

Giovanni Ferrer • Thesis Defense • July 15, 2026

FUNCTIONAL ANALYSIS IN THREE-DIMENSIONAL CATEGORY THEORY



Functional Analysis in Three-Dimensional Category Theory

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Abstract

☒ Functional analysis was developed in order to describe the mathematical foundations of quantum mechanics.

☒ There is a movement throughout mathematics of categorification as n -categories naturally describe n D structures.

☒ This leads us to develop tools for describing a theory of functional analysis in 3D category theory.

Higher functional analysis: Research program

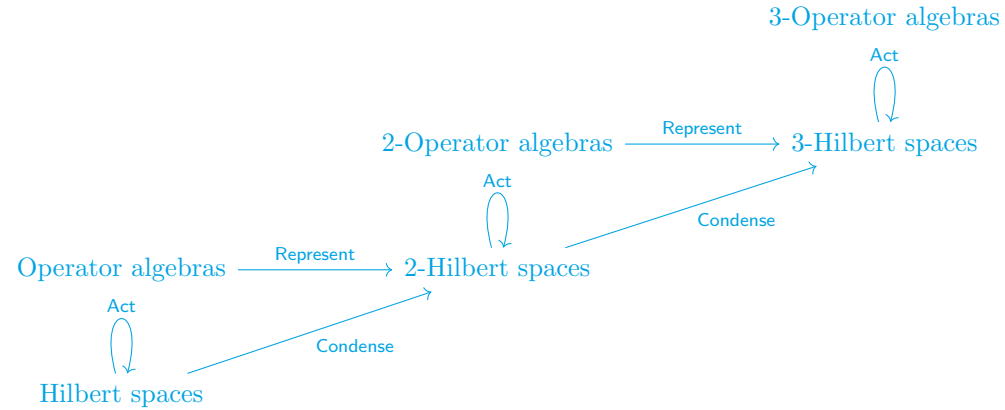
[F+10]

[F22] [F24] [FF26]

[CFHPS25] [FHPW26]

[FMPS26]

[UQSL]

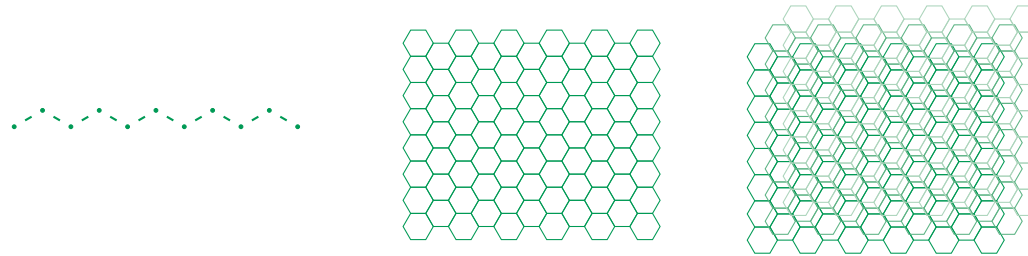


- ☒ [UQSL] = Unitary Quantum Symmetries Lite (F, Penneys, Kawagoe)
- ☒ [FHPW26] = Orthonormal bases for higher Hilbert spaces (F, Hungar, Penneys, Wesley)
- ☒ [FMPS26] = The many faces of higher Hilbert spaces (F, Müller, Penneys, Stehouwer)
- ☒ [FF26] = The homotopy 3-type of abelian C*-algebras (Faurot, F)
- ☒ [CFHPS25] = Manifestly unitary higher Hilbert spaces (Chen, F, Hungar, Penneys, Sanford)
- ☒ [F24] = Foundations for operator algebraic tricategories (F)
- ☒ [F+10] = Dagger n -categories (F + 10 wonderful human beings)
- ☒ [F22] = Gray-categories model algebraic tricategories (F)

Higher functional analysis: Motivation

Quantum symmetry. The classical symmetries of a set S are captured by the corresponding group of symmetries G : groups act on sets. In the same way, higher operator algebras $\mathcal{A}$ act on higher Hilbert spaces as so-called *quantum symmetries*.

Topological order. Higher operator algebras provide an algebraic structure for excitations in so-called *topologically ordered* systems, capturing the fusion, braiding, and commutation relations of the quasi-particle excitations (or anyons) in your material.



Topological Quantum Field Theory. Higher Hilbert spaces are expected to serve as the appropriate receptacles for fully-extended unitary topological quantum field theories (TQFTs).

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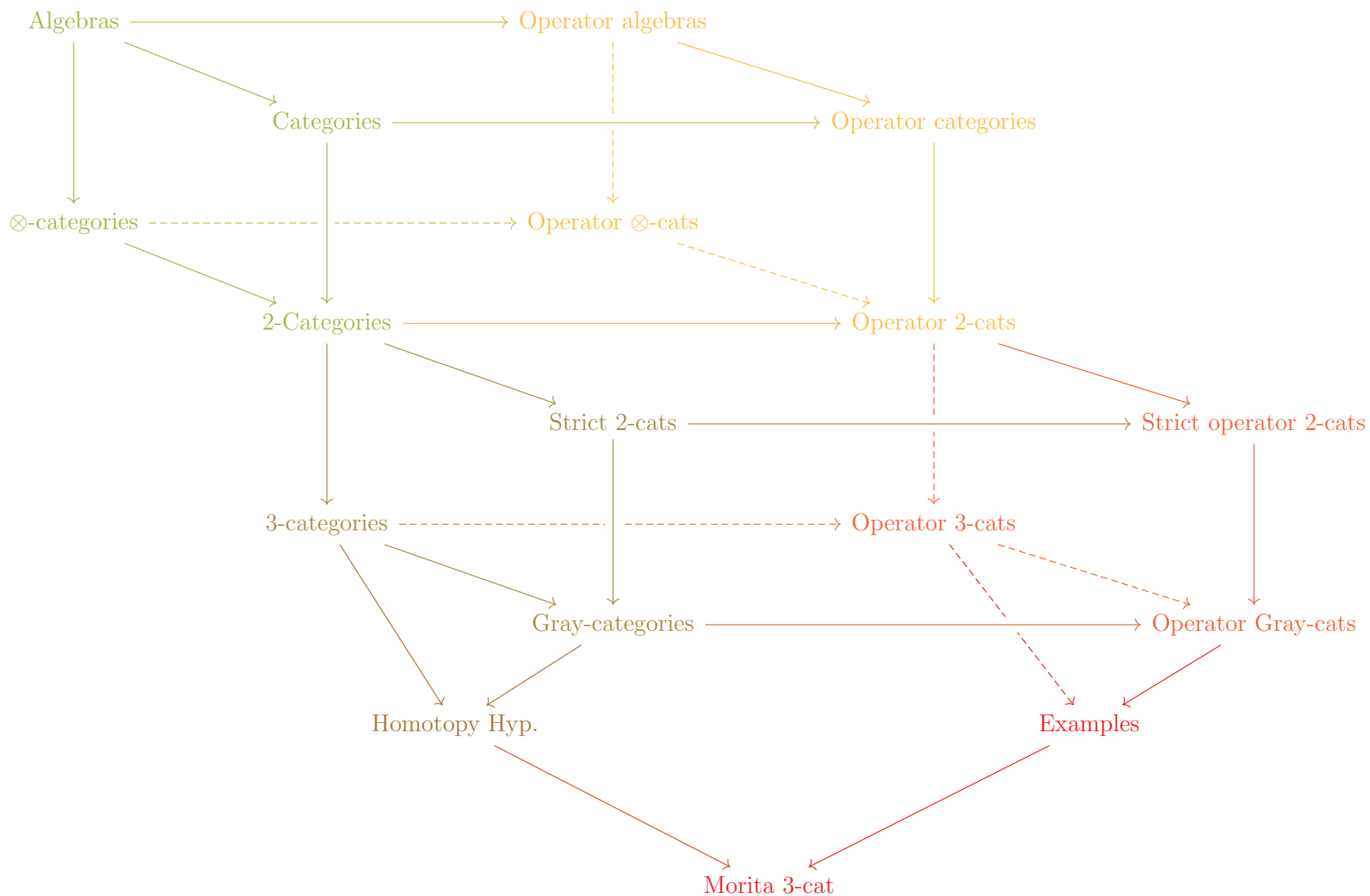
☒ Part §1: Introduce functional analytic and category theoretic preliminaries.

☒ Part §2: Finish establishing the homotopy hypothesis for homotopy 3-types.

☒ Part §3: Provide the foundations for operator 3-categories and an example of interest: $\mathbf{AbC}^*\mathbf{Alg}$.

☒ Part §4: Describe the homotopy 3-type associated to $\mathbf{AbC}^*\mathbf{Alg}$ in terms of topological and cohomological data.

