From C*-categories to W*-categories

Quantum Symmetry Seminar
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Abstract

In this talk, we will construct the free W*-category generated by a C*-category. This leads us to prove the Sherman-Takeda theorem along the way, which states that the double-dual of a C*-category agrees with the bicommutant of its universal representation.

Contents

1.1 C*-preliminaries

We begin by recalling the main objects of study.

Definition 1.1.1 († -category) — Let $\mathcal{A}$ be a linear category, i.e. a category where each hom set $\mathcal{A}(A \to B)$ is equipped with a vector space structure such that composition is bilinear. A dagger on $\mathcal{A}$ consists of a conjugate linear map $\dagger\colon \mathcal{A}(A \to B) \to \mathcal{A}(B \to A)$ for each pair of objects $A, B \in \mathcal{A}$ such that:

- $x^{\dagger\dagger} = x$ for every morphism x, and
- $(y \circ x)^\dagger = (A \xrightarrow{x} B \xrightarrow{y} C)^\dagger = A \xleftarrow{x^\dagger} B \xleftarrow{y^\dagger} C = x^\dagger \circ y^\dagger$ for each pair of composable morphisms.

A †-category is then a linear category together with a choice of dagger. For simplicity, we will further assume all †-categories admit orthogonal direct sums.

Remark 1.1.2. Note that being a †-category is not a *property* of a category, but extra *structure*.

Definition 1.1.3 (C*-category) — We say that a †-category is C* if the spectral norm, given on a morphism $x\colon A \to B$ by

$$\|x\| \coloneqq \sup\left\{|\lambda| \,\middle|\, (x^\dagger \circ x) - \lambda \mathrm{id}_A \text{ is not invertible}\right\},$$

is complete on each hom-space $\mathcal{A}(A \to B)$.

Remark 1.1.4. Note that being C* is a *property* of a †-category, and not extra *structure*.

Fact — The spectral norm on a †-category satisfies the so-called C*-identity

$$\|x\|^2 = \|x^\dagger \circ x\|.$$

Furthermore, any complete norm which satisfies the C*-identity must agree with the spectral norm.

Example 1.1.5 (C*-category) — The category Hilb of Hilbert spaces together with bounded linear maps (a.k.a. operators), where † is given by taking the adjoint of an operator. Furthermore, every norm-closed subcategory of Hilb forms a C*-category.

We now provide the structure preserving maps between †-categories and in particular C*-categories.

Definition 1.1.6 (†-functor) — We say that a functor $F\colon \mathcal{A} \to \mathcal{B}$ between †-categories $\mathcal{A}$ and $\mathcal{B}$ is a †-functor if it is linear on each hom-space $\mathcal{A}(A \to B)$ and †-preserving, i.e.

(†) $F(x)^\dagger = F(x^\dagger)$ for every morphism $x \in \mathcal{A}(A \to B)$.

Exercise 1.1.7. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a †-functor between C*-categories. Show that F is contractive, i.e. for every morphism $x \in \mathcal{A}(A \to B)$ we have $\|Fx\| \leq \|x\|$.

Exercise 1.1.8. Prove that C*-categories together with †-functors form a category, which we denote by $\mathrm{C}^*\mathsf{Cat}$.

Philosophy 1.1.9. As a digression, first recall how groups *are* symmetries in a sense. Indeed, Cayley's theorem tells us that every group can be realized as a group of symmetries (or permutations) on a set. In the very same sense, C*-categories *are* categories of operators on Hilbert spaces. This is made precise by the Gelfand-Naimark theorem and, in particular, the Gelfand-Naimark-Segal (GNS) construction.

Theorem (Gelfand-Naimark-Segal) — Every small C*-category $\mathcal{A}$ admits a monic[a] †-functor

$$\Upsilon\colon \mathcal{A} \to \mathsf{Hilb},$$

called the universal[b] representation of $\mathcal{A}$. This construction satisfies the following property:

- For objects $A, B \in \mathcal{A}$ and a functional $\varphi \in \mathcal{A}(A \to B)^*$, there exist $\xi \in \Upsilon(A)$ and $\eta \in \Upsilon(B)$ such that
$$\varphi(x) = \langle \Upsilon(x)\xi, \eta\rangle, \qquad \text{for all } x \in \mathcal{A}(A \to B).$$
In this case we will write $\varphi = \langle \Upsilon \cdot \xi, \eta\rangle$.

