HOCHSCHILD COHOMOLOGY AND DEFORMATIONS OF ALGEBRAS

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Abstract. Hochschild cohomology for algebras over a field was first introduced in 1945 by Gerhard Hochschild [5] and has since proved useful in the fields of algebraic geometry, category theory, functional analysis, and topology [8]. In 1964, algebraic deformation theory was introduced for associative algebras by Murray Gerstenhaber [2]. In his work [4], Gerstenhaber described the connection between the Hochschild cohomology of an algebra and its deformations. The goal of these notes is to introduce Hochschild cohomology and deformations of algebras in order to showcase some of the interactions between the two.

Contents

1. **Background**

We begin by recalling definitions and establishing notation. Further details may be found in [6] and [1].

Definition 1.1 (Algebra) — An (associative, unital) algebra over a commutative ring R is a R-module A together with an R-bilinear multiplication $\mu_A : A \times A \to A$ and a unit $\eta_A : R \to A$ such that the following diagrams commute:

$$\begin{tikzcd} A^3 \arrow[r, "\mu_A \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu_A"'] & A^2 \arrow[d, "\mu_A"] \\ A^2 \arrow[r, "\mu_A"'] & A \end{tikzcd} \qquad \begin{tikzcd} R \otimes_R A \arrow[r, "\eta_A \otimes \mathrm{id}_A"] \arrow[dr, equal] & A \otimes_R A \arrow[d] & A \otimes_R R \arrow[l, "\mathrm{id}_A \otimes \eta_A"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

In other words, an algebra over R is a monoid object in the category $R-\mathsf{mod}$ of module over R. Notice we will denote

$$\underbrace{A \times \ldots \times A}_{k \text{ times}} \text{ by } A^k \quad \text{and} \quad \underbrace{A \otimes_R \ldots \otimes_R A}_{k \text{ times}} \text{ by } A^{\otimes k}.$$

For $k = 0$, we adopt the convention $A^0 = A^{\otimes 0} = R$. We will also denote $1_A := \eta_A(1_k)$. We will also drop the subscripts on μ_A, η_A, and 1_A whenever it is clear from context. Throughout these notes, we will mostly concern ourselves with $\mathbb{C}$-algebras, that is, algebras over $\mathbb{C}$.

Example 1.2 ($\mathbb{C}$-Algebras) —

- $\mathrm{End}(V)$ for any $\mathbb{C}$-vector space $V \in \mathsf{Vect}$, so in particular, $\mathrm{Mat}_{n\times n}(\mathbb{C})$,
- $\mathbb{C}[x_1, \ldots, x_n]$ for $n \in \mathbb{N}$,
- $\mathbb{C}[[\hbar]]$, the algebra of formal power series over $\mathbb{C}$ with variable $\hbar$,
- Given a $\mathbb{C}$-algebra A, the *opposite algebra* A^{op} is given by $\mu_{A^{\mathrm{op}}}(a, b) := \mu_A(b, a)$.
- Given a $\mathbb{C}$-algebra A, the *enveloping algebra* of A is given by $A^e := A \otimes A^{\mathrm{op}}$.
- The group algebra $\mathbb{C}[G]$ of a group G,
- The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a complex Lie algebra $\mathfrak{g}$.

The organic notion of an algebra homomorphism $\varphi : A \to B$ is then a linear transformation making the following diagrams commute:

$$\begin{tikzcd} A \times A \arrow[r, "\mu_A"] \arrow[d, "\varphi\times\varphi"'] & A \arrow[d, "\varphi"] \\ B \times B \arrow[r, "\mu_B"'] & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\varphi"] \\ R \arrow[ur, "\eta_A"] \arrow[dr, "\eta_B"'] & \\ & B \end{tikzcd}$$

The direct sum, product, and tensor of algebras agree with their vector space counterparts, where additionally, multiplication is defined entry-wise, e.g. $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

It is then natural to consider representations of algebras, which leads us to the notion of a module over these structures.

Definition 1.3 (A-module) — A left A-module M over an $\mathbb{C}$-algebra A is a $\mathbb{C}$-vector space M together with a bilinear map $(a, m) \mapsto am$ from $A \times M$ to M such that

$$a(a'm) = (aa')m \quad \text{and} \quad 1m = m.$$

This is analogous with the notion of a left module over a ring, in the sense that a left A-module over an algebra A can be viewed as an algebra homomorphism $\alpha : A \to \mathrm{End}(M)$ where $\mathrm{End}(M) = \mathrm{Hom}(M, M)$ is the *algebra* of linear transformations from M to itself. The organic notion of a A-module morphism $\varphi : M \to M'$ is also analogous, being a linear transformation that makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha'"'] & \mathrm{End}(M) \arrow[d, "\varphi\circ-"] \\ \mathrm{End}(M') \arrow[r, "-\circ\varphi"'] & \mathrm{Hom}(M, M') \end{tikzcd}$$

In other words, $\varphi : M \to M'$ must satisfy $\varphi(am + m') = a\varphi(m_1) + \varphi(m_2)$ for every $a \in A$ and $m_1, m_2 \in M$.

Fact 4 ($A{-}\mathsf{mod}$) — The category $A{-}\mathsf{mod}$ of left A-modules together with A-module morphisms is Abelian.