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1 Preliminaries

1-categories

Definition — A 1-category $\mathcal{A}$ consists of the following data:

- ☒ A collection of objects, which we denote by $A, B, C, \dots \in \mathcal{A}$;
- ☒ For every pair of objects $A, B \in \mathcal{A}$, a collection $\mathcal{A}(A \rightarrow B)$ of morphisms;
- ☒ For every triple of objects $A, B, C \in \mathcal{A}$, a function

$$\begin{aligned} \mathcal{A}(A \rightarrow B) \times \mathcal{A}(B \rightarrow C) &\xrightarrow{\circ} \mathcal{A}(A \rightarrow C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

This data is subject to the following axioms:

- ☒ Composition is associative

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- ☒ Composition is unital, where we denote the identity on $A \in \mathcal{A}$ by id_A :

$$\text{id}_B \circ f = f \quad \text{and} \quad f \circ \text{id}_A = f.$$

$\mathcal{A}$ is $(\mathbb{C})$ -linear when each $\mathcal{A}(A \rightarrow B)$ is equipped with the structure of a (complex) vector space and $\circ$ is bilinear.

1-categories (data)

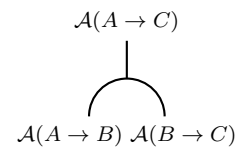


Figure 1: 2D string diagram for 1-composition in 1-categories

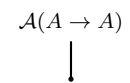


Figure 2: 2D string diagram for identities in 1-categories

1-categories (axioms)

$$\begin{array}{c} \mathcal{A}(A \rightarrow D) \\ \downarrow \\ \text{[Diagram: A line from } \mathcal{A}(A \rightarrow B) \text{ goes up, then a semi-circle over } \mathcal{A}(B \rightarrow C) \text{ and } \mathcal{A}(C \rightarrow D) \text{ joins it to } \mathcal{A}(A \rightarrow D).]} \\ \mathcal{A}(A \rightarrow B) \mathcal{A}(B \rightarrow C) \mathcal{A}(C \rightarrow D) \end{array} = \begin{array}{c} \mathcal{A}(A \rightarrow D) \\ \downarrow \\ \text{[Diagram: A line from } \mathcal{A}(A \rightarrow B) \text{ and } \mathcal{A}(B \rightarrow C) \text{ goes up, then a semi-circle over } \mathcal{A}(C \rightarrow D) \text{ joins it to } \mathcal{A}(A \rightarrow D).]} \\ \mathcal{A}(A \rightarrow B) \mathcal{A}(B \rightarrow C) \mathcal{A}(C \rightarrow D) \end{array}$$

Figure 3: Associativity axiom for categories

$$\begin{array}{c} \mathcal{A}(A \rightarrow B) \\ \downarrow \\ \text{[Diagram: A line from } \mathcal{A}(A \rightarrow B) \text{ goes up, then a semi-circle with a dot on the left input joins it to } \mathcal{A}(A \rightarrow B).]} \\ \mathcal{A}(A \rightarrow B) \end{array} = \begin{array}{c} \mathcal{A}(A \rightarrow B) \\ \downarrow \\ \mathcal{A}(A \rightarrow B) \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{A}(A \rightarrow B) \\ \downarrow \\ \text{[Diagram: A line from } \mathcal{A}(A \rightarrow B) \text{ goes up, then a semi-circle with a dot on the right output joins it to } \mathcal{A}(A \rightarrow B).]} \\ \mathcal{A}(A \rightarrow B) \end{array} = \begin{array}{c} \mathcal{A}(A \rightarrow B) \\ \downarrow \\ \mathcal{A}(A \rightarrow B) \end{array}$$

Figure 4: Left and right unit axioms for categories

1-categories

1-category $\mathcal{A}$	1-functor F	nat. trans. θ
hom set $\mathcal{A}(A \rightarrow B)$	map of homs F	naturality axiom
composition $\circ$	composability axiom	coherence theorem
associativity axiom	coherence theorem	coherence theorem
pentagon (coh. thm.)	coherence theorem	coherence theorem
3D associahedron (coh. thm.)	coherence theorem	coherence theorem

1-category $\mathcal{A}$				1-functor F		nat. trans. θ			
unit I_A				unitality axiom		coherence theorem			
left unit. axiom		right unit. axiom		coherence theorem	coherence theorem				
left unital (c. thm.)	mid. unital (c. thm.)	right unital (c. thm.)							
far left (c. thm.)	mid left (c. thm.)	mid right (c. thm.)	far right (c. thm.)						

Table 1: Data and axioms of one-dimensional category theory

2-categories

Definition — A 2-category $\mathcal{A}$ consists of the following data:

- ☒ Collection of *objects* of $\mathcal{A}$;
- ☒ 1-categories $\mathcal{A}(A \rightarrow B)$;
- ☒ 1-functors $\odot: \mathcal{A}(A \rightarrow B) \times \mathcal{A}(B \rightarrow C) \rightarrow \mathcal{A}(A \rightarrow C)$ called *1-composition*;
- ☒ 1-functors $I_A: * \rightarrow \mathcal{A}(A \rightarrow A)$ which picks out an identity 1-morphism id_A ;
- ☒ “*associator*” natural isomorphisms α ;
- ☒ *left and right “unitor”* natural isomorphisms λ and ρ .

We require this data satisfies the following axioms:

- ☒ 2-dimensional associahedron (pentagon) axiom (non-abelian 3-cocycle condition);
- ☒ Triangle axiom relating the left and right unitors with the associator (normalized cocycle condition).

2-categories (data)

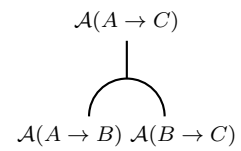


Figure 5: 2D string diagram for 1-composition in 2-categories

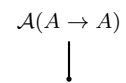


Figure 6: 2D string diagram for identities in 2-categories

2-categories (constraint data)

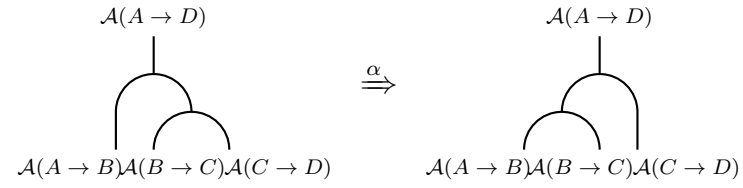


Figure 7: $(2 + 1)$ D string diagram for associators in 2-categories

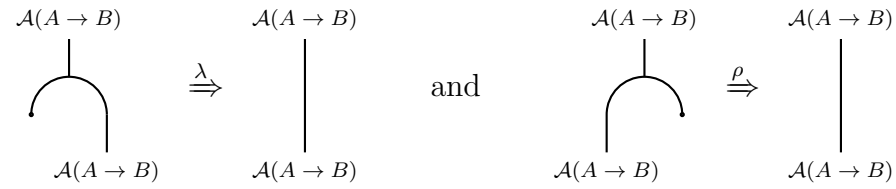


Figure 8: $(2 + 1)$ D string diagram for left and right unitors in 2-categories

2-categories (axioms)

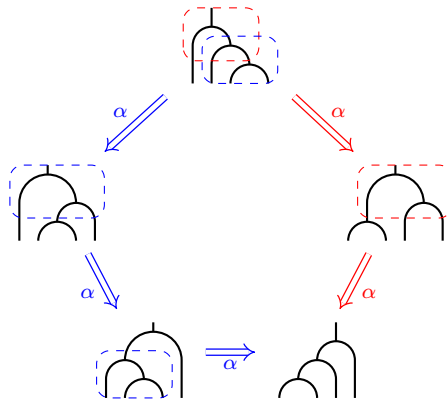


Figure 9: Associativity pentagon axiom for 2-categories

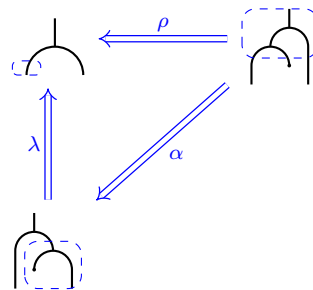


Figure 10: Middle unitality triangle axiom for 2-categories

2-categories (consequences)

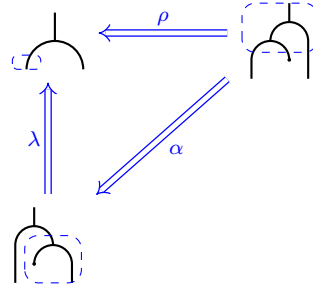


Figure 11: Middle unitality triangle axiom for 2-categories

At this category level, it is a coincidence that the following automatically commute in any 2-category.