[a] By monic, we mean faithful and injective on objects
[b] Capital upsilon stands for Υniversal.

Definition 1.1.10 — We define the C*-category $\mathsf{GNS}(\mathcal{A})$ to be the image of Υ in Hilb, for which $\Upsilon\colon \mathcal{A} \to \mathsf{GNS}(\mathcal{A})$ is an isomorphism of C*-categories.

Exercise 1.1.11. Note that each $x \in \mathsf{Hilb}(\mathcal{H} \to \mathcal{K})$ induces a bounded sesquilinear form $B_x\colon \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ given by

$$B_x(\xi, \eta) := \langle x\xi, \eta \rangle.$$

Show that the map $x \mapsto B_x$ is an isometric bijective correspondence between operators in $\mathsf{Hilb}(\mathcal{H} \to \mathcal{K})$ and bounded sesquilinear forms $\mathcal{H} \times \mathcal{K} \to \mathbb{C}$.

Hint: First show the map $x \mapsto B_x$ is isometric, then show surjectivity using the Riesz representation theorem.

1.2 W*-preliminaries

Definition 1.2.1 (W*-category) **—** We say a C*-category $\mathcal{A}$ is W* if each hom-space $\mathcal{A}(A \to B)$ admits a predual $\mathcal{A}(A \to B)_*$, i.e.

$$\mathcal{A}(A \to B) \cong \mathcal{A}(A \to B)_*^*.$$

Remark 1.2.2. Note that being W* is a *property* of a †-category, and not extra *structure*.

Example 1.2.3 (W*-category) **—** One can show Hilb is a W*-category. Furthermore, any †-subcategory of Hilb which is WOT-closed[a] forms a W*-category.

[a] We say that $x_\lambda \to x$ WOT in $\mathsf{Hilb}(\mathcal{H} \to \mathcal{K})$ if $\langle x_\lambda \xi, \eta \rangle \to \langle x\xi, \eta \rangle$ for every $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$.

Fact — Addition and scalar action are weak*-continuous in a W*-category, whereas composition is only separately weak*-continuous.

Definition 1.2.4 (Normal †-functor) **—** We say a †-functor $F\colon \mathcal{A} \to \mathcal{B}$ between W*-categories is *normal* if it is weak*-continuous on each hom-space.

Exercise 1.2.5. Prove that we may identify $\mathcal{A}(A \to B)_*$ with the functionals on $\mathcal{A}(A \to B)$ which are weak*-continuous. Deduce that $F\colon \mathcal{A} \to \mathcal{B}$ is normal if and only if precomposition by F restricts to a map

$$F^*\colon \mathcal{B}(FA \to FB)_* \to \mathcal{A}(A \to B)_* \quad \text{for every } A, B \in \mathcal{A}.$$

Fact — Fully faithful †-functors between W*-categories are automatically normal.

Exercise 1.2.6. Show that W*-categories together with normal †-functors form a category, which we denote by $\mathrm{W}^*\mathsf{Cat}$.

Definition 1.2.7 (Bicommutant) — Given any C*-subcategory $\mathcal{A}$ of Hilb, we may take its WOT-closure in Hilb, a W*-category which we will call the bicommutant $\mathcal{A}''$ of $\mathcal{A}$.

Remark 1.2.8. The previous definition is not how one would normally define the bicommutant. For the purposes of this talk, we use the von Neumann bicommutant theorem to define the bicommutant. We do however encourage the reader to look up the real definitions and the von Neumann bicommutant theorem, as it is a great story.

We now introduce an incredibly useful result in operator algebras which, fortunately, also holds for operator categories.

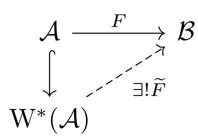
Theorem (Kaplansky Density Theorem) — For a subset $A \subset \mathsf{Hilb}(\mathcal{H} \to \mathcal{K})$, if $x \colon \mathcal{H} \to \mathcal{K}$ is in the SOT-closure[a] of A, then there exist $(x_\lambda) \subset A$ with $\|x_\lambda\| \leq \|x\|$ such that $x_\lambda \to x$ SOT.