Proof of Fact. Notice that $A{-}\mathsf{mod} = \mathcal{F}(\mathsf{B}A, \mathsf{Vect})$ where $\mathsf{B}A$ is the *delooping* of A, i.e. the linear category with a single object $\bullet$, $\mathrm{Hom}(\bullet, \bullet) = (A, +)$, and composition is given by multiplication in A. Since $\mathsf{Vect} = \mathbb{C} - \mathsf{mod}$ is Abelian, $\mathcal{F}(\mathsf{B}A, \mathsf{Vect})$ is Abelian. ∎

Exercise 1.1. Use the fact that $A{-}\mathsf{mod} = \mathcal{F}(\mathsf{B}A, \mathsf{Vect})$ to the deduce what the zero object, direct sums, kernels, and cokernels are in $A{-}\mathsf{mod}$.

Similarly, a *right* A-module is a $\mathbb{C}$-vector space M together with a bilinear map $(m, a) \mapsto ma$ from $M \times A$ such that

$$(ma)a' = m(aa') \quad \text{and} \quad m1 = m.$$

So a right A-module is nothing else than a *left* module over the opposite algebra A^{op}.

Definition 1.5 ($A - A$ bimodule) — An $A - A$ bimodule M is a $\mathbb{C}$-vector space M together with compatible left and right A-module structures, i.e.

$$(am)a' = a(ma').$$

Example 1.6 (Bimodules over Algebras) —

- Any $\mathbb{C}$-algebra A can be viewed as an $A - A$ bimodule, where the left and right actions are given by multiplying on the left and right respectively.
- More generally, any two-sided ideal B of A can also be endowed with an $A - A$ bimodule structure.
- $\prod_{i \in I} A$ and $\bigoplus_{i \in I} A$ are $A - A$ bimodules where the left and right actions are given coordinate-wise, i.e. $a(v_i)_{i \in I} a' := (a v_i a')_{i \in I}$.
- Given a $\mathbb{C}$-vector space V, $A \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} A$ can be equipped with a $A - A$ bimodule structure determined by $a(a' \otimes v \otimes b')b = (aa') \otimes v \otimes (bb')$. This is in fact the free $A - A$ bimodule generated by V. In particular, $A^{\otimes n+2}$ for $n \in \mathbb{N}$ can be equipped with an $A - A$ bimodule structure by setting $V = A^{\otimes n}$.

Similarly, $A - A$ bimodules are nothing more than left A^e-modules. Hence, Fact 4 yields that the category $\mathsf{Bim}_A = A^e - \mathsf{mod}$ of $A - A$ bimodules is also a (linear) Abelian category.

Proposition 1.7. $A^{\otimes n}$ are projective objects in Bim_A for $n \geq 2$.

Proof. As we saw in Example 1.6, for any $\mathbb{C}$-vector space V, $A \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} A$ is the free $A - A$ bimodule generated by V, i.e.

$$\operatorname{Hom}_{A^e}(A \otimes_{\mathbb{C}} V \otimes_C A, -) \cong \operatorname{Hom}_{\mathbb{C}}(V, U(-)),$$

where $U : A^e \to \mathsf{Vect}$ is the corresponding forgetful functor. Since every vector space is projective, $\operatorname{Hom}_{\mathbb{C}}(M, U(-))$ is exact, and thus $A \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} A$ is projective. Taking $V = A^{\otimes n-2}$ for $n \geq 2$ yields the desired result. $\square$

We will see later on that A is *not necessarily* projective as a bimodule over itself. We now introduce certain constructions once given an $A - A$ bimodule M:

Definition 1.8 (Center) — The center $Z_A(M)$ is defined as:

$$Z_A(M) := \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

When $M = A$, this corresponds to the center $Z(A)$ of an algebra:

$$Z(A) := Z_A(A) = \{a \in A \mid ba = ab \text{ for all } b \in A\}.$$

Notice that A can be viewed as a ring together with an $\mathbb{C}$-action. Then $Z(A)$ is precisely the center of A as a ring. Now recall the Leibniz (or product) rule for differentiation:

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g).$$

Generalizing the Leibniz rule to bimodules over algebras, this yields the following:

Definition 1.9 (Derivation) — We say that a $\mathbb{C}$-linear $\delta : A \to M$ is a derivation of A with values in M if

$$\delta(ab) = \delta(a)b + a\delta(b) \qquad \text{for every } a, b \in A.$$

The space of all such derivations is denoted by $\mathrm{Der}_A(M)$. When $M = A$, we say these are derivations on A and write $\mathrm{Der}(A) := \mathrm{Der}_A(A)$.

Example 1.10 (Derivation) — $\frac{d}{dx}$ is a derivation on $\mathbb{C}[x]$.

Exercise 1.2. Show that $\mathrm{Der}(\mathbb{C}[x]) = \mathbb{C}[x]\frac{d}{dx}$.
Hint: Show that any $\delta \in \mathrm{Der}(\mathbb{C}[x])$ is of the form $\delta(x)\frac{d}{dx}$.

Exercise 1.3. Let $[\delta, \delta'] := \delta \circ \delta' - \delta' \circ \delta$ for $\delta, \delta' \in \mathrm{Der}_A$. Show that Der_A equipped with this bracket $[\,\cdot\,,\,\cdot\,]$ is a Lie algebra.