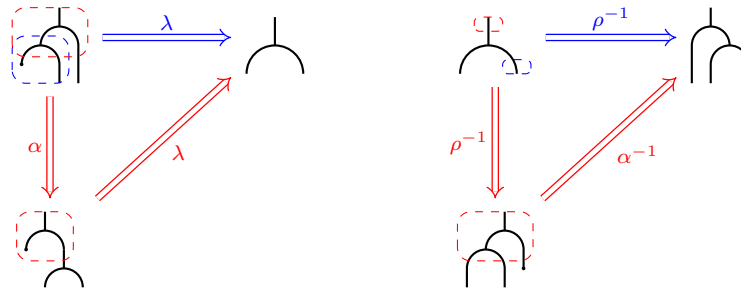


Figure 12: Commutative left and right unitality triangles in 2-categories

Thinking of $\alpha_{-, -, -}$ as a 3-cocycle, these commutative triangles witness the fact that α is normalized on the left, middle, and right entries respectively.

2-categories

2-category $\mathcal{A}$	2-functor F	2-nat. trans. θ	modification m
hom category $\mathcal{A}(A \rightarrow B)$	map of homs F	naturator θ	naturality axiom
1-composition $\odot$	compositor F^2	composability axiom	coherence theorem
associator α	associativity axiom	coherence theorem	coherence theorem
pentagon axiom	coherence theorem	coherence theorem	coherence theorem
3D associahedron (coh. thm.)	coherence theorem	coherence theorem	coherence theorem

2-category $\mathcal{A}$				2-functor F			2-nat. trans. θ			modification m			
unit I_A				unitor F^0			unitality axiom			coherence theorem			
left unitor l		right unitor r		left unit. axiom		right unit. axiom	coh. thm.		coh. thm.				
left unital (c. thm.)	mid. unital axiom		right unital (c. thm.)	coh. thm.	coh. thm.	coh. thm.							
far left (c. thm.)	mid left (c. thm.)	mid right (c. thm.)	far right (c. thm.)										

Table 2: Data and axioms of two-dimensional category theory

Strict 2-categories

☒ A 2-category is *strict* when 1-composition $\odot$ is associative and unital up to equality and the constraints α, λ, ρ are identities.

	2-category	strict 2-category
constraints	non-trivial	trivial

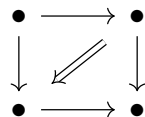
Table 3: Degrees of strictness for 2-categories

Theorem (Mac Lane) — Every 2-category is equivalent to a strict one.

$$\text{2-categories} \xrightarrow{\text{Yoneda}} \text{strict 2-categories}$$

☒ We may also talk about *strict* 2-functors, which have trivial constraints.

☒ For 2-functors of the form $F: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$, there is an in-between notion—we say F is *cubical* if it is strict in each entry and its compositor is partially trivial.



3-categories

Definition — A 3-category $\mathcal{A}$ consists of the following data:

- ☒ Collection of *objects* of $\mathcal{A}$;
- ☒ 2-categories $\mathcal{A}(A \rightarrow B)$
- ☒ 2-functors $\odot: \mathcal{A}(A \rightarrow B) \times \mathcal{A}(B \rightarrow C) \rightarrow \mathcal{A}(A \rightarrow C)$ called *1-composition*.
- ☒ 2-functors $I_A: * \rightarrow \mathcal{A}(A \rightarrow A)$.
- ☒ “*associator*” adjoint equivalences $\mathbf{a} = (a, a', \epsilon^a, \eta^a)$ where a is a 2-natural transformation.
- ☒ *left and right “unitor”* adjoint equivalences $\mathbf{l}$ and $\mathbf{r}$ where l and r are 2-natural transformations.
- ☒ “*pentagonator*” modifications.
- ☒ “*middle, left, and right “triangulator”*” modifications.

We require this data satisfies the following axioms:

- ☒ 3-dimensional associahedron axiom (non-abelian 4-cocycle condition);
- ☒ Two axioms relating the left, middle, right triangulators with the pentagonator (normalized cocycle condition).

3-categories (data)

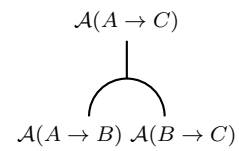


Figure 13: 2D string diagram for 1-composition in 3-categories

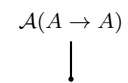


Figure 14: 2D string diagram for identities in 3-categories

3-categories (constraint data)

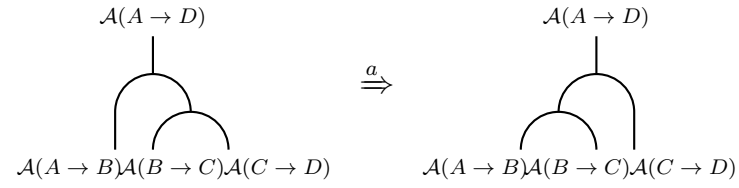


Figure 15: (2+1)D string diagram for associators in 3-categories



Figure 16: (2 + 1)D string diagrams for left and right unitors in 3-categories

3-categories (higher constraint data)

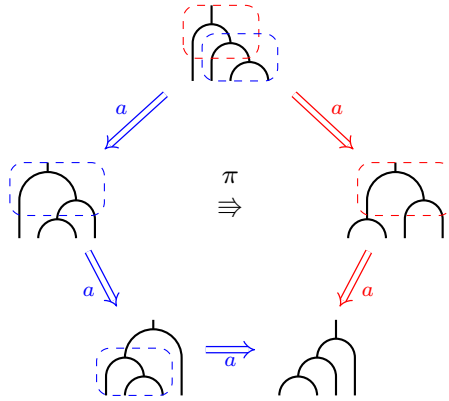


Figure 17: (2+2)D string diagram for pentagonators in 3-categories

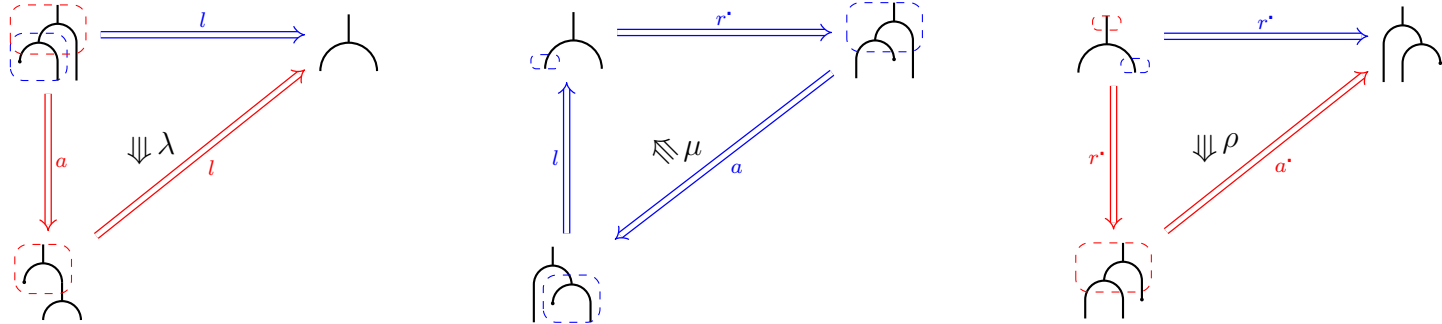


Figure 18: (2 + 2)D string diagrams for left, middle, and right triangulators in 3-categories

3-categories (axioms)

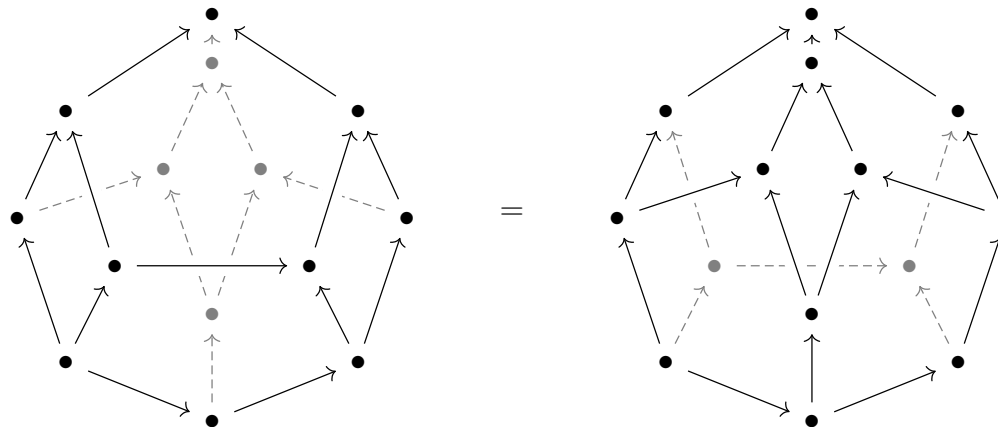


Figure 19: 3-dimensional associahedron axiom for 3-categories (non-abelian 4-cocycle condition)

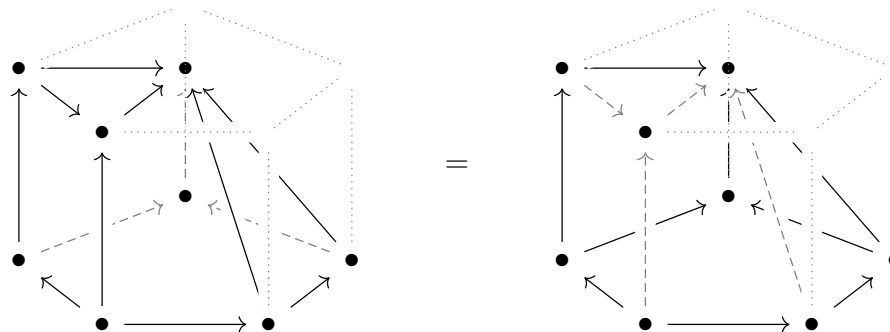


Figure 20: Two axioms relating triangulators and pentagonators (normalized 4-cocycle conditions).

