Hochschild Cohomology and Deformations of Algebras

Giovanni Ferrer

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1 Background

Definition 1.1 (Algebra) — An (associative, unital) algebra over a commutative ring R is a R-module A together with an R-bilinear multiplication $\mu_A : A \times A \to A$ and a unit $\eta_A : R \to A$ such that the following diagrams commute:

$$\begin{array}{ccc} A^3 & \xrightarrow{\mu_A \times \mathrm{id}} & A^2 \\ {\scriptstyle \mathrm{id}\times\mu_A}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ A^2 & \xrightarrow[\mu_A]{} & A \end{array} \qquad \begin{array}{ccccc} R \otimes_R A & \xrightarrow{\eta_A \otimes \mathrm{id}_A} & A \otimes_R A & \xleftarrow{\mathrm{id}_A \otimes \eta_A} & A \otimes_R R \\ & \| & \downarrow & \| & \\ & & A & & \end{array}$$

Example 1.2 ($\mathbb{C}$-Algebras) —

- $\mathrm{End}(V)$ for any $\mathbb{C}$-vector space $V \in \mathsf{Vect}$, so in particular, $\mathrm{Mat}_{n\times n}(\mathbb{C})$,
- $\mathbb{C}[x_1, \dots, x_n]$ for $n \in \mathbb{N}$,
- $\mathbb{C}[[\hbar]]$, the algebra of formal power series over $\mathbb{C}$ with variable $\hbar$,
- Given a $\mathbb{C}$-algebra A, the *opposite algebra* A^{op} is given by $\mu_{A^{\mathrm{op}}}(a, b) := \mu_A(b, a)$.
- Given a $\mathbb{C}$-algebra A, the *enveloping algebra* of A is given by $A^e := A \otimes A^{\mathrm{op}}$.
- The group algebra $\mathbb{C}[G]$ of a group G,
- The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a complex Lie algebra $\mathfrak{g}$.

- The organic notion of an algebra homomorphism $\varphi : A \to B$ is then a linear transformation making the following diagrams commute:

$$\begin{array}{ccc} A \times A & \xrightarrow{\mu_A} & A \\ {\scriptstyle \varphi\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ B \times B & \xrightarrow[\mu_B]{} & B \end{array} \qquad\qquad \begin{array}{ccc} & \overset{\eta_A}{\nearrow} & A \\ R & & \downarrow{\scriptstyle \varphi} \\ & \underset{\eta_B}{\searrow} & B \end{array}$$

- The direct sum, product, and tensor of algebras agree with their vector space counterparts, where additionally, multiplication is defined entry-wise, e.g. $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

- What about representations of these structures?

Definition 1.3 (A-module) — A left A-module M over an $\mathbb{C}$-algebra A is a $\mathbb{C}$-vector space M together with a bilinear map $(a, m) \mapsto am$ from $A \times M$ to M such that

$$a(a'm) = (aa')m \quad \text{and} \quad 1m = m.$$

- Similarly, a *right* A-module is a $\mathbb{C}$-vector space M together with a bilinear map $(m, a) \mapsto ma$ from $M \times A$ such that

$$(ma)a' = m(aa') \quad \text{and} \quad m1 = m.$$

- So a right A-module is nothing else than a *left* module over the opposite algebra A^{op}.

Definition 1.4 ($A - A$ bimodule) — An $A - A$ bimodule M is a $\mathbb{C}$-vector space M together with compatible left and right A-module structures, i.e.

$$(am)a' = a(ma').$$

Example 1.5 (Bimodules over Algebras) —

- Any $\mathbb{C}$-algebra A can be viewed as an $A - A$ bimodule, where the left and right actions are given by multiplying on the left and right respectively.
- More generally, any two-sided ideal B of A can also be endowed with an $A - A$ bimodule structure.
- $\prod_{i \in I} A$ and $\bigoplus_{i \in I} A$ are $A - A$ bimodules where the left and right actions are given coordinate-wise, i.e. $a(v_i)_{i \in I} a' := (a v_i a')_{i \in I}$.
- Given a $\mathbb{C}$-vector space V, $A \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} A$ can be equipped with a $A - A$ bimodule structure determined by $a(a' \otimes v \otimes b')b = (aa') \otimes v \otimes (bb')$.