[a]We say that $x_\lambda \to x$ SOT in $\mathsf{Hilb}(\mathcal{H} \to \mathcal{K})$ if $x_\lambda \xi \to x\xi$ in $\mathcal{K}$ for every $\xi \in \mathcal{H}$.

1.3 The double dual construction

For a C*-category $\mathcal{A}$, we wish to construct the enveloping W*-category $\mathrm{W}^*(\mathcal{A})$ together with a monic †-functor $\mathcal{A} \hookrightarrow \mathrm{W}^*(\mathcal{A})$, which satisfies the following universal property:

- For every †-functor $F \colon \mathcal{A} \to \mathcal{B}$ into a W*-category $\mathcal{B}$, there exists a unique normal extension making the following diagram commute:



Idea 1.3.1. We wish to find the "smallest" C*-category containing $\mathcal{A}$ which admits a predual. Recall that, for a Banach space A, there exists an organic inclusion $\mathrm{ev} \colon A \hookrightarrow A^{**}$ given by

$$\mathrm{ev}_a(\varphi) := \varphi(a) \quad \text{for } a \in A \text{ and } \varphi \in A^*.$$

Clearly A^{**} has a predual, namely A^*, and the Goldstine theorem tells us that A^{**} is "small" in a sense:

Theorem (Goldstine) — For a Banach space A, $\mathrm{ev}(A)$ is weak*-dense in A^{**}.

The idea is then to try to upgrade this double dual construction for C*-categories.

When $\mathcal{A}$ is a C*-category, consider the vector space enriched graph[1] $\mathcal{A}^{**}$ with vertices $\mathsf{Ob}\,\mathcal{A}$ and edges $\mathcal{A}^{**}(A \to B) := \mathcal{A}(A \to B)^{**}$. We define two so-called Arens compositions on $\mathcal{A}^{**}$, which equip $\mathcal{A}^{**}$ with the structure of a (linear) category.

Definition 1.3.2 (Arens composition) — For $\Phi \in \mathcal{A}^{**}(A \to B)$ and $\Psi \in \mathcal{A}^{**}(B \to C)$, we define the left and right Arens compositions $\circ_\ell$ and $\circ_r$ as follows:

(ℓ) For $\varphi \in \mathcal{A}^*(A \to C)$, we set $(\Psi \circ \Phi)(\varphi) := \Psi(\Phi \triangleright \varphi)$ where $\Phi \triangleright \varphi \in \mathcal{A}(B \to C)^*$ is given by:

- ($\triangleright$) For $b \in \mathcal{A}(B \to C)$, we set $(\Phi \triangleright \varphi)(b) := \Phi(\varphi \triangleleft b)$ where $\varphi \triangleleft b \in \mathcal{A}(A \to B)^*$ is given by:
  - ($\triangleleft$) For $a \in \mathcal{A}(A \to B)$, we set $(\varphi \triangleleft b)(a) := \varphi(b \circ a)$.

[1]By this, we simply mean a $(\mathsf{Ob}\,\mathcal{A} \times \mathsf{Ob}\,\mathcal{A})$-indexed collection of vector spaces.

More succinctly, $\Psi \circ_\ell \Phi$ is given by the following formula:

$$\begin{aligned}\Psi \circ_\ell \Phi &= \varphi \mapsto \Psi\Big(\Phi \triangleright \varphi\Big),\\ &= \varphi \mapsto \Psi\Big(b \mapsto \Phi\big(\varphi \triangleleft b\big)\Big),\\ &= \varphi \mapsto \Psi\Big(b \mapsto \Phi\big(a \mapsto \varphi(b \circ a)\big)\Big).\end{aligned}$$

(r) For $\varphi \in \mathcal{A}^*(A \to C)$, we set $(\Psi \circ_r \Phi)(\varphi) := \Phi(\varphi \triangleleft \Psi)$ where $\varphi \triangleleft \Psi \in \mathcal{A}(A \to B)^*$ is given by:

- ($\triangleleft$) For $a \in \mathcal{A}(A \to B)$, we set $(\varphi \triangleleft \Psi)(a) := \Psi(a \triangleright \varphi)$ where $a \triangleright \varphi \in \mathcal{A}(B \to C)^*$ is given by:
  - ($\triangleright$) For $b \in \mathcal{A}(B \to C)$, we set $(a \triangleright \varphi)(b) := \varphi(b \circ a)$.

