A duality formalism in the spirit of functional analysis: Worksheet

September 5, 2025

Abstract

The first goal of this worksheet is to generalize Grothendieck-Verdier categories in order to account for infinite dimensional spaces. We then study Hermitian objects internal to anti-involutive monoidal categories with such generalized duality theory. Our main goal is then to describe the internal Hilbert objects, for which we aim to show a corresponding the Riesz representation theorem.

1 Prerequisites on monoidal categories

Exercise 1.1. Write down a definition of monoidal category $(\mathcal{C}, \otimes, 1)$.

Note: There are many ways to "package" such a definition, but make sure to provide enough detail so that it is palatable to you!

Exercise 1.2. Describe the graphical calculus for a monoidal category $(\mathcal{C}, \otimes, 1)$.

Note: Organize this in the way you find most natural!

2 Generalized duality in monoidal categories

2.1 Preliminary definitions

Definition 2.1. For objects $W, X \in \mathcal{C}$ in a monoidal category $\mathcal{C}$, an *exponential* W^X or *the power of* W *for* X is an object representing the functor $\mathcal{C}(- \otimes X \to W)$, i.e.

$$\mathcal{C}(A \otimes X \to W) \cong \mathcal{C}(A \to W^X) \text{ naturally for all } A \in \mathcal{C}.$$

If X admits a power of W, we say that X *powers* W. Moreover, if $\mathcal{C}$ admits all powers of W, we say that W is powered (by $\mathcal{C}$) or $\mathcal{C}$ powers W.

In particular, for $W = 1$, we will denote the *power of unity* for X by $X^* := 1^X$.

Exercise 2.2. On the other hand, verify that W^1 always exists for every $W \in \mathcal{C}$.

Hint: Here, your intuition about numbers will be correct.

Remark 2.3. The notation W^X arises from the case when $\mathcal{C} = \mathsf{Set}$, where

$$W^X := \{f : X \to W\}$$

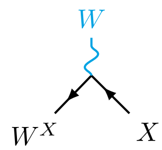
has cardinality $|W^X| = |W|^{|X|}$. Alternatively, some people use the term *internal hom* $\underline{Hom}(X, W)$ for the exponential W^X. Through this perspective, $\mathcal{C}$ powers W precisely when the *internal Yoneda construction* $\underline{Hom}(-, W)$ assembles into a contravariant functor $\mathcal{C}^{\text{op}} \to \mathcal{C}$ on all of $\mathcal{C}$.

Exercise 2.4. Fix an object $W \in \mathcal{C}$, which we will represent by

For $X \in \mathcal{C}$, show that the data of a power of W for X is equivalent to a pair

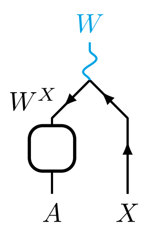
$$\left(W^X \in \mathcal{C}, \mathrm{ev}_X^W \in \mathcal{C}(W^X \otimes X \to W)\right)$$

where we represent W^X and ev_X^W diagramatically by:



such that the right Frobenius reciprocity maps ${}_A\lambda_X^W$ for $A \in \mathcal{C}$ are bijective, where:

${}_A\lambda_X^W :=$

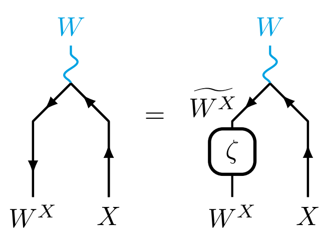


$: \mathcal{C}(A \to W^X) \to \mathcal{C}(A \otimes X \to W)$

Hint: First obtain the map ev_X^W *from the definition of a power* W^X*. Then verify that* ${}_A\lambda_X^W$ *is indeed bijective.*

Exercise 2.5. Adapt the previous diagrams to the case when $W = 1$ and $W^X = X^*$.