- Similarly, $A - A$ bimodules are nothing more than left A^e-modules.

Let M be an $A - A$ bimodule:

Definition 1.6 (Center) — The center $Z_A(M)$ is defined as:

$$Z_A(M) := \{m \in M \,|\, am = ma \text{ for all } a \in A\}\,.$$

When $M = A$, this corresponds to the center $Z(A)$ of an algebra:

$$Z(A) := Z_A(A) = \{a \in A \,|\, ba = ab \text{ for all } b \in A\}\,.$$

Recall the Leibniz (or product) rule for differentiation:

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g).$$

Definition 1.7 (Derivation) — We say that a $\mathbb{C}$-linear $\delta : A \to M$ is a derivation of A with values in M if

$$\delta(ab) = \delta(a)b + a\delta(b) \qquad \text{for every } a, b \in A.$$

The space of all such derivations is denoted by $\mathrm{Der}_A(M)$. When $M = A$, we say these are derivations on A and write $\mathrm{Der}(A) := \mathrm{Der}_A(A)$.

Example 1.8 (Derivation) — $\frac{d}{dx}$ is a derivation on $\mathbb{C}[x]$.

Exercise 1.1. Show that $\mathrm{Der}(\mathbb{C}[x]) = \mathbb{C}[x]\frac{d}{dx}$.
Hint: Show that any $\delta \in \mathrm{Der}(\mathbb{C}[x])$ is of the form $\delta(x)\frac{d}{dx}$.

2 Hochschild Cohomology

Definition 2.1 (Hochschild Complex) — We define the co-chain complex $C^\bullet(A; M)$ as:

$$C^k(A; M) := \mathrm{Hom}_{\mathbb{C}}(A^{\otimes k}, M) = \left\{ \begin{array}{c} \mathbb{C}\text{-multi-linear maps} \\ A^n \to M. \end{array} \right\}$$

with differentials $\mathsf{d}^k : C^k(A; M) \to C^{k+1}(A; M)$ given on $\xi : A^n \to M$ by:

$$\begin{aligned} \mathsf{d}^k \xi(a_0, \ldots, a_k) := a_0 \cdot \xi(a_1, \ldots, a_k) & \\ + \sum_{i=0}^{k-1} (-1)^{i+1} \xi(\ldots, a_i a_{i+1}, \ldots) & \\ + (-1)^{k+1} \xi(a_0, \ldots, a_{k-1}) \cdot a_k & \end{aligned}$$

Definition 2.2 (Hochschild Cohomology) — The k^{th} Hochschild Cohomology group of A valued in M is given by:

$$\mathsf{HH}^k(A; M) := H^k(C^\bullet(A; M)) = \frac{\mathrm{Ker}(\mathsf{d}^k)}{\mathrm{Im}(\mathsf{d}^{k-1})}.$$

When $M = A$ as a bimodule over itself, we write $\mathsf{HH}^k(A) := \mathsf{HH}^k(A; A)$.

Facts 2.3 (Hochschild Cohomology) —

- $\mathsf{HH}^0(A;M) = Z_A(M)$ as in Definition 1.6.

  Proof of Fact. Notice $\mathsf{HH}^0(A;M) = \ker(\mathsf{d}^0) = \{\xi : \mathbb{C} \to M \mid \mathsf{d}^0\xi = 0\}$ where

  $$\mathsf{d}^0\xi(a_0) = a_0\xi(1) - \xi(1)a_0.$$

  Identifying ξ with $\xi(1) \in M$, the statement is then clear. ■

- $\mathrm{Ker}(\mathsf{d}^1) = Z^1(C^\bullet(A;M)) = \mathrm{Der}_A(M)$ as in Definition 1.7.