More succinctly, $\Psi \circ_r \Phi$ is given by the following formula:

$$\begin{aligned}\Psi \circ_r \Phi &= \varphi \mapsto \Phi\Big(\varphi \triangleleft \Psi\Big),\\ &= \varphi \mapsto \Phi\Big(a \mapsto \Psi\big(a \triangleright \varphi\big)\Big),\\ &= \varphi \mapsto \Phi\Big(a \mapsto \Psi\big(b \mapsto \varphi(b \circ a)\big)\Big).\end{aligned}$$

Exercise 1.3.3. Prove that $\triangleright$ and $\triangleleft$ induce actions and deduce that $\circ_\ell$ and $\circ_r$ are bilinear.

Notice there exists an organic inclusion $\mathrm{ev}\colon \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ which acts as the identity on objects and is given on morphisms as follows:

- For $a \in \mathcal{A}(A \to B)$, we define $\mathrm{ev}(a) = \mathrm{ev}_a \in \mathcal{A}^{**}(A \to B)$ by

$$\mathrm{ev}_a(\varphi) := \varphi(a) \quad \text{for } \varphi \in \mathcal{A}(A \to B)^*.$$

Exercise 1.3.4. Show that ev is linear on each hom-space.

Lemma 1.3.5 — $\mathrm{ev}\colon \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ is a functor when we equip $\mathcal{A}^{**}$ with either Arens composition.

Proof of lemma. We will show our claim for the left Arens composition. Indeed, for composable morphisms x, y in $\mathcal{A}$, observe

$$\begin{aligned}\mathrm{ev}_y \circ_\ell \mathrm{ev}_x &= \varphi \mapsto \mathrm{ev}_y\Big(b \mapsto \mathrm{ev}_x\big(a \mapsto \varphi(b \circ a)\big)\Big)\\ &= \varphi \mapsto \mathrm{ev}_y\Big(b \mapsto \varphi(b \circ x)\Big)\\ &= \varphi \mapsto \varphi(y \circ x)\\ &= \mathrm{ev}_{y \circ x}.\end{aligned}$$

Hence, ev is composition preseving. For a morphism $\Phi \in \mathcal{A}^{**}(A \to B)$, we have

$$\begin{aligned}
\mathrm{ev}_{\mathrm{id}_B} \circ_\ell \Phi &= \varphi \mapsto \mathrm{ev}_{\mathrm{id}_B}\Big(b \mapsto \Phi\big(a \mapsto \varphi(b \circ a)\big)\Big) \\
&= \varphi \mapsto \Phi\big(a \mapsto \varphi(\mathrm{id}_B \circ a)\big) \\
&= \varphi \mapsto \Phi\big(a \mapsto \varphi(a)\big) \\
&= \varphi \mapsto \Phi(\varphi) \\
&= \Phi, \\
\Phi \circ_\ell \mathrm{ev}_{\mathrm{id}_A} &= \varphi \mapsto \Phi\Big(b \mapsto \mathrm{ev}_{\mathrm{id}_A}\big(a \mapsto \varphi(b \circ a)\big)\Big) \\
&= \varphi \mapsto \Phi\big(b \mapsto \varphi(b \circ \mathrm{id}_A)\big) \\
&= \varphi \mapsto \Phi\big(b \mapsto \varphi(b)\big) \\
&= \varphi \mapsto \Phi(\varphi) \\
&= \Phi.
\end{aligned}$$

Therefore $\mathrm{ev}_{\mathrm{id}_B} = \mathrm{id}_{\mathrm{ev}_B} = \mathrm{id}_B$ in $\mathcal{A}^{**}$. ■

Exercise 1.3.6. Prove that $\mathrm{ev}\colon \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ is a functor when we equip $\mathcal{A}^{**}$ with the right Arens composition.