The following definitions and exercises were adapted from Dr. Penneys' 8800 notes [7].

Definition 1.11 (Simplicial Category) — The simplicial category Δ has objects $[n] := \{0 \leq 1 \leq \ldots \leq n\}$ for $n \in \mathbb{N}$ and weakly order preserving functions as morphisms.

Exercise 1.4. Prove that Δ has the following presentation by generators and relations:

- Generators: for all n, we have $\delta_i : [n-1] \to [n]$ for $0 \leq i \leq n$ and $\sigma_i : [n+1] \to [n]$ for $0 \leq i \leq n$.
- Relations: $\delta_j\delta_i = \delta_i\delta_{j-1}$ if $i < j$, $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$ if $i \leq j$, and

$$\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1\ . \\ \delta_{i-1}\sigma_j & \text{if } i > j+1. \end{cases}$$

Hint: Send δ_i to the map which skips i and σ_i to the map which maps i and $i+1$ to i.

The maps δ_i are called *face maps*, and can be viewed as the inclusion of the $n-1$ simplex into the n-simplex as the fact which does not include the vertex i. The maps σ_i are called *degeneracies*.

Definition 1.12 (Simplicial Object) — Given a category $\mathcal{C}$, a simplicial object $\mathcal{X}_\bullet$ in $\mathcal{C}$ is a presheaf on Δ valued in $\mathcal{C}$, i.e. a functor $\mathcal{X}_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$.

Equivalently, a simplicial set associates an object $\mathcal{X}_n \in \mathcal{C}$ to each $n \in \mathbb{N}$, and includes maps $d_i : \mathcal{X}^n \to \mathcal{X}^{n-1}$ and $s_j : \mathcal{X}^n \to \mathcal{X}^{n+1}$ for $0 \leq i \leq n$ satisfying the (opposite) relations as in Exercise 1.4.

Example 1.13 (Simplicial Object in Bim_A) —

- $\mathcal{X}_n := A^{\otimes n+2} \in \mathsf{Bim}_A$ for each $n \in \mathbb{N}$,
- $d_i : A^{\otimes n+2} \to A^{\otimes n+1}$ for $0 \leq i \leq n$ determined by

$$d_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

- $s_i : A^{\otimes n+2} \to A^{\otimes n+3}$ determined by

$$s_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1}$$

Exercise 1.5. Verify that Example 1.13 is indeed a simplicial object.
Hint: Use Exercise 1.4.

Proposition 1.14. Given a simplicial object $\mathcal{X}_\bullet$ in an Abelian category, one can define an *exact* sequence $(\mathcal{X}_\bullet, \partial_\bullet)$ by defining $\partial_n : \mathcal{X}_n \to \mathcal{X}_{n-1}$ as $\partial_n := \sum_{i=0}^n (-1)^i d_i$.

Proof. Recall the (opposite) relations in Exercise 1.4 and observe

$$\begin{aligned}
\partial_n \circ \partial_{n+1} &= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} d_i d_j = \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} d_i d_j \\
&= \sum_{j=1}^{n+1} \sum_{i=0}^{j-1} (-1)^{i+j} d_i d_j + \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i+j} d_i d_j \\
&= \sum_{j=1}^{n+1} \sum_{i=0}^{j-1} (-1)^{i+j} d_{j-1} d_i + \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i+j} d_i d_j \\
&= \sum_{i=1}^{n+1} \sum_{j=0}^{i-1} (-1)^{i+j} d_{i-1} d_j + \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i+j} d_i d_j \\
&= \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{i+j+1} d_i d_j + \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i+j} d_i d_j \\
&= \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i+j+1} d_i d_j + \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i+j} d_i d_j \\
&= 0.
\end{aligned}$$

Thus $(\mathcal{X}_\bullet, \partial_\bullet)$ is a chain complex. Now notice that $\frac{1}{2}s_j : \mathcal{X}_j \to \mathcal{X}_{j+1}$ with

$$\frac{1}{2} d_{j+1} \circ s_j + \frac{1}{2} s_{j-1} \circ d_j = \frac{1}{2}\mathrm{id}_{\mathcal{X}_n} + \frac{1}{2}\mathrm{id}_{\mathcal{X}_n} = \mathrm{id}_{\mathcal{X}_n}.$$

In other words, $\frac{1}{2}s_\bullet$ is a chain homotopy and $\mathrm{id}_{\mathcal{X}_\bullet}$ is null-homotopic. However this only occurs when $\mathcal{X}_\bullet$ is acyclic. $\square$

2. **Hochschild Cohomology**

Given a $\mathbb{C}$-algebra A, we saw in Example 1.13 how to construct a simplicial object in $A-\mathsf{mod}$ based only on A and its tensor powers. Proposition 1.14 then organically gives rise to an exact sequence. This sequence may be slightly extended (with $\to 0$) to obtain:

Definition 2.1 (Bar Complex) — For a $\mathbb{C}$-algebra A, we define the bar (or standard) complex (or resolution) of A as:

$$\cdots \xrightarrow{\partial_3} A^{\otimes 4} \xrightarrow{\partial_2} A^{\otimes 3} \xrightarrow{\partial_1} A^{\otimes 2} \to 0$$

where $\partial_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n}(-1)^n a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$.

