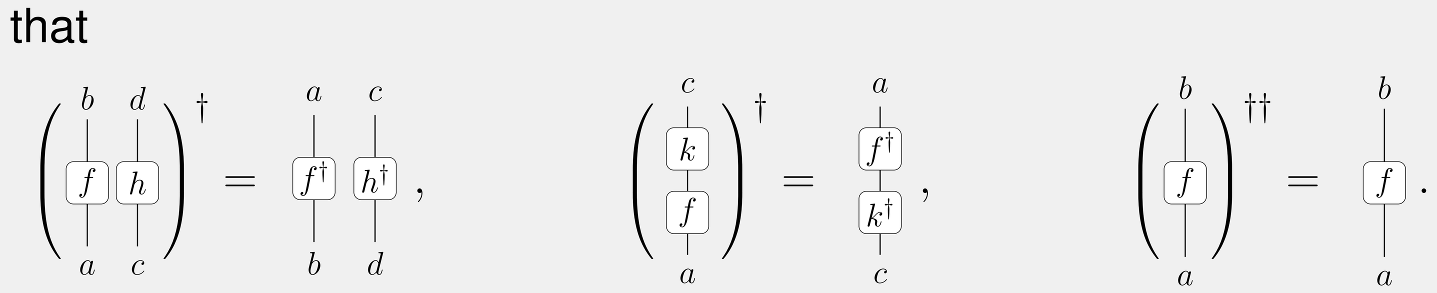


- A *unit* object $1_{\mathcal{C}} \in \mathcal{C}$ and *identity* morphisms $1_a \in \mathcal{C}(a \to a)$:



  One tends to omit instances of $1_{\mathcal{C}}$ and 1_a.

- For all $a, b \in \mathcal{C}$, a conjugate-linear map $\dagger\colon \mathcal{C}(a \to b) \to \mathcal{C}(b \to a)$ such that



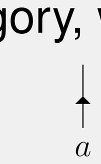
  We say f is unitary when $f^\dagger = f^{-1}$.

- The operation $\otimes$ on objects is *associative* and *unital* up to chosen natural unitaries, for which we also require some axioms.

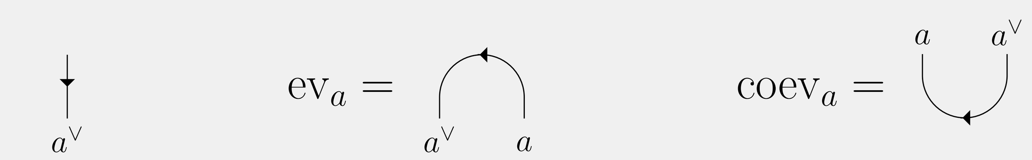
Unitary fusion categories

Rigidity

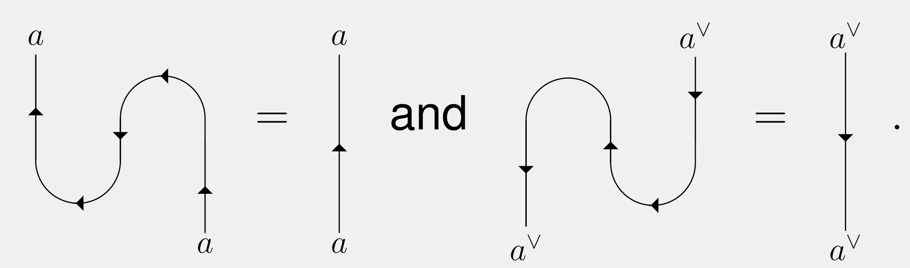
In a *rigid* unitary monoidal category, we orient strings:



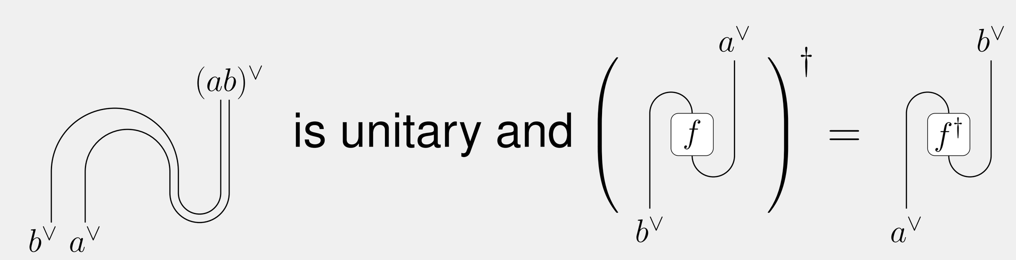
For each $a \in \mathcal{C}$, we choose a dual $a^\vee \in \mathcal{C}$ together with morphisms:



such that



We require that



Cauchy complete

$\mathcal{C}$ is *Cauchy complete* if:

- every projection splits, and
- finite orthogonal direct sums of objects exist.