Just as the composition in $\mathcal{A}$ induces structure on $\mathcal{A}^{**}$, the dagger in $\mathcal{A}$ induces the following mapping.

Definition 1.3.7 — We define a conjugate-linear contravariant map $\dagger\colon \mathcal{A}^{**}(A \to B) \to \mathcal{A}^{**}(B \to A)$ as follows:

(†) For $\Phi \in \mathcal{A}^{**}(A \to B)$, we define $\Phi^\dagger \in \mathcal{A}^{**}(B \to A)$ by

$$\Phi^\dagger(\varphi) \coloneqq \overline{\Phi(\varphi^\dagger)} \quad \text{for } \varphi \in \mathcal{A}(B \to A)^*,$$

where $\varphi^\dagger \in \mathcal{A}(A \to B)^*$ is given by $\varphi^\dagger(a) \coloneqq \overline{\varphi(a^\dagger)}$ for $a \in \mathcal{A}(A \to B)$.

More succinctly, $\Phi^\dagger$ is given by the following formula:

$$\Phi^\dagger = \varphi \mapsto \overline{\Phi\big(a \mapsto \overline{\varphi(a^\dagger)}\big)}.$$

Exercise 1.3.8. Show that † is conjugate-linear and $\Phi^{\dagger\dagger} = \Phi$ for every $\Phi \in \mathcal{A}^{**}(A \to B)$.

Exercise 1.3.9. Prove that † is weak*-continuous on each hom-space.

We now relate the Arens compositions via the following identity.

Lemma 1.3.10 — For $\Phi \in \mathcal{A}^{**}(A \to B)$ and $\Psi \in \mathcal{A}^{**}(B \to C)$,

$$(\Psi \circ_\ell \Phi)^\dagger = \Phi^\dagger \circ_r \Psi^\dagger.$$

Proof of lemma. Observe

$$
\begin{aligned}
(\Psi \circ_\ell \Phi)^\dagger &= \varphi \mapsto \overline{(\Psi \circ_\ell \Phi)\big(\varphi^\dagger\big)} \\
&= \varphi \mapsto \overline{\Psi\Big(b \mapsto \Phi\big(a \mapsto \varphi^\dagger(b \circ a)\big)\Big)} \\
&= \varphi \mapsto \overline{\Psi\Big(b \mapsto \Phi\big(a \mapsto \overline{\varphi(a^\dagger \circ b^\dagger)}\big)\Big)} \\
&= \varphi \mapsto \overline{\Psi\Big(b \mapsto \Phi\big((b^\dagger \triangleright \varphi)^\dagger\big)\Big)} \\
&= \varphi \mapsto \overline{\Psi\Big(b \mapsto \overline{\Phi^\dagger(b^\dagger \triangleright \varphi)}\Big)} \\
&= \varphi \mapsto \Psi^\dagger\Big(b \mapsto \Phi^\dagger(b \triangleright \varphi)\Big) \\
&= \Phi^\dagger \circ_r \Psi^\dagger.
\end{aligned}
$$

■

1.4 The main theorem

From our previous result, we see that if the Arens compositions on $\mathcal{A}^{**}$ agree then:

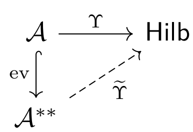
- $\mathcal{A}^{**}$ forms a †-category, and
- $\mathrm{ev}\colon \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ is then a †-functor since, for every morphism x in $\mathcal{A}$, we have

$$
\mathrm{ev}_x^\dagger = \varphi \mapsto \overline{\mathrm{ev}_x\left(a \mapsto \overline{\varphi(a^\dagger)}\right)} = \varphi \mapsto \overline{\overline{\varphi(x^\dagger)}} = \mathrm{ev}_{x^\dagger}\,.
$$

We will now show this is always the case for C*-categories, after which we will prove $\mathcal{A}^{**}$ satisfies the universal property required of the W*-envelope of $\mathcal{A}$.

Theorem 1.4.1. For a C*-category $\mathcal{A}$, the left and right Arens compositions on $\mathcal{A}^{**}$ coincide. Furthermore, these serve to equip $\mathcal{A}^{**}$ with the structure of a W*-category.