By Proposition 1.7, each $A^{\otimes n}$ is projective. Moreover $\operatorname{Coker} \partial_1 \cong A$ since the following chain is exact:

$$A^{\otimes 3} \xrightarrow{\partial_1} A^{\otimes 2} \xrightarrow{\pi} A \to 0 \quad \text{where } \pi(a \otimes b) := ab.$$

Indeed, it is clear that $\pi \circ \partial_1 = 0$ since $a \otimes b \otimes c \overset{\partial_1}{\longmapsto} (ab) \otimes c - a \otimes (bc) \overset{\pi}{\mapsto} abc - abc = 0$ and π is onto since $a \otimes 1 \overset{\pi}{\mapsto} a$ for every $a \in A$. Then, if $\sum a_i \otimes b_i \in \ker(\pi)$, then $\sum 1 \otimes (a_i b_i) = 0$ and $\sum a_i \otimes b_i = \sum (1a_i) \otimes b_i - 1 \otimes (a_i b_i) \in \operatorname{im}(\partial_1)$, implying $\ker(\pi) = \operatorname{im}(\partial_1)$. So the bar complex is actually a *projective resolution* of A. This then allows us to compute:

Definition 2.2 (Hochschild Cohomology v.1) — $\mathsf{HH}^k(A; M) := \operatorname{Ext}^k_{A^e}(A, M)$.

Indeed, $\operatorname{Ext}^k_{A^e}(A, M)$ can be computed as the k^{th} cohomology group of the co-chain complex

$$0 \to \operatorname{Hom}_{A^e}(A^{\otimes 2}, M) \xrightarrow{-\circ \partial_1} \operatorname{Hom}_{A^e}(A^{\otimes 3}, M) \xrightarrow{-\circ \partial_2} \operatorname{Hom}_{A^e}(A^{\otimes 4}, M) \xrightarrow{-\circ \partial_3} \cdots$$

Furthermore, we saw in Example 1.6 and Proposition 1.7 how $A^{\otimes n+2}$ are the free $A - A$ bi-modules generated by $A^{\otimes n}$ for $n \in \mathbb{N}$. More specifically, $\operatorname{Hom}_{A^e}(A^{\otimes n+2}, M) \cong \operatorname{Hom}_{\mathbb{C}}(A^{\otimes n}, M)$ via the maps

$$f \in \operatorname{Hom}_{A^e}(A^{\otimes n+2}, M) \mapsto \big(a_1 \otimes \cdots \otimes a_n \mapsto f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)\big)$$
$$g \in \operatorname{Hom}_{\mathbb{C}}(A^{\otimes n}, M) \mapsto \big(a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_0 g(a_1 \otimes \cdots \otimes a_n) a_{n+1}\big).$$

We may thus provide a more explicit presentation of Hochschild Cohomology in the spirit of Dr. Gautam's notes [3]:

Definition 2.3 (Hochschild Complex) — We define the co-chain complex $C^\bullet(A; M)$ as:

$$C^k(A; M) := \operatorname{Hom}_{\mathbb{C}}(A^{\otimes k}, M) = \left\{ \begin{array}{c} \mathbb{C}\text{-multi-linear maps} \\ A^n \to M. \end{array} \right\}$$

with differentials $\mathsf{d}^k : C^k(A; M) \to C^{k+1}(A; M)$ given on $\xi : A^n \to M$ by:

$$\begin{aligned} \mathsf{d}^k \xi(a_0, \ldots, a_k) := {} & a_0 \cdot \xi(a_1, \ldots, a_k) \\ & + \sum_{i=0}^{k-1} (-1)^{i+1} \xi(\ldots, a_i a_{i+1}, \ldots) \\ & + (-1)^{k+1} \xi(a_0, \ldots, a_{k-1}) \cdot a_k \end{aligned}$$

Definition 2.4 (Hochschild Cohomology v.2) — The k^{th} Hochschild Cohomology group of A valued in M is given by:

$$\mathsf{HH}^k(A;M) := H^k(C^\bullet(A;M)) = \frac{\mathrm{Ker}(\mathsf{d}^k)}{\mathrm{Im}(\mathsf{d}^{k-1})}.$$

When $M = A$ as a bimodule over itself, we write $\mathsf{HH}^k(A) := \mathsf{HH}^k(A;A)$.

Facts 2.5 (Hochschild Cohomology) —

- $\mathsf{HH}^0(A;M) = Z_A(M)$ as in Definition 1.8.

  Proof of Fact. Notice $\mathsf{HH}^0(A;M) = \ker(\mathsf{d}^0) = \{\xi : \mathbb{C} \to M \mid \mathsf{d}^0\xi = 0\}$ where
  $$\mathsf{d}^0\xi(a_0) = a_0\xi(1) - \xi(1)a_0.$$
  Identifying ξ with $\xi(1) \in M$, the statement is then clear. ■
- $\mathrm{Ker}(\mathsf{d}^1) = Z^1(C^\bullet(A;M)) = \mathrm{Der}_A(M)$ as in Definition 1.9.