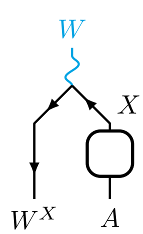
Exercise 2.6. Show that the power W^X of W for X is unique up to unique isomorphism (whenever it exists). That is, if W^X and $\widetilde{W^X}$ are powers of W for X, there exists a unique isomorphism $\zeta\colon W^X \to \widetilde{W^X}$ with



Definition 2.7. A monoidal category $\mathcal{C}$ is called *closed* if every $W \in \mathcal{C}$ is powered.

Definition 2.8. Similarly, fix $W \in \mathcal{C}$ and consider some $X \in \mathcal{C}$ which admits a power of W. The left Frobenius reciprocity maps ${}^W\lambda_A^X$ are given by

${}^W\lambda_A^X :=$



$: \mathcal{C}(A \to X) \to \mathcal{C}(X^* \otimes A \to 1)$

We then say:

- W^X is *faithful* or *separating* if ${}^W\lambda_A^X$ is injective for every $A \in \mathcal{C}$, and
- W^X is *full* if ${}^W\lambda_A^X$ is surjective for every $A \in \mathcal{C}$.
- A logarithm $\log_W(X)$ of W for X is an object X_* with $W^{\log_W(X)} \cong X$. In the case when $W = 1$, we denote the logarithm of unity $\log_1(X)$ for X by X_*.

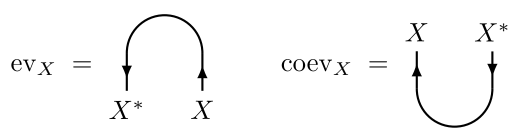
2.2 Powers of unity vs. dualizability

In this section, we contrast this new notion of powers of unity against the well-studied notion of duals. In particular, this discussion relates to the case when $W = 1$.

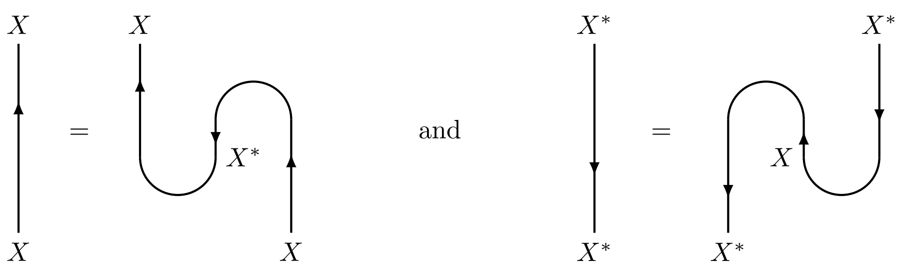
Definition 2.9. A *dual* for an object $X \in \mathcal{C}$ consists of a tuple

$$(X^*, \mathrm{ev}_X \in \mathcal{C}(X^* \otimes X \to 1), \mathrm{coev}_X \in \mathcal{C}(1 \to X \otimes X^*))$$

where we represent X^*, ev_X, and coev_X diagramatically by:

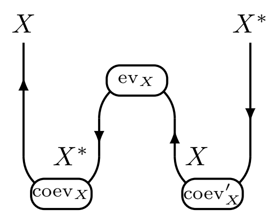


such that the *zig-zag relations* hold:



Exercise 2.10. Show that coev existing is actually a property of the pair (X^*, ev_X).

Hint: Suppose there exist $\mathrm{coev}, \mathrm{coev}' \in \mathcal{C}(1 \to X \otimes X^*)$*. Show that* $\mathrm{coev} = \mathrm{coev}'$ *by considering the following morphism:*



Exercise 2.11. Prove that the following are equivalent for a pair (X^*, ev_X).

- There exists $\mathrm{coev} \in \mathcal{C}(1 \to X \otimes X^*)$ satisfying the zigzag relations;
- the left and right Frobenius reciprocity maps ${}_A^B\lambda_X$ for $A, B \in \mathcal{C}$ are bijective, where

  [[Todo: Include diagram]]

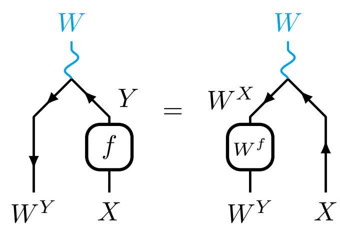
- (X^*, ev_X) is a power of unity such that the [[Check: left or right]] Frobenius reciprocity maps are bijective

From this we conclude that being a dual is a property of a power of unity (X^*, ev_X), i.e. dualizability is a stricter notion while powers of unity are more general.