3-categories

3-category $\mathcal{A}$	3-functor F	3-nat. trans. θ	3-modification m	perturbation σ
hom 2-category $\mathcal{A}(A \rightarrow B)$	map of homs F	naturator θ	naturator θ	naturality axiom
1-composition $\odot$	tensorator F^2	compositor θ^2	composability axiom	coherence theorem
associator a	associator F^a	associativity axiom	coherence theorem	coherence theorem
pentagonator π	pentagonity axiom	coherence theorem	coherence theorem	coherence theorem
3D associahedron axiom	coherence theorem	coherence theorem	coherence theorem	coherence theorem

3-category $\mathcal{A}$				3-functor F				3-nat. trans. θ			3-modification m			perturbation σ			
unit I_A				unitor F^0				unitor θ^0			unitality axiom			coherence theorem			
left unitor l		right unitor r		left unitor F^l		right unitor F^r		left unit. axiom	right unit. axiom		coh. thm.	coh. thm.					
left unitor λ	middle unitor μ		right unitor ρ	left unital (c. thm.)	mid. unital axiom		right unital (c. thm.)	coh. thm.	coh. thm.	coh. thm.							
far left (c. thm.)	mid left axiom	mid right axiom	far right (c. thm.)	c. thm.	c. thm.	c. thm.	c. thm.										

Table 4: Data and axioms of three-dimensional category theory

Gray-categories

⊠ Not every 3-category is equivalent to a strict one (the obstruction is the existence of braidings in 3D).

	3-category	cubical 3-category	Gray-category	strict 3-category
hom 2-cats	weak	strict	strict	strict
units	weak	strict	strict	strict
composition	weak	cubical	cubical	strict
constraints	nontrivial	nontrivial	trivial	trivial

Table 5: Degrees of strictness for 3-categories

Theorem ([GPS95]) — Every 3-category is equivalent to a Gray-category.

$$\text{3-categories} \xrightarrow{\text{locally strictify}} \text{cubical 3-categories} \xrightarrow{\text{Yoneda}} \text{Gray-categories} \text{ ----- } \text{strict 3-categories}$$

Gray-categories

☒ Let $2\mathbf{Cat}_{\text{st}}$ be the 1-category of strict 2-categories and strict 2-functors.

☒ There is a Gray tensor product $\mathcal{A} \boxtimes \mathcal{B}$ of strict 2-categories which represents cubical functors $F: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$.

☒ The symmetric monoidal 1-category **Gray** is the category $(2\mathbf{Cat}_{\text{st}}, \boxtimes)$.

Definition — A Gray-category is a category enriched in **Gray** in the sense of [Kel05].

☒ We denote the 1-category of Gray-categories and Gray-functors by **GrayCat**.

2 Gray-categories model 3-categories

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(3\text{Cat})$

Acknowledgements

✂ This part is based on [Fer22].

✂ The author would like to thank David Penneys for his guidance in this project and for many helpful conversations.

✂ The author would also like to thank Nick Gurski, Niles Johnson, and David Reutter for useful suggestions.

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(\text{3Cat})$

$\text{GrayCat}[\mathcal{W}^{-1}]$

We first discuss the model category structure on GrayCat from [Lac11].

Definition — A Gray-functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between Gray-categories is called:

- ☒ a *weak equivalence* if it is an equivalence of $\mathcal{A}$ and $\mathcal{B}$ as 3-categories,
- ☒ a *fibration* if it is a fibration on each hom-2-category and satisfies a so-called biadjoint-biequivalence-lifting property,
- ☒ in particular, a *trivial fibration* if it is surjective on objects, full at all levels, and faithful at the top level,
- ☒ a *cofibration* if it has the left lifting property against trivial fibrations:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{a} & \mathcal{A}' \\ F \downarrow & \nearrow \ell & \downarrow F' \\ \mathcal{B} & \xrightarrow{b} & \mathcal{B}' \end{array}$$

We denote by $\mathcal{W}$ the collection of weak equivalences in GrayCat .

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(3\text{Cat})$

$\text{GrayCat}[\mathcal{W}^{-1}]$

We denote by $\mathcal{W}$ the collection of weak equivalences in GrayCat .

Definition — The category $\text{GrayCat}[\mathcal{W}^{-1}]$ is the localization of GrayCat at the weak equivalences, which is determined by the following universal property [GZ67]:

☒ If $F : \text{GrayCat} \rightarrow \mathcal{D}$ maps weak equivalences to isomorphisms, F uniquely factors through $\text{GrayCat}[\mathcal{W}^{-1}]$, i.e., there exists a unique $\Phi : \text{GrayCat}[\mathcal{W}^{-1}] \rightarrow \mathcal{D}$ s.t.

$$\begin{array}{ccc} \text{GrayCat} & \xrightarrow{F} & \mathcal{D} \\ \pi \downarrow & \nearrow \exists! \Phi & \\ \text{GrayCat}[\mathcal{W}^{-1}] & & \end{array}$$

Remark. In [GZ67], an explicit construction is provided in which an object of $\text{GrayCat}[\mathcal{W}^{-1}]$ is simply a **Gray**-category and a morphism from $\mathcal{A} \rightarrow \mathcal{B}$ in $\text{GrayCat}[\mathcal{W}^{-1}]$ is a “zigzag”, i.e. a finite chain of **Gray**-categories and **Gray**-functors

$$\mathcal{A} \xleftarrow{w_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{w_2} X_3 \xrightarrow{f_2} \dots \xleftarrow{w_n} X_{2n-1} \xrightarrow{f_n} \mathcal{B}$$

where the $\leftarrow$ and $\rightarrow$ alternate and each morphism w_i “in the wrong direction” is a weak equivalence.

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(\mathbf{3Cat})$

$\text{ho}(\mathbf{3Cat})$

The notation $\text{ho}(\mathbf{3Cat})$ is short for the homotopy (1-)category of 3-categories.

Definition — We denote by $\text{ho}(\mathbf{3Cat})$ the 1-category whose

- ☒ objects are **Gray**-categories;
- ☒ morphisms are natural equivalence classes of 3-functors.

☒ Since each 3-category is equivalent to a **Gray**-category [Gur13, §10.4], $\text{ho}(\mathbf{3Cat})$ is equivalent to the 1-category of 3-categories and natural equivalence classes of 3-functors.

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(\mathbf{3Cat})$

Importance

Definition — A homotopy 3-type is a topological space X whose homotopy groups $\pi_n(X)$ vanish for $n > 3$.

☒ Given a homotopy 3-type X , there is a corresponding 3-category $\Pi_3(X)$ of points, paths, homotopy, and higher homotopies.

☒ A 3-category whose 1-morphisms, 2-morphisms, and 3-morphisms are invertible is known as a 3-groupoid.

Corollary (Homotopy hypothesis for 3-types) — $\text{ho}(\mathbf{3Type}) \cong \text{ho}(\mathbf{3Grp})$

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(\mathbf{3Cat})$

$\text{GrayCat}[\mathcal{W}^{-1}] \rightarrow \mathbf{ho}(\mathbf{3Cat})$

We will use the *universal property* of localizations.

Definition —

☒ Note there is a functor mapping weak equivalences to isomorphisms:

$$\text{GrayCat} \xrightarrow{Q} \text{ho}(\mathbf{3Cat})$$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad} & \mathcal{C} \\ F \downarrow & & \downarrow [F] \\ \mathcal{D} & \xrightarrow{\quad} & \mathcal{D} \end{array}$$

☒ The universal property of $\text{GrayCat}[\mathcal{W}^{-1}]$ induces a unique functor $\Phi: \text{GrayCat}[\mathcal{W}^{-1}] \rightarrow \text{ho}(\mathbf{3Cat})$:

$$\begin{array}{ccc} \text{GrayCat} & \xrightarrow{Q} & \text{ho}(\mathbf{3Cat}) \\ \pi \downarrow & \nearrow \exists! \Phi & \\ \text{GrayCat}[\mathcal{W}^{-1}] & & \end{array}$$

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(\mathbf{3Cat})$

$\text{GrayCat}[\mathcal{W}^{-1}] \leftarrow \mathbf{ho}(\mathbf{3Cat})$

We will use the tool of *cofibrant replacement* for **Gray**-categories.