  Proof of Fact. Notice $\mathsf{d}^1\xi(a_0,a_1) = a_0\xi(a_1) - \xi(a_0a_1) + \xi(a_0)a_1$. ■

Example 2.6 (Hochschild Cohomology) — Let $A = \mathbb{C}[x]$ and let us compute $\mathsf{HH}^k(A)$.

- $\mathsf{HH}^0(A) = Z(A) = \mathbb{C}[x]$.
- $\mathsf{HH}^1(A) = \ker(\mathsf{d}^1) = \mathrm{Der}(A) = \mathbb{C}[x]\frac{d}{dx}$ by Exercise 1.2.
- $\mathsf{HH}^k(A) = 0$ for $k \geq 2$ since
  $$0 \to \mathbb{C}[x]^{\otimes 2} \xrightarrow{(x\otimes 1 - 1\otimes x)\cdot} \mathbb{C}[x]^{\otimes 2} \to 0$$
  is a projective resolution of A, where the map $(x\otimes 1 - 1\otimes x)\cdot$ is given by multiplication with the element $(x\otimes 1 - 1\otimes x)$.

Exercise 2.1. Verify that $0 \to \mathbb{C}[x]^{\otimes 2} \xrightarrow{(x\otimes 1 - 1\otimes x)\cdot} \mathbb{C}[x]^{\otimes 2} \xrightarrow{\pi} \mathbb{C}[x] \to 0$ is exact.

Exercise 2.2.🌶 Let $A = \mathbb{C}[x,y]$. We can view $\frac{\mathsf{d}}{\mathsf{d}x}$ and $\frac{\mathsf{d}}{\mathsf{d}y}$ as acting formally on A. Show that:

- $\mathsf{HH}^0(A) = A$,
- $\mathsf{HH}^1(A) = A\frac{\mathsf{d}}{\mathsf{d}x} \oplus A\frac{\mathsf{d}}{\mathsf{d}y}$,
- $\mathsf{HH}^2(A) = A(\frac{\mathsf{d}}{\mathsf{d}x} \wedge \frac{\mathsf{d}}{\mathsf{d}y})$,
- $\mathsf{HH}^k(A) = 0$ for $k \geq 3$.

Hint: Read up on the Hochschild-Kostant-Rosenberg Theorem.

In light of Example 2.6 and Exercise 2.2, A is not necessarily a projective $A-A$ bimodule since $\mathrm{Ext}^1_{A^e}(A,M) \neq 0$ when $A = \mathbb{C}[x]$ or $\mathbb{C}[x,y]$. In other words, A is a projective $A-A$ bimodule only when A has acyclic Hochschild cohomology.

One could also use the bar complex to compute:

Definition 2.7 (Hochschild *homology* of A valued in M) — $\mathsf{HH}_\bullet(A;M) := \mathrm{Tor}^\bullet_{A^e}(A,M)$

However, we will not investigate this any further in these notes.

3. **Deformations of Algebras**

Let A be a $\mathbb{C}$-algebra, consider the $\mathbb{C}$-vector space $A[[\hbar]] = A \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$. Notice A is included in $A[[\hbar]]$, and $A[[\hbar]]$ projects down into A via the map $\mathrm{ev}_0 : A[[\hbar]] \twoheadrightarrow A$ determined by $\hbar \mapsto 0$.

$$\begin{tikzcd} A[[\hbar]] \arrow[d, bend left, two heads, "\mathrm{ev}_0"] \\ A \arrow[u, bend left, hook] \end{tikzcd}$$

The aim of deformation theory is to equip $A[[\hbar]]$ with a multiplication compatible with the multiplication on A, enhancing $A[[\hbar]]$ into a $\mathbb{C}[[\hbar]]$-algebra.

Definition 3.1 (Deformation of an Algebra) — By a deformation of A, we mean an associative $\mathbb{C}[[\hbar]]$-bilinear $* : A[[\hbar]]^2 \to A[[\hbar]]$ such that the following diagrams commutes

$$\begin{tikzcd} A[[\hbar]] \times A[[\hbar]] \arrow[r, "*"] & A[[\hbar]] \arrow[d, two heads, "\mathrm{ev}_0"] \\ A \times A \arrow[u, hook] \arrow[r] & A \end{tikzcd}$$

Observe that such a multiplication would be uniquely determined by its action on elements of the form $a \otimes b$ for $a, b \in A$. Furthermore, this action can be decomposed uniquely into a family of $\mathbb{C}$-bilinear maps $\{\mu_n : A \times A \to A\}_{n\in\mathbb{N}}$ such that

$$a * b = \sum_{n \geq 0} \hbar^n \mu_n(a, b).$$

In particular, notice that $\mu_0(a, b) = ab$. The associativity of $*$ can then be expressed in terms of the maps $\{\mu_n\}_{n\in\mathbb{N}}$ as follows:

$$\sum_{k+\ell=n} \mu_k(\mu_\ell(a, b), c) = \sum_{i+j=n} \mu_i(a, \mu_j(b, c)) \quad \text{for } a, b, c \in A \text{ and } n \in \mathbb{N}. \tag{3.1}$$

One can also consider deformations of A over $\mathbb{C}[\hbar]$, which are multiplications on $A[\hbar]$ which turn $A[\hbar]$ into a $\mathbb{C}[\hbar]$-algebra. When $*$ is a finite deformation on $A[[\hbar]]$, that is, when $a * b$ is just a polynomial in $A[\hbar] \subseteq A[[\hbar]]$ it is usual to restrict oneself to work on $A[\hbar]$.