  Proof of Fact. For $\xi : A \to M$, notice

  $$\mathsf{d}^1\xi(a_0, a_1) = a_0\xi(a_1) - \xi(a_0a_1) + \xi(a_0)a_1.$$ ■

Example 2.4 (Hochschild Cohomology) —

- $\mathsf{HH}^0(\mathbb{C}[\hbar]) = Z(\mathbb{C}[\hbar]) = \mathbb{C}[\hbar]$.
- $\mathsf{HH}^1(\mathbb{C}[\hbar]) = \ker(\mathsf{d}^1) = \mathrm{Der}(\mathbb{C}[\hbar]) = \mathbb{C}[\hbar]\frac{d}{d\hbar}$ by Exercise 1.1.
- $\mathsf{HH}^k(\mathbb{C}[\hbar]) = 0$ for $k \geq 2$.

3 Deformations of Algebras

Let A be a $\mathbb{C}$-algebra, consider the $\mathbb{C}$-vector space $A[[\hbar]] = A \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$.

$$\begin{array}{c} A[[\hbar]] \\ \uparrow \ \ \downarrow \mathrm{ev}_0 \\ A \end{array}$$

- The aim of deformation theory is to equip $A[[\hbar]]$ with a multiplication compatible with the multiplication on A, enhancing $A[[\hbar]]$ into a $\mathbb{C}[[\hbar]]$-algebra.

Definition 3.1 (Deformation of an Algebra) — By a deformation of A, we mean an associative $\mathbb{C}[[\hbar]]$-bilinear $* : A[[\hbar]]^2 \to A[[\hbar]]$ such that the following diagrams commutes

$$\begin{array}{ccc} A[[\hbar]] \times A[[\hbar]] & \xrightarrow{\ *\ } & A[[\hbar]] \\ \uparrow & & \downarrow \mathrm{ev}_0 \\ A \times A & \longrightarrow & A \end{array}$$

Definition 3.2 (Deformation of an Algebra) — By a deformation of A, we mean an associative $\mathbb{C}[[\hbar]]$-bilinear $* : A[[\hbar]]^2 \to A[[\hbar]]$ such that the following diagrams commutes

$$\begin{array}{ccc} A[[\hbar]] \times A[[\hbar]] & \xrightarrow{\;*\;} & A[[\hbar]] \\ \uparrow & & \downarrow \mathrm{ev}_0 \\ A \times A & \longrightarrow & A \end{array}$$

- Observe that such a multiplication would be uniquely determined by its action on A.
- This action can be decomposed uniquely into a family of $\mathbb{C}$-bilinear maps $\{\mu_n : A \times A \to A\}_{n\in\mathbb{N}}$ such that

$$a * b = \sum_{n\geq 0} \hbar^n \mu_n(a, b).$$

- Notice $\mu_0(a, b) = ab$.
- The associativity of $*$ can then be expressed in terms of the maps $\{\mu_n\}_{n\in\mathbb{N}}$:

$$\sum_{k+\ell=n} \mu_k(\mu_\ell(a, b), c) \;=\; \sum_{i+j=n} \mu_i(a, \mu_j(b, c)) \quad \text{for } a, b, c \in A \text{ and } n \in \mathbb{N}. \tag{3.1}$$

Example 3.3 (Deformations of Algebras) —

- For any algebra A, the *trivial deformation* is given by $\mu_k = 0$ for $k \geq 1$. Notice that $a * b = ab$ for every $a, b \in A$.
- For $A = \mathbb{C}[x, y]$, the deformation $*$ on $A[\hbar]$ (or $A[[\hbar]]$) determined by

$$x^n * x^m = x^{n+m}, \qquad y^n * y^m = y^{n+m} \qquad x^n * y^m = x^n y^m \qquad y * x = xy + \hbar$$