Proof. Without loss of generality, we may assume $\mathcal{A}$ is small as compositions coincide if and only if they agree on each small subcategory. For objects $A, B \in \mathcal{A}$ and vectors $\xi \in \Upsilon(A)$, $\eta \in \Upsilon(B)$, recall we denote the functional $x \mapsto \langle \Upsilon(x)\xi, \eta\rangle$ in $\mathcal{A}(A \to B)^*$ by $\langle \Upsilon \cdot \xi, \eta\rangle$. We will extend the universal representation $\Upsilon\colon \mathcal{A} \to \mathsf{Hilb}$ along ev



such that for each $\Phi \in \mathcal{A}^{**}(A \to B)$ we have

$$
\langle \widetilde{\Upsilon}(\Phi)\xi, \eta\rangle = \Phi(\langle \Upsilon \cdot \xi, \eta\rangle) \quad \text{for all } \xi \in \Upsilon(A) \text{ and } \eta \in \Upsilon(B).
$$

Indeed, notice $\Phi(\langle \Upsilon \cdot \xi, \eta\rangle)$ is linear in ξ, antilinear in η and

$$
|\Phi(\langle \Upsilon \cdot \xi, \eta\rangle)| \le \|\Phi\| |\langle \Upsilon \cdot \xi, \eta\rangle| \le \|\Phi\| \|\xi\| \|\eta\|.
$$

By Exercise 1.1.11, there exists a unique operator $\widetilde{\Upsilon}(\Phi)\colon \Upsilon(A) \to \Upsilon(B)$ with $\|\widetilde{\Upsilon}(\Phi)\| \le \|\Phi\|$ satisfying the desired identity. For $x \in \mathcal{A}(A \to B)$, we see that

$$
\langle \widetilde{\Upsilon}(\mathrm{ev}_x)\xi, \eta\rangle = \mathrm{ev}_x(\langle \Upsilon \cdot \xi, \eta\rangle) = \langle \Upsilon(x)\xi, \eta\rangle \qquad \text{for all } \xi \in \Upsilon(A) \text{ and } \Upsilon(B),
$$

which implies $\widetilde{\Upsilon}(\mathrm{ev}_x) = \Upsilon(x)$. Thus $\widetilde{\Upsilon} \circ \mathrm{ev} = \Upsilon$.

We now inspect various properties of the map $\widetilde{\Upsilon}$:

(∘) For $a \in \mathcal{A}(A \to B)$, $b \in \mathcal{A}(B \to C)$, and $\xi \in \Upsilon(A)$, $\eta \in \Upsilon(C)$, we have

$$\langle \Upsilon \cdot \xi, \eta \rangle \triangleleft b = \langle \Upsilon \cdot \xi, \Upsilon(b)^\dagger \eta \rangle, \quad \text{and} \quad a \triangleright \langle \Upsilon \cdot \xi, \eta \rangle = \langle \Upsilon \cdot \Upsilon(a)\xi, \eta \rangle.$$

Indeed,

$$\begin{aligned} \big(\langle \Upsilon \cdot \xi, \eta \rangle \triangleleft b\big)(a) &= \langle \Upsilon \cdot \xi, \eta \rangle(ba) = \langle \Upsilon(ba)\xi, \eta \rangle \\ &= \langle \Upsilon(b)\Upsilon(a)\xi, \eta \rangle = \langle \Upsilon(a)\xi, \Upsilon(b)^\dagger \eta \rangle \\ &= \langle \Upsilon \cdot \xi, \Upsilon(b)^\dagger \eta \rangle(a), \end{aligned}$$

$$\begin{aligned} \big(a \triangleright \langle \Upsilon \cdot \xi, \eta \rangle\big)(b) &= \langle \Upsilon \cdot \xi, \eta \rangle(ba) = \langle \Upsilon(ba)\xi, \eta \rangle \\ &= \langle \Upsilon(b)\Upsilon(a)\xi, \eta \rangle = \langle \Upsilon \cdot \Upsilon(a)\xi, \eta \rangle(b). \end{aligned}$$