Example 3.2 (Deformations of Algebras) —

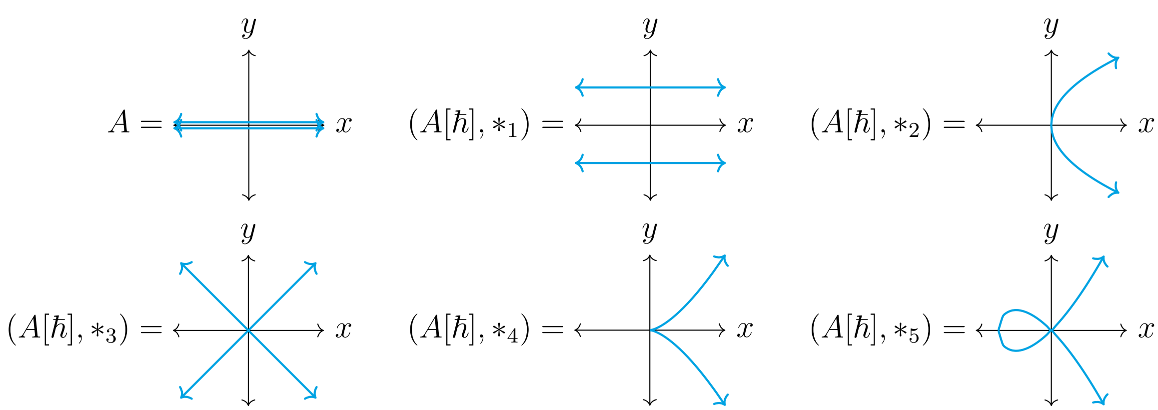
- For any algebra A, the *trivial deformation* is given by $\mu_k = 0$ for $k \geq 1$. Notice that $a * b = ab$ for every $a, b \in A$.
- For $A = \mathbb{C}[x, y]$, [9] defines the deformation $*$ on $A[\hbar]$ (or $A[[\hbar]]$) determined by

$$x^n * x^m = x^{n+m}, \qquad y^n * y^m = y^{n+m} \qquad x^n * y^m = x^n y^m \qquad y * x = xy + \hbar$$

  Notice that $(A, *) \cong \mathbb{C}[x, y, \hbar]/(yx - xy - \hbar)$. When we set $\hbar = 1$, this is known as the *Weyl algebra* and usually y is denoted by $\frac{d}{dx}$ or ∂_x.

- For $A = \mathbb{C}\langle x, y\rangle/(y^2)$, [2] defines the deformations $*_1, \ldots, *_5$ on $A[\hbar]$ determined by $x^n *_i x^m = x^{n+m}$, $x^n *_i y = y *_i x^n = x^n y$ and
$$\begin{aligned} \overline{y} *_1 \overline{y} &= t^2, \\ \overline{y} *_2 \overline{y} &= \overline{x}t, \\ \overline{y} *_3 \overline{y} &= \overline{x}^2 t^2, \\ \overline{y} *_4 \overline{y} &= \overline{x}^3 t, \\ \overline{y} *_5 \overline{y} &= (\overline{x}^2 + \overline{x}^3)t. \end{aligned}$$
Notice that:
$$\begin{aligned} (A[\hbar], *_1) &\cong \mathbb{C}\langle x, y, t\rangle/(y^2 - t^2) \\ (A[\hbar], *_2) &\cong \mathbb{C}\langle x, y, t\rangle/(y^2 - xt) \\ (A[\hbar], *_3) &\cong \mathbb{C}\langle x, y, t\rangle/(y^2 - x^2t^2) \\ (A[\hbar], *_4) &\cong \mathbb{C}\langle x, y, t\rangle/(y^2 - x^3t) \\ (A[\hbar], *_5) &\cong \mathbb{C}\langle x, y, t\rangle/(y^2 - x^2t - x^3t) \end{aligned}$$

Recall that $A = \mathbb{C}[x, y]/(y^2)$ can be visualized as a "double line" on the x-axis. Then, the deformations of A in Example 3.2 may be visualized for some fixed value of $\hbar$ as follows:



We may also compare deformations of a particular algebra, the organic notion being an $\mathbb{C}[[\hbar]]$-algebra homomorphism $\varphi : (A[[\hbar]], *) \to (A[[\hbar]], *')$ making the following diagram commute:

$$\begin{tikzcd} (A[[\hbar]], *) \arrow[r, "\varphi"] & (A[[\hbar]], *') \arrow[d, two heads, "\mathrm{ev}_0"] \\ A \arrow[u, hook] \arrow[r, equal] & A \end{tikzcd}$$

From this, we obtain the following:

Definition 3.3 (Equivalence of Deformations) — We say $*$ and $*'$ are equivalent if there exists a bijective $\mathbb{C}[[\hbar]]$-linear $\varphi : A[[\hbar]] \to A[[\hbar]]$ such that

- $\varphi(a) = a \pmod{\hbar}$ for all $a \in A$,
- $\varphi(a * b) = \varphi(a) *' \varphi(b)$ for all $a, b \in A$.