Proposition (Cofibrant replacement in GrayCat) —

- ☒ For every $\mathcal{D} \in \text{GrayCat}$, there is a cofibrant $\widehat{\mathcal{D}} \in \text{GrayCat}$ together with an “evaluation” **Gray**-functor $\text{ev}_{\mathcal{D}}: \widehat{\mathcal{D}} \rightarrow \mathcal{D}$ which is a trivial fibration.
- ☒ For every $\mathcal{A}, \mathcal{B} \in \text{GrayCat}$ and 3-functor $F: \mathcal{A} \rightarrow \mathcal{B}$, there is a **Gray**-functor $\widehat{F}: \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{B}}$.

This construction satisfies the following properties:

- (1) For every 3-functor $F: \mathcal{A} \rightarrow \mathcal{B}$, the diagram

$$\begin{array}{ccc} \widehat{\mathcal{A}} & \xrightarrow{\widehat{F}} & \widehat{\mathcal{B}} \\ \text{ev}_{\mathcal{A}} \downarrow & & \downarrow \text{ev}_{\mathcal{B}} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$
 weakly commutes.
- (2) When $F: \mathcal{A} \rightarrow \mathcal{B}$ is a **Gray**-functor, the diagram in (1) strictly commutes.
- (3) $\widehat{\text{id}_{\mathcal{A}}} = \text{id}_{\widehat{\mathcal{A}}}$.
- (4) If $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{D}$ are 3-functors, then $\widehat{G \circ F} \cong \widehat{G} \circ \widehat{F}$.
- (5) In (4), if either F or G is a **Gray**-functor, then $\widehat{G \circ F} = \widehat{G} \circ \widehat{F}$.

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(\mathbf{3Cat})$

$\text{GrayCat}[\mathcal{W}^{-1}] \leftarrow \mathbf{ho}(\mathbf{3Cat})$

We will use the tool of *cofibrant replacement* for **Gray**-categories.

Proposition (Cofibrant replacement in **GrayCat**) —

- ☒ For every $\mathcal{D} \in \text{GrayCat}$, there is a cofibrant $\widehat{\mathcal{D}} \in \text{GrayCat}$ together with an “evaluation” **Gray**-functor $\text{ev}_{\mathcal{D}}: \widehat{\mathcal{D}} \rightarrow \mathcal{D}$ which is a trivial fibration.
- ☒ For every $\mathcal{A}, \mathcal{B} \in \text{GrayCat}$ and 3-functor $F: \mathcal{A} \rightarrow \mathcal{B}$, there is a **Gray**-functor $\widehat{F}: \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{B}}$.

This construction satisfies the following properties:

- (1) For every 3-functor $F: \mathcal{A} \rightarrow \mathcal{B}$, the diagram
- $$\begin{array}{ccc} \widehat{\mathcal{A}} & \xrightarrow{\widehat{F}} & \widehat{\mathcal{B}} \\ \text{ev}_{\mathcal{A}} \downarrow & & \downarrow \text{ev}_{\mathcal{B}} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array} \text{ weakly commutes.}$$

Definition — Define $\Psi: \text{ho}(\mathbf{3Cat}) \rightarrow \text{GrayCat}[\mathcal{W}^{-1}]$ to be the identity on objects and, for a 3-functor $F: \mathcal{A} \rightarrow \mathcal{B}$, define

$$\Psi([F]) := \mathcal{A} \xleftarrow{\text{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{F}} \widehat{\mathcal{B}} \xrightarrow{\text{ev}_{\mathcal{B}}} \mathcal{B}.$$

Theorem ([Fer22]) — $\text{GrayCat}[\mathcal{W}^{-1}] \cong \text{ho}(\mathbf{3Cat})$

Additional comments

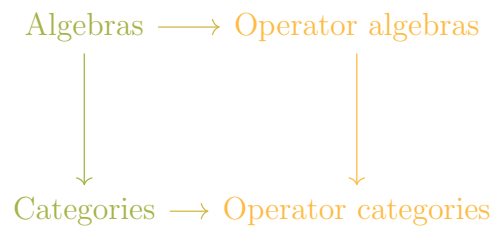
☒ We use Properties $(\widehat{1})$ – $(\widehat{5})$ together with another tool—the existence of *path objects*—to show $\Psi : \text{ho}(\mathbf{3Cat}) \rightarrow \text{GrayCat}[\mathcal{W}^{-1}]$ is well-defined.

☒ We only need Properties $(\widehat{1})$ – $(\widehat{5})$ to show Ψ is indeed a functor;

☒ We then use Properties $(\widehat{1})$ – $(\widehat{5})$ together with the universal property of localizations are used to show Φ and Ψ are inverses.

1 Preliminaries (Reloaded)

Operator algebras



Definition — An algebra consists of:

- vector space A ;
- multiplication $m: A \otimes A \rightarrow A$;
- unit $i: \mathbb{C} \rightarrow A$.

We require this data to satisfy associativity and left/right unitality conditions.

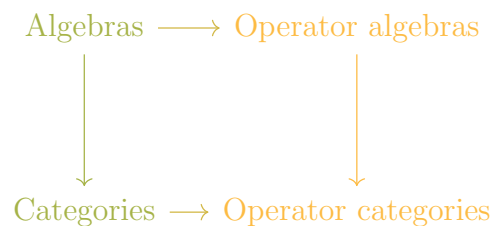
Definition — A module M_A for an algebra A consists of:

- vector space M ;
- action $a: M \otimes A \rightarrow M$;

We require this data to satisfy composability and unitality conditions.

- Every algebra gives rise to a category—namely, its category of modules $\mathbf{Mod}(A)$.

Operator algebras



Definition — A C*-algebra consists of:

- ☒ algebra A ;
- ☒ involution $\dagger: \overline{A}^{\text{op}} \rightarrow A$;
- ☒ norm $\|\cdot\|: A \rightarrow [0, \infty)$.

We require this data to satisfy the C*-axiom $\|a\| = \|a^\dagger a\|^{1/2}$ for $a \in A$.

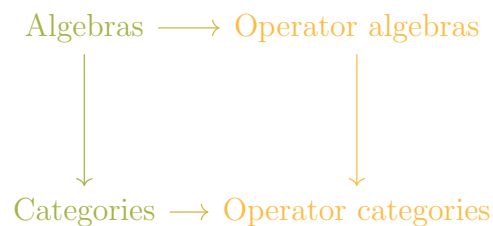
Definition — A Hilbert C*-module M_A for A consists of:

- ☒ module M_A ;
- ☒ A -valued inner product $\langle \cdot | \cdot \rangle_A$:

We require this data to be complete with respect to the norm $\|m\|_M := \|\langle m | m \rangle_A\|^{1/2}$ for $m \in M$.

☒ Every C*-algebra gives rise to a C*-category—namely, its C*-category of Hilbert C*-modules $\text{Mod}^\dagger(A)$.

Operator categories



Definition — A C*-category consists of:

- ☒ category $\mathcal{A}$;
- ☒ involution $\dagger: \overline{\mathcal{A}(A \rightarrow B)} \rightarrow \mathcal{A}(B \rightarrow A)$;
- ☒ norm $\|\cdot\|: \mathcal{A}(A \rightarrow B) \rightarrow [0, \infty)$.

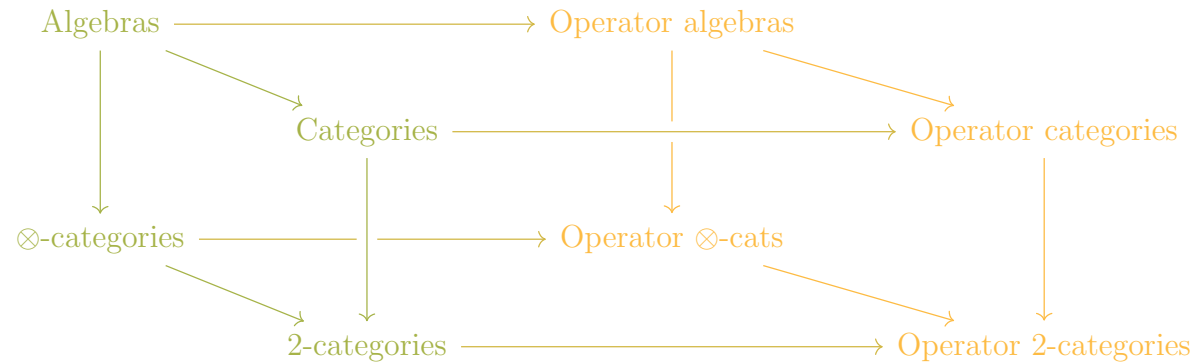
We require this data to satisfy the C*-axiom $\|f\| = \|f^\dagger \circ f\|^{1/2}$ for $f \in \mathcal{A}(A \rightarrow B)$ and a technical spectral condition.