Simple objects and semisimple categories

- $c \in \mathcal{C}$ is *simple* when $\mathcal{C}(c \to c) = \mathbb{C}\,1_c$.
- $\mathcal{C}$ is *semisimple* when every $a \in \mathcal{C}$ is an orthogonal direct sum of simple objects.
- $\mathcal{C}$ is *finitely* semisimple if there are only finitely many simples.

A unitary monoidal category which is rigid, Cauchy complete, and semisimple is said to be (multi)fusion/tensor when:

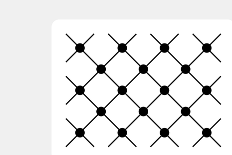
$\mathcal{C}$ is \ $1_{\mathcal{C}}$ is	simple	not simple
finitely semisimple	fusion	multifusion
not finitely semisimple	tensor	multitensor

Levin-Wen model

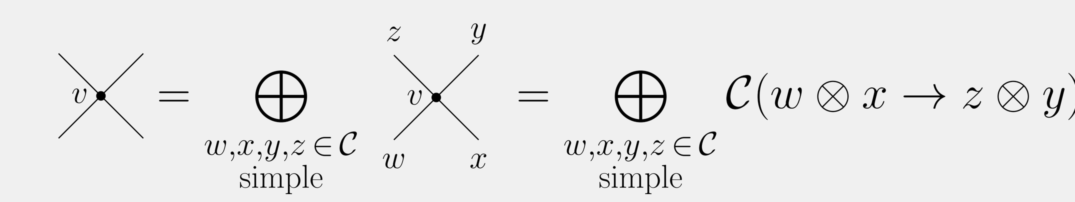
The Levin-Wen model showcases how category theoric constructions give rise to emergent quantum objects.
Physically, local energetic constraints on certain topological phases cause microscopic degrees of freedom to organize into effective extended objects called string-nets.
Where kinetic energy dominates local energetic constraints, the string-net condenses, filling all space and creating a doubled topological phase. Unitary fusion categories define and classify those phases.

Lattice and total Hilbert space

Consider a rectangular lattice:



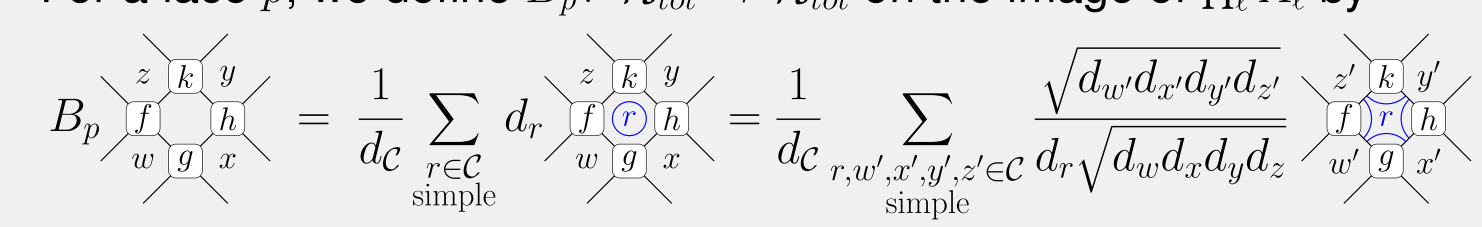
For each vertex v in the lattice, we associate a Hilbert space:



The total Hilbert space is $\mathcal{H}_{tot} = \bigotimes_v \mathcal{H}_v$.

Edge and face terms

- For an edge ℓ, the linear transformation $A_\ell\colon \mathcal{H}_{tot} \to \mathcal{H}_{tot}$ enforces that the simples labelling ℓ match.
- For a face p, we define $B_p\colon \mathcal{H}_{tot} \to \mathcal{H}_{tot}$ on the image of $\prod_\ell A_\ell$ by



Hamiltonian and ground state space

The string-net local Hamiltonian is a linear transformation

$H\colon \mathcal{H}_{tot} \to \mathcal{H}_{tot}$ given by $H = -\sum_\ell A_\ell - \sum_p B_p$.

The eigenspace in $\mathcal{H}_{tot}$ corresponding to the smallest eigenvalue of H is called the *ground state space*.

Quasiparticle excitations

- A unit vector in $\mathcal{H}_{tot}$ is said to be a *localized excitation* if is in the ground state space of every $-A_\ell$ and $-B_p$ except in a small region of the lattice.
- These localized excitations are the emergent quasiparticles, or *anyons* of the theory, and are given by the Drinfeld center $Z(\mathcal{C})$.
- The Drinfeld center is a categorification of the center of a group.

Reference: David Penneys, MATH 8800 course notes, OSU Spring 2021