  Notice that $(A, *) = \mathbb{C}\langle x, y, \hbar\rangle/(yx - xy - \hbar)$. When we set $\hbar = 1$, this is known as the *Weyl algebra* and usually y is denoted by $\frac{d}{dx}$ or ∂_x.
- For $A = \mathbb{C}[x, y]/(y^2)$, the deformations $*_1, \ldots, *_5$ on $A[\hbar]$ determined by $x^n *_i x^m = x^{n+m}$, $x^n *_i y = y *_i x^n = x^n y$ and

$$\begin{aligned}
\overline{y} *_1 \overline{y} &= \hbar^2, \\
\overline{y} *_2 \overline{y} &= \overline{x}\hbar, \\
\overline{y} *_3 \overline{y} &= \overline{x}^2\hbar^2, \\
\overline{y} *_4 \overline{y} &= \overline{x}^3\hbar, \\
\overline{y} *_5 \overline{y} &= (\overline{x}^2 + \overline{x}^3)\hbar.
\end{aligned}$$

  Notice that:

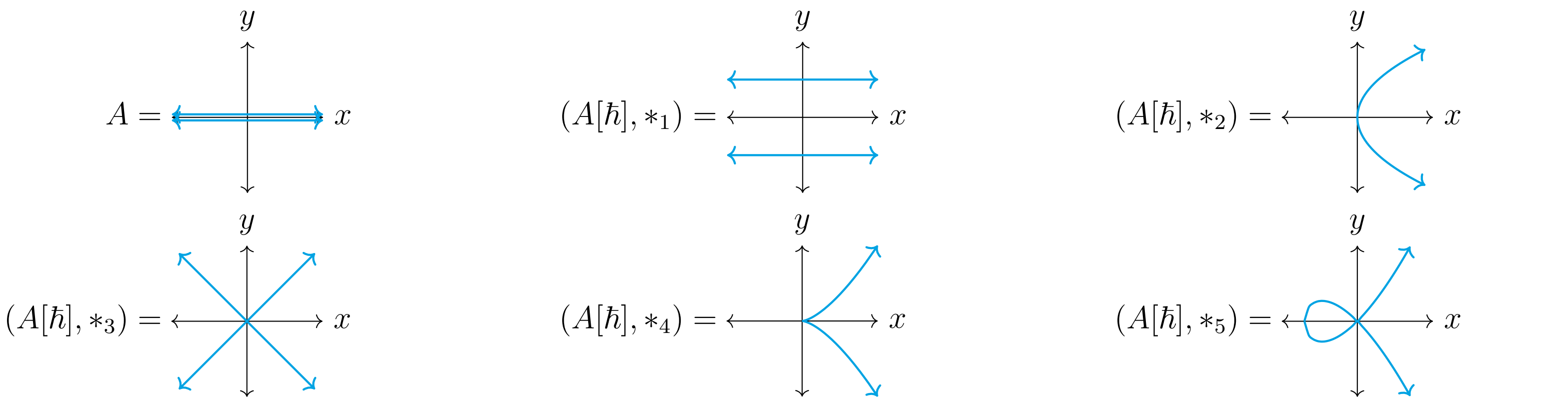
$$\begin{aligned}
(A[\hbar], *_1) &= \mathbb{C}\langle x, y, \hbar\rangle/(y^2 - \hbar^2) \\
(A[\hbar], *_2) &= \mathbb{C}\langle x, y, \hbar\rangle/(y^2 - x\hbar) \\
(A[\hbar], *_3) &= \mathbb{C}\langle x, y, \hbar\rangle/(y^2 - x^2\hbar^2) \\
(A[\hbar], *_4) &= \mathbb{C}\langle x, y, \hbar\rangle/(y^2 - x^3\hbar) \\
(A[\hbar], *_5) &= \mathbb{C}\langle x, y, \hbar\rangle/(y^2 - x^2\hbar - x^3 t)
\end{aligned}$$