Example — $\text{Mod}^\dagger(A)$ is a C*-category. In particular, $\text{Mod}^\dagger(\mathbb{C}) = \text{Hilb}$ is a C*-category.

☒ In this way, one “operatorifies” the theory of 1-categories:



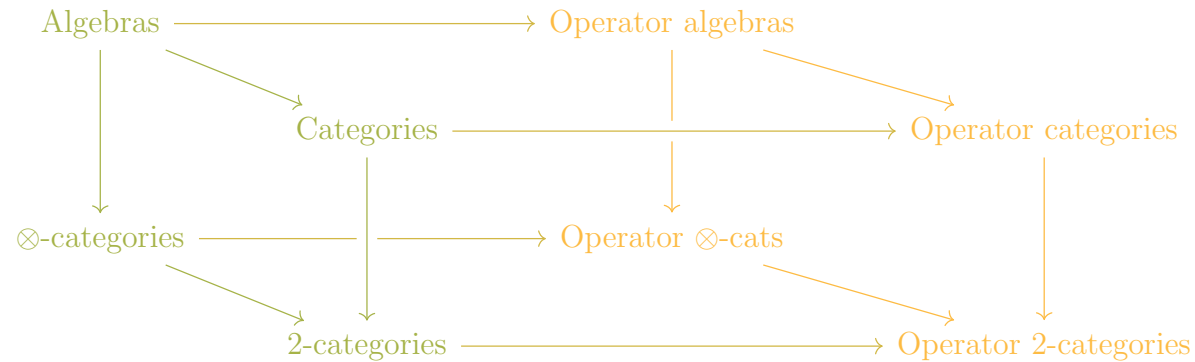
Operator 2-categories



Definition — A 2-category $\mathcal{A}$ consists of the following data:

- ☒ Collection of *objects* of $\mathcal{A}$;
- ☒ 1-categories $\mathcal{A}(A \rightarrow B)$;
- ☒ 1-functors $\odot: \mathcal{A}(A \rightarrow B) \times \mathcal{A}(B \rightarrow C) \rightarrow \mathcal{A}(A \rightarrow C)$ called *1-composition*;
- ☒ 1-functors $I_A: * \rightarrow \mathcal{A}(A \rightarrow A)$ which picks out an identity 1-morphism id_A ;
- ☒ “*associator*” natural isomorphisms α ;
- ☒ *left and right “unitor”* natural isomorphisms λ and ρ .

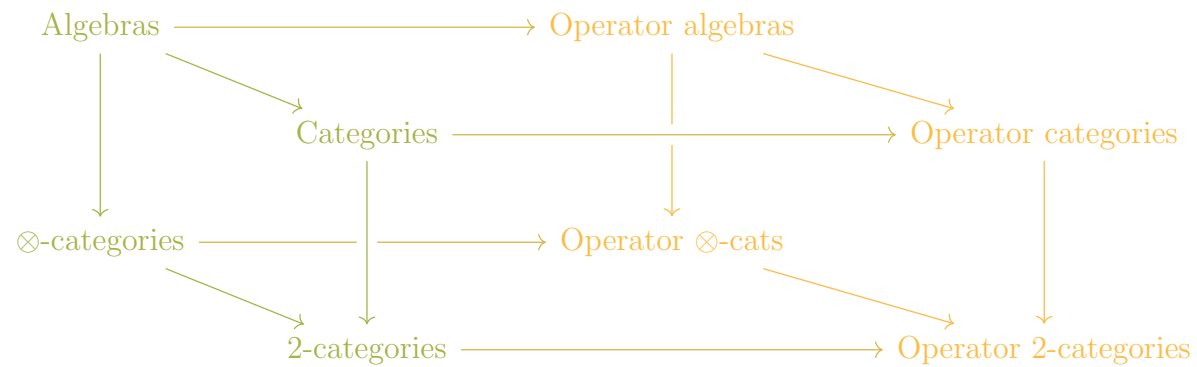
Operator 2-categories



Definition — A C^* -2-category $\mathcal{A}$ consists of the following data:

- ☒ Collection of *objects* of $\mathcal{A}$;
- ☒ C^* -categories $\mathcal{A}(A \rightarrow B)$;
- ☒ $\dagger$ -functors $\odot: \mathcal{A}(A \rightarrow B) \times \mathcal{A}(B \rightarrow C) \rightarrow \mathcal{A}(A \rightarrow C)$ called *1-composition*;
- ☒ $\dagger$ -functors $I_A: * \rightarrow \mathcal{A}(A \rightarrow A)$ which picks out an identity 1-morphism id_A ;
- ☒ **unitary** “*associator*” natural isomorphisms α ;
- ☒ **unitary** *left and right “unitor”* natural isomorphisms λ and ρ .

Operator 2-categories



☒ In this way, one “operatorifies” the theory of 2-categories:

$$\begin{array}{lcl}
 \text{2-category} & \rightsquigarrow & \text{C}^*\text{-2-category} \\
 \text{2-functor} & \rightsquigarrow & \dagger\text{-2-functor} \\
 \text{2-natural transformation} & \rightsquigarrow & \dagger\text{-2-natural transformation} \\
 \text{modification} & \rightsquigarrow & \text{uniformly bounded modification}
 \end{array}$$

3 Foundations for operator 3-categories

Theorem ([Fer24]) — Every C^* -3-category is equivalent to a $C^*\mathbf{Gray}$ -category.

Acknowledgments

✂ This part is based on [Fer24].

✂ The author would like to thank David Penneys and Corey Jones for many useful conversations.

✂ This work was supported by NSF DMS grants 1654159 and 2154389.

Theorem ([Fer24]) — Every C^* -3-category is equivalent to a $C^*\text{Gray}$ -category.

C^* -3-categories

Definition — A 3-category $\mathcal{A}$ consists of the following data:

- ☒ Collection of *objects* of $\mathcal{A}$;
- ☒ 2-categories $\mathcal{A}(A \rightarrow B)$
- ☒ 2-functors $\odot: \mathcal{A}(A \rightarrow B) \times \mathcal{A}(B \rightarrow C) \rightarrow \mathcal{A}(A \rightarrow C)$ called *1-composition*.
- ☒ 2-functors $I_A: * \rightarrow \mathcal{A}(A \rightarrow A)$.
- ☒ “*associator*” adjoint equivalences $\mathbf{a} = (a, a^*, \epsilon^a, \eta^a)$ where a is a 2-natural transformation.
- ☒ *left and right “unitor”* adjoint equivalences $\mathbf{l}$ and $\mathbf{r}$ where l and r are 2-natural transformations.
- ☒ “*pentagonator*” modifications.
- ☒ “*middle, left, and right “triangulator”*” modifications.

Theorem ([Fer24]) — Every C^* -3-category is equivalent to a $C^*\text{Gray}$ -category.

C^* -3-categories

Definition — A C^* -3-category $\mathcal{A}$ consists of the following data:

- ☒ Collection of *objects* of $\mathcal{A}$;
- ☒ C^* -2-categories $\mathcal{A}(A \rightarrow B)$
- ☒ $\dagger$ -2-functors $\odot: \mathcal{A}(A \rightarrow B) \times \mathcal{A}(B \rightarrow C) \rightarrow \mathcal{A}(A \rightarrow C)$ called *1-composition*.
- ☒ $\dagger$ -2-functors $I_A: * \rightarrow \mathcal{A}(A \rightarrow A)$.
- ☒ **unitary** “*associator*” adjoint equivalences $\mathbf{a} = (a, a^*, \epsilon^a, \eta^a)$ where a is a $\dagger$ -2-natural transformation.
- ☒ **unitary** *left and right “unitor”* adjoint $\dagger$ -equivalences $\mathbf{l}$ and $\mathbf{r}$ where l and r are $\dagger$ -2-natural transformations.
- ☒ **unitary** “*pentagonator*” modifications.
- ☒ **unitary** “*middle, left, and right “triangulator”*” modifications.

Theorem ([Fer24]) — Every C^* -3-category is equivalent to a $C^*\text{Gray}$ -category.

$C^*\text{Gray}$ -categories

▣ We denote the 1-category of strict C^* -2-categories and strict $\dagger$ -2-functors by $C^*2\text{Cat}_{\text{st}}$.

▣ For strict C^* -2-categories $\mathcal{A}$ and $\mathcal{B}$, we construct $\mathcal{A} \boxtimes_{\max} \mathcal{B}$ an operator algebraic version of the Gray tensor product.

▣ $\mathcal{A} \boxtimes_{\max} \mathcal{B}$ represents cubical $\dagger$ -2-functors;

▣ $\mathcal{A} \boxtimes_{\max} \mathcal{B}$ also categorifies the maximal tensor product $\otimes_{\max}$ of C^* -algebras.

Proposition — $C^*\text{Gray} := (C^*2\text{Cat}_{\text{st}}, \boxtimes_{\max})$ forms a closed symmetric monoidal category.