Observe that such a map is uniquely determined by its action on A. Furthermore, this action can be decomposed uniquely into a family of $\mathbb{C}$-linear maps $\{\varphi_n : A \to A\}_{n\in\mathbb{N}}$ such that

$$\varphi(a) = \sum_{n\geq 0} \hbar^n \varphi_n(a).$$

In particular, notice that $\varphi_0 = \mathrm{id}_A$. The fact that φ preserves the deformations of A can then be expressed in terms of the maps $\{\varphi_n\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\mu'_n\}_{n\in\mathbb{N}}$ as follows:

$$\sum_{k+\ell=n} \varphi_k(\mu_\ell(a,b)) = \sum_{i+j+k=n} \mu'_i(\varphi_j(a), \varphi_k(b)) \quad \text{for } a,b \in A \text{ and } n \in \mathbb{N}. \tag{3.2}$$

Definition 3.4 (Rigidity) — We say that A is rigid if every deformation of A is equivalent to the trivial deformation.

4. **Some Results**

The goal of this section is to explore how the Hochschild cohomology of an algebra A may cause obstructions to the possible ways one can deform A. This section closely follows sections 3 and 4 of [3].

Proposition 4.1. If $\mathsf{HH}^1(A) = 0$, then any auto-equivalence of the trivial deformation of A is inner. That is, if $I : A[[\hbar]] \to A[[\hbar]]$ is a $\mathbb{C}[[\hbar]]$-linear algebra isomorphism such that $I = \sum \hbar^n I_n$ on A with $\{I_k : A \to A\}_{k\geq 1}$ and $I_0 = \mathrm{id}_A$, there exists some invertible $\widetilde{a} \in A[[\hbar]]$ such that $I(x) = \widetilde{a}x\widetilde{a}^{-1}$.

Proof. Let $N \geq 1$ be the smallest index such that $I_N \neq 0$. Since $I(ab) = I(a)I(b) = \sum_n \sum_{k+\ell=n} \hbar^n I_k(a)I_\ell(b)$, comparing the coefficients of $\hbar^N$ yields:

$$I_N(ab) = \sum_{k+\ell=N} I_k(a)I_\ell(b) = aI_N(b) + I_N(a)b.$$

Thus I_N is a derivation, i.e. $\mathsf{d}^1 I_N = 0$ as we saw in Facts 2.5. Since $\mathsf{HH}^1(A) = 0$, i.e. $\ker(\mathsf{d}^1) = \mathrm{im}(\mathsf{d}^0)$, there exists some $\xi_N \in C^1(A;A) \cong A \ni \xi_N(1)$ such that $\mathsf{d}^0\xi_N = I_N$. In particular, $I_N(x) = x\xi_N(1) - \xi_N(1)x$. Note that $1 - \hbar^N\xi_N(1)$ is invertible with

$$\big(1 - \hbar^N\xi_N(1)\big)^{-1} = \sum \hbar^{Nk}\xi_N(1)^k.$$

So define

$$\begin{aligned}
\widetilde{I}(x) &= \big(1 - \hbar^N\xi_N(1)\big)^{-1} I(x)\big(1 - \hbar^N\xi_N(1)\big) \\
&= \big(1 + \hbar^N\xi_N(1) + O(\hbar^{2N})\big)\big(x + \hbar^N I_N(x) + O(\hbar^{N+1})\big)\big(1 - \hbar^N\xi_N(1)\big) \\
&= x + \hbar^N\big(I_N(x) - x\xi_N(1) + \xi_N(1)x\big) + O(\hbar^{N+1}) \\
&= x + O(\hbar^{N+1}).
\end{aligned}$$

The result then follows by induction on N and setting $\widetilde{a} := \prod_N 1 - \hbar^N\xi_N(1)$. $\square$

Proposition 4.2. If $\mathsf{HH}^2(A) = 0$, then A is rigid.

Proof. Let $*$ be a deformation of A. We will proceed by constructing a family of deformations $*_1, *_2, *_3, \dots$ equivalent to $*$ such that $a *_k b = ab + O(\hbar^k)$, i.e. for $\{\mu_n^{(k)} : A^2 \to A\}_{n\in\mathbb{N}}$ corresponding to $*_k$, we have $\mu_i^{(k)} = 0$ whenever $1 < i < k$. Notice that $*_1 := *$ already satisfies this property. Now suppose we have constructed $*_N$, where $*_N$ is equivalent to $*$ and

$$a *_N b = ab + O(\hbar^N) = ab + \hbar^N \mu_N^{(N)}(a,b) + O(\hbar^{N+1}).$$

Then the associativity equation for $n = N$ yields

$$\begin{aligned} \mu_N^{(N)}(a,b)c + \mu_N^{(N)}(ab,c) &= \sum_{k+\ell=N} \mu_k^{(N)}(\mu_\ell^{(N)}(a,b),c) \\ &\underset{(3.1)}{=} \sum_{i+j=N} \mu_i^{(N)}(a,\mu_j^{(N)}(b,c)) = a\mu_N^{(N)}(b,c) + \mu_N^{(N)}(a,bc). \end{aligned}$$