Example 3.4 —

$$A = \mathbb{C}[x,y]/(y^2) \qquad (A[\hbar],*_1) = \mathbb{C}\langle x,y,\hbar\rangle/(y^2-\hbar^2) \qquad (A[\hbar],*_2) = \mathbb{C}\langle x,y,\hbar\rangle/(y^2-x\hbar)$$

$$(A[\hbar],*_3) = \mathbb{C}\langle x,y,\hbar\rangle/(y^2-x^2\hbar^2) \qquad (A[\hbar],*_4) = \mathbb{C}\langle x,y,\hbar\rangle/(y^2-x^3\hbar) \qquad (A[\hbar],*_5) = \mathbb{C}\langle x,y,\hbar\rangle/(y^2-x^2\hbar-x^3\hbar)$$

- Recall that $A = \mathbb{C}[x,y]/(y^2)$ can be visualized as a "double line" on the x-axis.
- Then, the deformations of A in Example 3.3 may be visualized for some fixed value of $\hbar$ as follows:



- We may also compare deformations of a particular algebra, the organic notion being an $\mathbb{C}[[\hbar]]$-algebra homomorphism $\varphi : (A[[\hbar]], *) \to (A[[\hbar]], *')$ such that:

$$\begin{array}{ccc} (A[[\hbar]], *) & \xrightarrow{\varphi} & (A[[\hbar]], *') \\ \uparrow & & \downarrow \mathrm{ev}_0 \\ A & = & A \end{array}$$

Definition 3.5 (Equivalence of Deformations) — We say $*$ and $*'$ are equivalent if there exists a bijective $\mathbb{C}[[\hbar]]$-linear $\varphi : A[[\hbar]] \to A[[\hbar]]$ such that

- $\varphi(a) = a \pmod{\hbar}$ for all $a \in A$,
- $\varphi(a * b) = \varphi(a) *' \varphi(b)$ for all $a, b \in A$.

Definition 3.6 (Equivalence of Deformations) — We say $*$ and $*'$ are equivalent if there exists a bijective $\mathbb{C}[[\hbar]]$-linear $\varphi : A[[\hbar]] \to A[[\hbar]]$ such that

- $\varphi(a) = a \pmod{\hbar}$ for all $a \in A$,
- $\varphi(a * b) = \varphi(a) *' \varphi(b)$ for all $a, b \in A$.

- Such a map is uniquely determined by its action on A.
- This action can be decomposed uniquely into a family of $\mathbb{C}$-linear maps $\{\varphi_n : A \to A\}_{n\in\mathbb{N}}$ such that

$$\varphi(a) = \sum_{n\geq 0} \hbar^n \varphi_n(a).$$

- Notice $\varphi_0 = \mathrm{id}_A$.
- The fact that φ preserves the deformations of A can then be expressed in terms of the maps $\{\varphi_n\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\mu'_n\}_{n\in\mathbb{N}}$ as follows:

$$\sum_{k+\ell=n} \varphi_k(\mu_\ell(a,b)) = \sum_{i+j+k=n} \mu'_i(\varphi_j(a), \varphi_k(b)) \quad \text{for } a,b \in A \text{ and } n \in \mathbb{N}. \tag{3.2}$$

Definition 3.7 (Rigidity) — We say that A is rigid if every deformation of A is equivalent to the trivial deformation.

4 Some Results

Proposition 4.1. If $\mathsf{HH}^1(A) = 0$, then any auto-equivalence of the trivial deformation of A is inner.

That is, if $I : A[[\hbar]] \to A[[\hbar]]$ is a $\mathbb{C}[[\hbar]]$-linear algebra isomorphism such that $I = \sum \hbar^n I_n$ on A with $\{I_k : A \to A\}_{k\geq 1}$ and $I_0 = \mathrm{id}_A$, then there exists some invertible $\widetilde{a} \in A[[\hbar]]$ such that $I(x) = \widetilde{a}x\widetilde{a}^{-1}$.