(∘) Thus, for $\Phi \in \mathcal{A}(A \to B)$ and $\Psi \in \mathcal{A}(B \to C)$, we have

$$\Phi \triangleright \langle \Upsilon \cdot \xi, \eta \rangle = \langle \Upsilon \cdot \widetilde{\Upsilon}(\Phi)\xi, \eta \rangle \quad \text{and} \quad \langle \Upsilon \cdot \xi, \eta \rangle \triangleleft \Psi = \langle \Upsilon \cdot \xi, \widetilde{\Upsilon}(\Psi)^\dagger \eta \rangle$$

Indeed,

$$\begin{aligned} \big(\Phi \triangleright \langle \Upsilon \cdot \xi, \eta \rangle\big)(b) &= \Phi\big(\langle \Upsilon \cdot \xi, \eta \rangle \triangleleft b\big) = \Phi\big(\langle \Upsilon \cdot \xi, \Upsilon(b)^\dagger \eta \rangle\big) \\ &= \langle \widetilde{\Upsilon}(\Phi)\xi, \Upsilon(b)^\dagger \eta \rangle = \langle \Upsilon(b)\widetilde{\Upsilon}(\Phi)\xi, \eta \rangle \\ &= \big(\langle \Upsilon \cdot \widetilde{\Upsilon}(\Phi)\xi, \eta \rangle\big)(b), \end{aligned}$$

$$\begin{aligned} \big(\langle \Upsilon \cdot \xi, \eta \rangle \triangleleft \Psi\big)(a) &= \Psi\big(a \triangleright \langle \Upsilon \cdot \xi, \eta \rangle\big) = \Psi\big(\langle \Upsilon \cdot \Upsilon(a)\xi, \eta \rangle\big) \\ &= \langle \widetilde{\Upsilon}(\Psi)\Upsilon(a)\xi, \eta \rangle = \langle \Upsilon(a)\xi, \widetilde{\Upsilon}(\Psi)^\dagger \eta \rangle \\ &= \big(\langle \Upsilon \cdot \xi, \widetilde{\Upsilon}(\Psi)^\dagger \eta \rangle\big)(a). \end{aligned}$$

(∘) Therefore

$$\widetilde{\Upsilon}(\Psi \circ_\ell \Phi) = \widetilde{\Upsilon}(\Psi) \circ \widetilde{\Upsilon}(\Phi) = \widetilde{\Upsilon}(\Psi \circ_r \Phi).$$

Indeed,

$$\begin{aligned} \langle \widetilde{\Upsilon}(\Psi \circ_\ell \Phi)\xi, \eta \rangle &= (\Psi \circ_\ell \Phi)(\langle \Upsilon \cdot \xi, \eta \rangle) = \Psi(\Phi \triangleright \langle \Upsilon \cdot \xi, \eta \rangle) \\ &= \Psi(\langle \Upsilon \cdot \widetilde{\Upsilon}(\Phi)\xi, \eta \rangle) = \langle \widetilde{\Upsilon}(\Psi)\widetilde{\Upsilon}(\Phi)\xi, \eta \rangle, \end{aligned}$$

$$\begin{aligned} \langle \widetilde{\Upsilon}(\Psi \circ_r \Phi)\xi, \eta \rangle &= (\Psi \circ_r \Phi)(\langle \Upsilon \cdot \xi, \eta \rangle) = \Phi(\langle \Upsilon \cdot \xi, \eta \rangle \triangleleft \Psi) \\ &= \Phi(\langle \Upsilon \cdot \xi, \widetilde{\Upsilon}(\Psi)^\dagger \eta \rangle) = \langle \widetilde{\Upsilon}(\Phi)\xi, \widetilde{\Upsilon}(\Psi)^\dagger \eta \rangle \\ &= \langle \widetilde{\Upsilon}(\Psi)\widetilde{\Upsilon}(\Phi)\xi, \eta \rangle. \end{aligned}$$