2.3 Powers and morphisms

In a previous section, we defined what it means for an object W in a monoidal category $\mathcal{C}$ to be powered. In this section, we will investigate what happens at the level of morphisms.

Exercise 2.12. When $X, Y \in \mathcal{C}$ power W and $f \in \mathcal{C}(X \to Y)$, induce a map $W^f \in \mathcal{C}(W^Y \to W^X)$ determined by:



Verify that the map $f \mapsto W^f$ is injective if and only if W^Y is faithful. Similarly, verify $f \mapsto W^f$ is surjective if and only if W^Y is full.

Definition 2.13. When W is powered, $W^\bullet$ assembles into a functor, and we may interpret the previous exercise as follows.

- $W^\bullet$ is faithful when every power of W is faithful, and say $\mathcal{C}$ *faithfully* powers W;
- $W^\bullet$ full when every power of unity W is full, and say $\mathcal{C}$ *fully* powers W;
- $W^\bullet$ is essentially surjective when $\mathcal{C}$ admits logarithms of W.

In the case when $W = 1$, we denote this functor by $(-)^*$.

Definition 2.14. A *Grothendieck-Verdier category* $(\mathcal{C}, W)$ is a monoidal category with a choice of powered $W \in \mathcal{C}$ such that $W^\bullet$ is an equivalence. In the special case when $W = 1$, we say that $(\mathcal{C}, 1)$ or $\mathcal{C}$ is an *r-category.*[1]

Exercise 2.15. Show that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence[2] if and only if F is faithful, full, and essentially surjective.

Deduce that $(\mathcal{C}, W)$ is a GV-category if and only if $\mathcal{C}$ fully faithfully powers W and admits logarithms for W.

Hint: The first claim is a classic result from category theory, where one must use the Axiom of Choice to build a weak inverse $G \colon \mathcal{D} \to \mathcal{C}$.

Exercise 2.16. Let $\mathcal{C} = \mathsf{Vec}$, the category of all (not necessarily finite dimensional) vector spaces.

(a) Show that $(-)^*$ is faithful.

(b) On the other hand, show that $(-)^*$ is not full. More specifically, when is V^* full for $V \in \mathsf{Vec}$?

(c) When does V_* exist for $V \in \mathsf{Vec}$?

(d) Deduce that $\mathsf{Vec}_{f.d.}$, the category of finite dimensional vector spaces, is an r-category.

Hint: Recall that a vector space V is uniquely determined by its cardinality $\dim V$*. What is the cardinality*[3] *of V^* when* $\dim V < \infty$*? What goes wrong when V is infinite dimensional?*

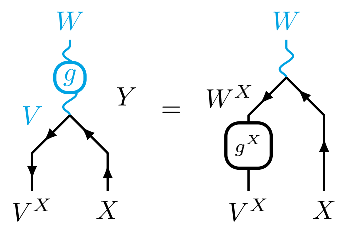
[1] Here the r stands for *rigid*, which is the usual term for a category with duality.
[2] Recall that $F \colon \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence* if there exists $G \colon \mathcal{D} \to \mathcal{C}$ with $G \circ F \cong \mathrm{id}_\mathcal{C}$ and $F \circ G \cong \mathrm{id}_\mathcal{D}$.
[3] This problem might require you to read up on infinite cardinals a bit!

Exercise 2.17. Let $\mathcal{C} = \mathsf{TVS}$, the category of topological vector spaces[4]. Recall that for any vector space V, one may equip V with the trivial topology to turn it into a topological vector space. Show that $(-)^*$ is not faithful.

Hint: Verify that $V^* = 0$ *for every* $V \in \mathsf{Vec}$.

We have now investigated how powers behave with morphisms in the top component $W^\bullet$. Let us now divert our attention to the bottom component $\bullet^X$.