Proof. Let $N \geq 1$ be the smallest index such that $I_N \neq 0$.

- Since $I(ab) = I(a)I(b) = \sum_n \sum_{k+\ell=n} \hbar^n I_k(a)I_\ell(b)$, comparing the coefficients of $\hbar^N$ yields:

$$I_N(ab) = \sum_{k+\ell=N} I_k(a)I_\ell(b) = aI_N(b) + I_N(a)b.$$

- Thus I_N is a derivation, i.e. $\mathsf{d}^1 I_N = 0$ as we saw in Facts 2.3.
- Since $\mathsf{HH}^1(A) = 0$, i.e. $\ker(\mathsf{d}^1) = \mathrm{im}(\mathsf{d}^0)$, there exists some $\xi_N \in C^1(A; A) \cong A \ni \xi_N(1)$ such that $\mathsf{d}^0\xi_N = I_N$.
- In particular, $I_N(x) = x\xi_N(1) - \xi_N(1)x$.
- Note that $1 - \hbar^N\xi_N(1)$ is invertible with

$$\big(1 - \hbar^N\xi_N(1)\big)^{-1} = \sum_{k\geq 0} \hbar^{Nk}\xi_N(1)^k.$$

- So define:

$$\begin{aligned}\widetilde{I}(x) &= \big(1 - \hbar^N\xi_N(1)\big)^{-1} I(x)\big(1 - \hbar^N\xi_N(1)\big)\\ &= \big(1 + \hbar^N\xi_N(1) + O(\hbar^{2N})\big)\big(x + \hbar^N I_N(x) + O(\hbar^{N+1})\big)\big(1 - \hbar^N\xi_N(1)\big)\\ &= x + \hbar^N\big(I_N(x) - x\xi_N(1) + \xi_N(1)x\big) + O(\hbar^{N+1})\\ &= x + O(\hbar^{N+1}).\end{aligned}$$

The result then follows by induction on N and noting $\left(\prod_N 1 - \hbar^N\xi_N(1)\right)^{-1} I(x)\left(\prod_N 1 - \hbar^N\xi_N(1)\right) = x$. $\square$

Proposition 4.2. If $\mathsf{HH}^2(A) = 0$, then A is rigid.

Proof. Let $*$ be a deformation of A. We will proceed by constructing a family of deformations $*_1, *_2, *_3, \ldots$ equivalent to $*$ such that $a *_k b = ab + O(\hbar^k)$, i.e. for $\{\mu_n^{(k)} : A^2 \to A\}_{n\in\mathbb{N}}$ corresponding to $*_k$, we have $\mu_i^{(k)} = 0$ whenever $1 < i < k$.

- Notice that $*_1 := *$ already satisfies this property.
- Now suppose we have constructed $*_N$, where $*_N$ is equivalent to $*$ and

$$a *_N b = ab + O(\hbar^N) = ab + \hbar^N \mu_N^{(N)}(a, b) + O(\hbar^{N+1}).$$

- Then the associativity equation for $n = N$ yields

$$\begin{aligned} \mu_N^{(N)}(a, b)c + \mu_N^{(N)}(ab, c) &= \sum_{k+\ell=N} \mu_k^{(N)}(\mu_\ell^{(N)}(a, b), c) \\ &\underset{(3.1)}{=} \sum_{i+j=N} \mu_i^{(N)}(a, \mu_j^{(N)}(b, c)) = a\mu_N^{(N)}(b, c) + \mu_N^{(N)}(a, bc). \end{aligned}$$