So $\mathsf{d}^2\mu_N^{(N)}(a,b,c) = a\mu_N^{(N)}(b,c) - \mu_N^{(N)}(ab,c) + \mu_N^{(N)}(a,bc) - \mu_N^{(N)}(a,b)c = 0$. Since $\mathsf{HH}^2(A) = 0$, i.e. $\ker \mathsf{d}^2 = \operatorname{im} \mathsf{d}^1$, there exists some $f_N \in C^1(A;A)$ such that $\mu_N^{(N)} = \mathsf{d}^1 f_N$. In particular, $f_N : A \to A$ is a $\mathbb{C}$-linear map such that

$$\mu_N^{(N)}(a,b) = af_N(b) - f_N(ab) + f_N(a)b. \tag{4.1}$$

Now consider the $\mathbb{C}[[\hbar]]$-linear map $\varphi_N : A[[\hbar]] \to A[[\hbar]]$ determined by $\varphi_N(a) := a + \hbar^N f_N(a)$ for every $a \in A$.

Claim 4.2.1 — φ_N is bijective.

Proof of Claim. To see that φ_N is injective, suppose

$$\varphi_N\left(\sum a_k\hbar^k\right) = \sum a_k\hbar^k + \sum f_N(a_k)\hbar^{k+N} = \sum_{k=0}^{N-1} a_k\hbar^k + \sum_{k\ge N} \hbar^k(a_k + f_N(a_{k-N})) = 0.$$

Then $a_k = 0$ for $1 \le k \le N-1$ and $a_k + f_N(a_{k-N}) = 0$ for every $k \ge N$. By induction we see that $a_k = 0$ for every k, so $\sum a_k\hbar^k = 0$. To see that φ_N is surjective, it suffices to show $A \subset \operatorname{im}\varphi_N$. So consider some $a \in A$ and observe

$$\varphi_N\left(\sum(-1)^k f_N^k(a)\hbar^{Nk}\right) = \sum(-1)^k f_N^k(a)\hbar^{Nk} + \sum(-1)^k f_N^{k+1}(a)\hbar^{Nk+N} = a. \qquad \blacksquare$$

We can then define $*_{N+1}$ by transporting $*_N$ via φ_N, i.e. $a *_{N+1} b := \varphi_N(\varphi_N^{-1}(a) *_N \varphi_N^{-1}(b))$. It is clear by construction that $*_{N+1}$ is a deformation equivalent to $*_N$, and hence to $*$.

Claim 4.2.2 — $a *_{N+1} b = ab + O(\hbar^{N+1})$.

Proof of Claim. We will compare two ways of computing $a *_{N+1} b$, namely by using the definition of $*_{N+1}$ and the linearity of φ_N

$$\begin{aligned} \varphi_N(a) *_{N+1} \varphi_N(b) = \varphi_N(a *_N b) &= \varphi_N\big(ab + \hbar^N\mu_N^{(N)}(a,b) + O(\hbar^{n+1})\big) \\ &= ab + \hbar^N\big(f_N(ab) + \mu_N^{(N)}(a,b)\big) + O(\hbar^{N+1}); \end{aligned}$$

and by distributing over $*_{N+1}$,

$$\begin{aligned}\varphi_N(a) *_{N+1} \varphi_N(b) &= \big(a + \hbar^N f_N(a)\big) *_{N+1} \big(b + \hbar^N f_N(b)\big) \\ &= \big(a *_{N+1} b\big) + \hbar^N\big(f_N(a) *_{N+1} b\big) + \hbar^N\big(a *_{N+1} f_N(b)\big) \\ &\quad + \hbar^{2N}\big(f_N(a) *_{N+1} f_N(b)\big) \\ &= ab + \left(\sum_{k=1}^{N-1} \hbar^k \mu_k^{(N+1)}(a,b)\right) + \hbar^N\big(\mu_N^{(N+1)}(a,b) + f_N(a)b + af_N(b)\big) \\ &\quad + O(\hbar^{N+1}).\end{aligned}$$

Comparing each coefficient of $\hbar^k$ for $1 \leq k \leq N$, we find that $\mu_k^{(N)}(a,b) = 0$ for $1 \leq k < N-1$ and

$$f_N(ab) + \mu_N^{(N)}(a,b) = \mu_N^{(N+1)}(a,b) + f_N(a)b + af_N(b)$$

By Equation 4.1, we also obtain $\mu_N^{(N+1)}(a,b) = 0$. ■

We conclude that $*$ is equivalent to the trivial deformation via the map φ (which is to be thought of as "$\cdots \circ \varphi_2 \circ \varphi_1$") given on $a \in A$ by

$$\varphi(a) := \sum_{k=1}^{\infty} \hbar^k \sum_{\substack{i_\ell > \cdots > i_1 \\ i_\ell + \cdots + i_1 = k}} f_{i_\ell} \circ \cdots \circ f_{i_1}(a).$$ □

This result allows us to establish the following:

Corollary 4.3 — Every deformation of $\mathbb{C}[x]$ is equivalent to the trivial one.

Proof of Fact. We saw in Example 2.6 that $\mathsf{HH}^2(\mathbb{C}[x]) = 0$. ■

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