Definition — A $C^*\text{Gray}$ -category is a category enriched in $C^*\text{Gray}$.

Theorem ([Fer24]) — Every C^* -3-category is equivalent to a C^* Gray-category.

Locally strictifying

Proposition (Cofibrant replacement in C^*2Cat) —

☀ For every C^* -2-category $\mathcal{C}$, there exists a strict C^* -2-category $\widehat{\mathcal{C}}$ together with an epic 2-equivalence $ev_{\mathcal{C}}: \widehat{\mathcal{C}} \rightarrow \mathcal{C}$.

☀ For each $\dagger$ -2-functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between C^* -2-categories, there exists a strict $\dagger$ -2-functor $\widehat{F}: \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$.

This construction satisfies the following properties:

(1) For every 2-functor $F: \mathcal{C} \rightarrow \mathcal{D}$, there is a unitary natural equivalence

$$\begin{array}{ccc} \widehat{\mathcal{C}} & \xrightarrow{\widehat{F}} & \widehat{\mathcal{D}} \\ ev_{\mathcal{C}} \downarrow & \swarrow & \downarrow ev_{\mathcal{D}} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

($\widehat{\times}$) There exists a cubical $\dagger$ -2-functor $\widehat{\mathcal{C}} \times \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C} \times \mathcal{D}}$ compatible with the ev maps.

Proposition (Transport of structure) — Given a C^* -3-category $\mathcal{A}$, there is an equivalent *cubical* C^* -3-category $\mathcal{A}_{\boxtimes}$ given by:

☀ Replacing each C^* -2-category $\mathcal{A}(A \rightarrow B)$ with the strict C^* -2-category $\widehat{\mathcal{A}(A \rightarrow B)}$;

☀ Replacing $\dagger$ -2-functor $\odot: \mathcal{A}(A \rightarrow B) \times \mathcal{A}(B \rightarrow C) \rightarrow \mathcal{A}(A \rightarrow C)$ with the cubical $\dagger$ -2-functor

$$\widehat{\mathcal{A}(A \rightarrow B)} \times \widehat{\mathcal{A}(B \rightarrow C)} \rightarrow \widehat{\mathcal{A}(A \rightarrow B)} \times \widehat{\mathcal{A}(B \rightarrow C)} \xrightarrow{\widehat{\odot}} \widehat{\mathcal{A}(A \rightarrow C)}$$

Theorem ([Fer24]) — Every C^* -3-category is equivalent to a $C^*\mathbf{Gray}$ -category.

Yoneda Embedding Theorem

Lemma — Given a C^* -3-category $\mathcal{B}$ and a $C^*\mathbf{Gray}$ -category $\mathcal{C}$, there is a $C^*\mathbf{Gray}$ -category $\mathbf{Fun}(\mathcal{B} \rightarrow \mathcal{C})$ of $\dagger$ -3-functors, $\dagger$ -3-natural transformations, $\dagger$ -3-modifications, and uniformly bounded natural transformations.

Theorem — The Yoneda embedding for a cubical C^* -3-category $\mathcal{B}$, given on objects by

$$B \mapsto \mathcal{B}(- \rightarrow B),$$

assembles into a fully-faithful $\dagger$ -3-functor

$$\mathcal{Y} : \mathcal{B} \rightarrow \mathbf{Fun}(\mathcal{B}^{\mathrm{1op}} \rightarrow C^*\mathbf{2Cat}_{\mathrm{strict}}).$$

Conclusion

Theorem ([Fer24]) — Every C*-3-category is equivalent to a C*Gray-category.

Proof. We see $\mathcal{A}$ is equivalent to a sub C*-3-category of a C*Gray-category:

$$\mathcal{A} \xrightarrow{\sim} \mathcal{A}_{\boxtimes} \xhookrightarrow{\mathfrak{J}} \mathrm{Fun}(\mathcal{A}_{\boxtimes}^{\mathrm{1op}} \rightarrow \mathrm{C}^*\mathbf{2Cat}_{\mathrm{st}})$$

□

Example: $\mathbf{AbC}^*\mathbf{Alg}$

Example — We construct a C^* -3-category $\mathbf{AbC}^*\mathbf{Alg}$ whose:

- ☒ Objects are abelian C^* -algebras $C(T)$;
- ☒ 1-morphisms ${}_{\phi}A_{\psi}: C(T) \rightarrow C(S)$ are C^* -algebras A with $*$ -homomorphisms $\phi: C(T) \rightarrow Z(A)$ and $\psi: C(S) \rightarrow Z(A)$;
- ☒ 2-morphisms ${}_AM_B: {}_{\phi}A_{\psi} \Rightarrow {}_{\mu}B_{\nu}$ are A - B C^* -correspondences, i.e. a Hilbert C^* -modules M_B over B with a left A -action $A \otimes M \rightarrow M$ by adjointable B -module maps, compatible with the central maps from $C(T)$ and $C(S)$.
- ☒ 3-morphisms $x: {}_AM_B \rightarrow {}_AN_B$ are adjointable A - B -bimodule maps, and define $x^{\dagger}$ to be its adjoint.

We now describe the compositions in $\mathbf{AbC}^*\mathbf{Alg}$:

- ☒ 1-composition is given by the relative maximal tensor product ${}_{\phi}A_{\psi} \otimes_{C(T)} {}_{\mu}B_{\nu}$;
- ☒ 2-composition is given by the relative tensor product of C^* -correspondences ${}_AM \otimes_B N_C$;
- ☒ 3-composition is given by function composition $x \circ y$.

Finally, we describe the units in $\mathbf{AbC}^*\mathbf{Alg}$:

- ☒ For objects, $\mathrm{id}_{C(T)} = \mathrm{id}_{C(T)} \mathrm{id}$;
- ☒ For 1-morphisms, $\mathrm{id}_{{}_{\phi}A_{\psi}} = {}_AA_A$. In particular $\mathrm{id}_{\mathrm{id}_{C(T)}} = {}_{C(T)}C(T)_{C(T)}$.
- ☒ For 2-morphisms, $\mathrm{id}_{{}_AM_B} = \mathrm{id}_M$. In particular, $\mathrm{id}_{\mathrm{id}_{\mathrm{id}_{C(T)}}} = \mathrm{id}_{C(T)}$.

4 The homotopy 3-type of abelian C^* -algebras

Theorem ([FF26]) — The homotopy groups of $\mathbf{AbC}^*\mathbf{Alg}$ are:

$$\begin{aligned}\pi_0(\mathbf{AbC}^*\mathbf{Alg}) &\cong \mathbf{CHaus}/\mathbf{Homeo}, \\ \pi_1(\mathbf{AbC}^*\mathbf{Alg}, C(T)) &\cong H_{\mathrm{tor}}^3(T; \mathbb{Z}) \rtimes \mathbf{Homeo}(T), \\ \pi_2(\mathbf{AbC}^*\mathbf{Alg}, C(T)) &\cong \mathbf{Pic}(T), \\ \pi_3(\mathbf{AbC}^*\mathbf{Alg}, C(T)) &\cong \mathbf{Hom}(T; \mathbb{C}^\times).\end{aligned}$$

Acknowledgements

☒ This part is based on joint work with Gregory Faurot [FF26].

☒ The authors would like to thank Corey Jones for initially suggesting this problem.

☒ We also thank David Penneys and Nick Gurski for numerous helpful conversations.

☒ The authors were partially supported by NSF grant DMS-2154389.

Postnikov data for homotopy 3-types

The data of a homotopy 3-type X is classified by:

- ☒ the set $\pi_0(X)$ of connected components; and for each $p \in \pi_0(X)$,
- ☒ the fundamental groups $\pi_1(X; p)$, $\pi_2(X; p)$, $\pi_3(X; p)$ where the latter two are abelian;
- ☒ the actions $\rho_i: \pi_1(X; p) \otimes_{\mathbb{Z}} \pi_i(X; p) \rightarrow \pi_i(X; p)$ on $\pi_2(X; p)$ and $\pi_3(X; p)$;
- ☒ the Whitehead product $[-, -]: \pi_2(X; p) \otimes_{\mathbb{Z}} \pi_2(X; p) \rightarrow \pi_3(X; p)$;
- ☒ the k -invariants $k_3 \in H^3(\pi_1(X; p); \pi_2(X; p))$ and $k_4 \in H^4(X_2; \pi_3(X; p))$ where X_2 is the homotopy 2-type associated to X .

As homotopy 3-types are in correspondence with 3-groupoids by the homotopy hypothesis, we may compare their data as follows.