- So $\mathsf{d}^2\mu_N^{(N)}(a, b, c) = a\mu_N^{(N)}(b, c) - \mu_N^{(N)}(ab, c) + \mu_N^{(N)}(a, bc) - \mu_N^{(N)}(a, b)c = 0.$
- Since $\mathsf{HH}^2(A) = 0$, i.e. $\ker \mathsf{d}^2 = \operatorname{im} \mathsf{d}^1$, there exists some $f_N \in C^1(A; A)$ such that $\mu_N^{(N)} = \mathsf{d}^1 f_N$.
- In particular, $f_N : A \to A$ is a $\mathbb{C}$-linear map such that

$$\mu_N^{(N)}(a, b) = af_N(b) - f_N(ab) + f_N(a)b. \tag{4.1}$$

Now consider the $\mathbb{C}[[\hbar]]$-linear map $\varphi_N : A[[\hbar]] \to A[[\hbar]]$ determined by $\varphi_N(a) := a + \hbar^N f_N(a)$ for every $a \in A$.

Claim 4.2.1 — φ_N is bijective.

Proof of Claim. We'll skip this, but one can find the proof in the notes. ∎

- We can then define $*_{N+1}$ by transporting $*_N$ via φ_N, i.e.

$$a *_{N+1} b := \varphi_N(\varphi_N^{-1}(a) *_N \varphi_N^{-1}(b)).$$

- It is clear by construction that $*_{N+1}$ is a deformation equivalent to $*_N$, and hence to $*$.

Claim 4.2.2 — $a *_{N+1} b = ab + O(\hbar^{N+1})$.

Proof of Claim. We will compare two ways of computing $a *_{N+1} b$, namely

- by using the definition of $*_{N+1}$ and the linearity of φ_N:

$$\begin{aligned}\varphi_N(a) *_{N+1} \varphi_N(b) = \varphi_N(a *_N b) &= \varphi_N\big(ab + \hbar^N \mu_N^{(N)}(a,b) + O(\hbar^{n+1})\big) \\ &= ab + \hbar^N\big(f_N(ab) + \mu_N^{(N)}(a,b)\big) + O(\hbar^{N+1});\end{aligned}$$

- and by distributing over $*_{N+1}$,

$$\begin{aligned}\varphi_N(a) *_{N+1} \varphi_N(b) &= \big(a + \hbar^N f_N(a)\big) *_{N+1} \big(b + \hbar^N f_N(b)\big) \\ &= \big(a *_{N+1} b\big) + \hbar^N\big(f_N(a) *_{N+1} b\big) + \hbar^N\big(a *_{N+1} f_N(b)\big) + \hbar^{2N}\big(f_N(a) *_{N+1} f_N(b)\big) \\ &= ab + \left(\sum_{k=1}^{N-1} \hbar^k \mu_k^{(N+1)}(a,b)\right) + \hbar^N\big(\mu_N^{(N+1)}(a,b) + f_N(a)b + af_N(b)\big) + O(\hbar^{N+1}).\end{aligned}$$

Comparing each coefficient of $\hbar^k$ for $1 \le k \le N$, we find that $\mu_k^{(N)}(a,b) = 0$ for $1 \le k < N-1$ and

$$f_N(ab) + \mu_N^{(N)}(a,b) = \mu_N^{(N+1)}(a,b) + f_N(a)b + af_N(b)$$

By Equation 4.1, we also obtain $\mu_N^{(N+1)}(a,b) = 0$. ∎

- We conclude that $*$ is equivalent to the trivial deformation via the map φ (which is to be thought of as "$\cdots \circ \varphi_2 \circ \varphi_1$") given on $a \in A$ by:

$$\varphi(a) := \sum_{k=1}^{\infty} \hbar^k \sum_{\substack{i_\ell > \cdots > i_1 \\ i_\ell + \cdots + i_1 = k}} f_{i_\ell} \circ \cdots \circ f_{i_1}(a). \qquad \square$$

Corollary 4.3 — Every deformation of $\mathbb{C}[x]$ is equivalent to the trivial one.