Exercise 2.18. Suppose X powers $V, W \in \mathbb{C}$. For $g \in \mathcal{C}(V, W)$, induce a map $g^X \in \mathcal{C}(V^X \to W^X)$ determined by:



Verify that if X powers every $W \in \mathcal{C}$, then $\bullet^X$ assembles into a functor $\mathcal{C} \to \mathcal{C}$.

Note: This is also known as the covariant internal Yoneda embedding $\underline{Hom}(-, X)$. *Given our set-up, it is more awkward to determine when this one is faithful, full, or essentially surjective. But, as we will in general be focused on a specific choice of* W, *this will not be an issue!*

2.4 Powers and tensors

In this section, we divert our focus to how powers interact with the tensor product $\otimes$ on $\mathcal{C}$. This topic is a bit more delicate, as we will see.

Exercise 2.19. Suppose $\mathcal{C}$ admits all powers.

(a) Construct a map $(W^V)^X \to W^{X \otimes Y}$.

Hint: What relation is this map uniquely determined by?

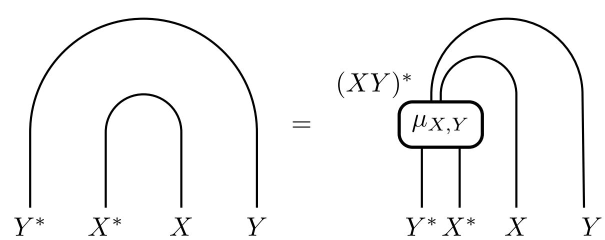
(b) On the other hand, construct a map $W^X \otimes W^* \otimes W^V \to W^{V \otimes X}$.

Hint: What relation is this map uniquely determined by?

(c) Suppose we are given a choice of "covector" $\varphi \in \mathcal{C}(W \to 1)$. Construct a "vector" $\varphi^* \in \mathcal{C}(1 \to W^*)$ and use it to build a map $W^X \otimes W^V \to W^{V \otimes X}$.

Note: In general, a choice of covector on W *is a bit awkward. One way to get around this is to consider the canonical case when* $W = 1$ *and* $\varphi = \mathrm{id}_1$.

Definition 2.20. If $X, Y \in \mathcal{C}$ admit powers of unity, Exercise 2.19 guarantees a canonical map $\mu_{X,Y} : Y^* \otimes X^* \to (X \otimes Y)^*$ which is uniquely determined by:



[4]Recall that a topological vector space is a vector space V equipped with a topology on V such that the operations $+$ and $\triangleright$ are continuous.

This map is in general *not* an isomorphism. In fact, the failure of μ being an isomorphism measures non-dualizability. We will make this precise in the following proposition.

Proposition 2.21. *Suppose $\mathcal{C}$ admits powers of unity. Then*

(a) If X is dualizable, then $\mu_{X,Y}$ is an isomorphism for every $Y \in \mathcal{C}$.

(b) If $\mu_{Y,X}$ is an isomorphism for every $Y \in \mathcal{C}$, then X^* is dualizable.

Exercise 2.22. In this Exercise, we give an outline for how to prove Proposition 2.21.

(a) [[Todo: Include hints]]

(b) [[Todo: Include hints]]

Remark 2.23. The data μ of these morphisms $\mu_{X,Y}$ for every $X, Y \in \mathcal{C}$ is what is known as a *tensorator* for the functor $(-)^*$. In a later section, we will describe tensorators for the functor $W^\bullet$ in the presence of a braiding on $\mathcal{C}$. Without such a braiding, we still have the following result for $W = 1$.

Exercise 2.24. [[Todo: Naturality, associativity, and unitality]]

Exercise 2.25 (Hard)**.** [[Question: If X^* and Y^* are faithful/full, is $\mu_{X,Y}$ mono/epic? When does this hold? When do the converses hold?]]

Remark 2.26. [[Todo: More generally, tensorators for $W^\bullet$ yield algebra structures on W, namely $\mu_{1,1} \colon W^1 \otimes W^1 \to W^1$. For the converse, it seems that we need a braiding. Not sure how to make this work without it]]