Homotopy 3-type X	3-groupoid $\mathcal{A}$
Points $p \in X$	Objects $A \in \mathcal{A}$
$\pi_0(X)$	set of equivalence classes of objects
$\pi_1(X; p)$ Multiplication	$\mathcal{A}(A \rightarrow A)/\text{equivalence}$ $\odot$ for 1-morphisms
$\pi_2(X; p)$ Multiplication	$\mathcal{A}(\text{id}_A \Rightarrow \text{id}_A)/\text{isomorphism}$ $\odot$ for 2-morphisms
$\pi_3(X; p)$ Multiplication	$\mathcal{A}(\text{id}_{\text{id}_A} \Rrightarrow \text{id}_{\text{id}_A})$ $\circ$ for 3-morphisms
ρ_2	$\odot$ for 2-morphisms
ρ_3	$\odot$ for 3-morphisms
$[-, -]$	interchanger
k_3	associator
k_4	pentagonator

$\pi_0(\mathbf{AbC}^*\mathbf{Alg})$

☒ For a 1-isomorphism ${}_{\varphi}A_{\psi}$ from $C(T)$ to $C(S)$, we prove the following result.

Lemma — $\phi: C(T) \rightarrow Z(A)$ and $\psi: C(S) \rightarrow Z(A)$ are *-isomorphisms.

☒ In particular $T \cong S$, which immediately implies the following result.

Theorem — The collection of equivalence classes of objects in $\mathbf{AbC}^*\mathbf{Alg}$ is given by

$$\pi_0(\mathbf{AbC}^*\mathbf{Alg}) \cong \pi_0(\mathbf{CHaus}) = \{\text{compact Hausdorff spaces}\}/\text{homeomorphism}.$$

$$\pi_1(\mathbf{AbC}^*\mathbf{Alg}; C(T))$$

▣ For a 1-automorphism ${}_{\phi}A_{\psi}$ of $C(T)$, we prove the following results.

Lemma — $\phi: C(T) \rightarrow Z(A)$ and $\psi: C(T) \rightarrow Z(A)$ are $*$ -isomorphisms.

Lemma — ${}_{\phi}A_{\psi} \cong {}_{\phi}Z(A)_{\psi} \otimes_T {}_{\psi}A_{\psi} \cong {}_{\psi^{-1} \circ \phi}C(T)_{\text{id}} \otimes_T {}_{\psi}A_{\psi}$.

▣ When $\phi = \psi$, we further prove the following.

Lemma — A is a continuous-trace C^* -algebra over T .

Lemma — The Dixmier–Douady class $\delta(A)$ lies in $H_{\text{tor}}^3(T; \mathbb{Z})$. Conversely, given any $\delta \in H_{\text{tor}}^3(T; \mathbb{Z})$, there is a 1-automorphism ${}_{\psi}B_{\psi}$ of $C(T)$ with $\delta(B) = \delta$.

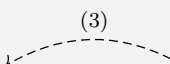
▣ We will identify ${}_{\phi}A_{\psi} \cong {}_{\psi^{-1} \circ \phi}C(T)_{\text{id}} \otimes {}_{\psi}A_{\psi}$ with the pair $\psi^{-1} \circ \phi \in \text{Homeo}(T)$ and $\delta(A) \in H_{\text{tor}}^3(T; \mathbb{Z})$.

$$\pi_1(\mathbf{AbC}^*\mathbf{Alg}; C(T))$$

▣ We will identify ${}_{\phi}A_{\psi} \cong {}_{\psi^{-1} \circ \phi}C(T)_{\mathrm{id}} \otimes {}_{\psi}A_{\psi}$ with the pair $\psi^{-1} \circ \phi \in \mathrm{Homeo}(T)$ and $\delta(A) \in H_{\mathrm{tor}}^3(T; \mathbb{Z})$.

Theorem ([FF26]) — There is a split short exact sequence

$$0 \longrightarrow H_{\mathrm{tor}}^3(T; \mathbb{Z}) \xrightarrow{(1)} \pi_1(\mathbf{AbC}^*\mathbf{Alg}, T) \xrightarrow{(2)} \mathrm{Homeo}(T) \longrightarrow 0.$$



▣ In particular, $\pi_1(\mathbf{AbC}^*\mathbf{Alg}, T) = H_{\mathrm{tor}}^3(T; \mathbb{Z}) \rtimes \mathrm{Homeo}(T)$.

$$\pi_2(\mathbf{AbC}^*\mathbf{Alg}; C(T))$$

Theorem — (Serre-Swan) There is a unitary monoidal equivalence

$$\mathbf{Hilb}(T) \cong \mathbf{Mod}_{\mathbf{fgp}}^\dagger(C(T))$$

between the C^* - $\otimes$ -categories of Hermitian bundles over T and finitely generated projective Hilbert C^* -modules over $C(T)$.

☒ This equivalence provides a correspondence between Hermitian *line* bundles over T and $\otimes$ -invertible modules.

Definition — For $T \in \mathbf{CHaus}$, its Picard group $\mathbf{Pic}(T)$ is given by isomorphism classes of complex line bundles over T .

Theorem ([FF26]) — $\pi_2(\mathbf{AbC}^*\mathbf{Alg}; C(T)) \cong \mathbf{Pic}(T)$.

☒ If T has the homotopy type of a CW-complex, $\pi_2(\mathbf{AbC}^*\mathbf{Alg}; C(T)) \cong H^2(T; \mathbb{Z})$.

$$\pi_3(\mathbf{AbC}^*\mathbf{Alg}; C(T))$$

Theorem ([FF26]) — $\pi_3(\mathbf{AbC}^*\mathbf{Alg}, C(T)) \cong \mathrm{Hom}(T \rightarrow \mathbb{C}^\times).$

Proof. Recall that $\mathrm{id}_{\mathrm{id}_T} = C(T)$ as a Hilbert C^* -bimodule over itself. Since $C(T)$ is abelian and unital,

$$\mathrm{End}_{(C(T)C(T)C(T))} \cong Z(C(T)) = C(T).$$

Hence, the invertible maps are given by $\pi_3(\mathbf{AbC}^*\mathbf{Alg}, C(T)) \cong C(T)^\times = \mathrm{Hom}(T; \mathbb{C}^\times).$ □

☒ If we restrict ourselves to consider only the *unitary* 3-morphisms in $\mathbf{AbC}^*\mathbf{Alg}$, there is an exact sequence

$$0 \rightarrow \mathrm{Hom}(T; \mathbb{Z}) \rightarrow \mathrm{Hom}(T; \mathbb{R}) \rightarrow \pi_3(\mathbf{AbC}^*\mathbf{Alg}; C(T)) \rightarrow H^1(T; \mathbb{Z}) \rightarrow 0$$

Conclusion

Theorem ([FF26]) — The homotopy groups of $\mathbf{AbC}^*\mathbf{Alg}$ are:

$$\begin{aligned}\pi_0(\mathbf{AbC}^*\mathbf{Alg}) &\cong \mathbf{CHaus}/\mathbf{Homeo}, \\ \pi_1(\mathbf{AbC}^*\mathbf{Alg}, C(T)) &\cong H_{\mathrm{tor}}^3(T; \mathbb{Z}) \rtimes \mathbf{Homeo}(T), \\ \pi_2(\mathbf{AbC}^*\mathbf{Alg}, C(T)) &\cong \mathbf{Pic}(T), \\ \pi_3(\mathbf{AbC}^*\mathbf{Alg}, C(T)) &\cong \mathbf{Hom}(T; \mathbb{C}^\times).\end{aligned}$$

☒ Fun observation:

$$\begin{aligned}\pi_1(\mathbf{AbC}^*\mathbf{Alg}; C(T)) &\propto H^3(T; \mathbb{Z}) \\ \pi_2(\mathbf{AbC}^*\mathbf{Alg}; C(T)) &\propto H^2(T; \mathbb{Z}) \\ \pi_3(\mathbf{AbC}^*\mathbf{Alg}; C(T)) &\propto H^1(T; \mathbb{Z})\end{aligned}$$

References

- [Fer22] Giovanni Ferrer, *Gray-categories model algebraic tricategories*, Theory Appl. Categ. **38** (2022), Paper No. 29, 1136–1155. MR 4483628
- [Fer24] Giovanni Ferrer, *Foundations for operator algebraic tricategories*, [arXiv:2404.05193](#), 2024.
- [FF26] Giovanni Ferrer and Gregory Faurot, *The homotopy 3-type of abelian C^* -algebras*, [arXiv:2603.29985](#), 2026.
- [GPS95] Robert Gordon, Anthony John Power, and Ross Street, *Coherence for tricategories*, American Mathematical Society, 1995.
- [Gur13] Nick Gurski, *Coherence in Three-Dimensional Category Theory*, Cambridge Tracts in Mathematics, Cambridge University Press, 2013.
- [GZ67] Peter Gabriel and Michel Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete **35** (1967), no. 2, 6–21.
- [Kel05] G.M. Kelly, *Basic concepts of enriched category theory*, Reprints in Theory and Applications of Categories **10** (2005), 1–136.
- [Lac11] Stephen Lack, *A Quillen model structure for Gray-categories*, Journal of K-Theory **8** (2011), no. 2, 183–221.