Thus, in order to show $\circ_\ell = \circ_r$, it suffices to show that $\widetilde{\Upsilon}$ is faithful. Suppose $\Phi \in \mathcal{A}^{**}(A \to B)$ satisfies $\widetilde{\Upsilon}(\Phi) = 0$ and consider some $\varphi \in \mathcal{A}(A \to B)^*$. As a property of the universal representation of $\mathcal{A}$, we know there exist $\xi \in \Upsilon(A), \eta \in \Upsilon(B)$ such that

$$\varphi = \langle \Upsilon \cdot \xi, \eta \rangle$$

But now observe

$$\Phi(\varphi) = \Phi(\langle \Upsilon \cdot \xi, \eta \rangle) = \langle \widetilde{\Upsilon}(\Phi)\xi, \eta \rangle = 0.$$

Hence $\Phi = 0$, $\widetilde{\Upsilon}$ is faithful, and we conclude that the Arens compositions agree on $\mathcal{A}^{**}$.

We will now prove the †-category $\mathcal{A}^{**}$ together with the composition $\circ := \circ_\ell = \circ_r$ is C*, and by construction it will immediately follow that $\mathcal{A}^{**}$ is W*. Due to the following argument, it suffices to show $\widetilde{\Upsilon}$ is †-preserving and is isometric[2]:

$$\|\Phi\| = \|\widetilde{\Upsilon}(\Phi)\| = \|\widetilde{\Upsilon}(\Phi)^\dagger \circ \Upsilon(\Phi)\|^{1/2} = \|\widetilde{\Upsilon}(\Phi^\dagger \circ \Phi)\|^{1/2} = \|\Phi^\dagger \circ \Phi\|^{1/2}.$$

To see that $\widetilde{\Upsilon}$ is †-preserving:

(†) Notice $\langle \Upsilon \cdot \xi, \eta\rangle^\dagger = \langle \Upsilon \cdot \eta, \xi\rangle$ since

$$\big(\langle \Upsilon \cdot \xi, \eta\rangle^\dagger\big)(a) = \overline{\langle \Upsilon \cdot \xi, \eta\rangle(a^\dagger)} = \overline{\langle \Upsilon(a^\dagger)\xi, \eta\rangle} = \langle \Upsilon(a)\eta, \xi\rangle = \big(\langle \Upsilon \cdot \eta, \xi\rangle\big)(a).$$

(†) Therefore $\widetilde{\Upsilon}(\Phi^\dagger) = \widetilde{\Upsilon}(\Phi)^\dagger$ since

$$\langle \widetilde{\Upsilon}(\Phi^\dagger)\xi, \eta\rangle = \Phi^\dagger(\langle \Upsilon \cdot \xi, \eta\rangle) = \overline{\Phi(\langle \Upsilon \cdot \xi, \eta\rangle^\dagger)} = \overline{\Phi(\langle \Upsilon \cdot \eta, \xi\rangle)} = \overline{\langle \widetilde{\Upsilon}(\Phi)\eta, \xi\rangle} = \langle \widetilde{\Upsilon}(\Phi)^\dagger \xi, \eta\rangle.$$

Before checking that $\widetilde{\Upsilon}$ is isometric, we will verify it is weak*-WOT continuous. Indeed, suppose $\Phi_\lambda \to \Phi$ weak* in $\mathcal{A}^{**}(A \to B)$ and observe

$$\langle \widetilde{\Upsilon}(\Phi_\lambda)\xi, \eta\rangle = \Phi_\lambda(\langle \Upsilon \cdot \xi, \eta\rangle) \to \Phi(\langle \Upsilon \cdot \xi, \eta\rangle) = \langle \widetilde{\Upsilon}(\Phi)\xi, \eta\rangle, \quad \text{for all } \xi \in \Upsilon(A) \text{ and } \eta \in \Upsilon(B).$$

Hence $\widetilde{\Upsilon}(\Phi_\lambda) \to \widetilde{\Upsilon}(\Phi)$ WOT.

To see that $\widetilde{\Upsilon}$ is isometric, consider some $\Phi \in \mathcal{A}^{**}(A \to B)$.

($\leq$) We already saw that $\|\widetilde{\Upsilon}(\Phi)\| \leq \|\Phi\|$ by construction.

($\geq$) Let $\varepsilon > 0$. As a property of the universal representation of $\mathcal{A}$, we know there exist $\xi \in \Upsilon(A), \eta \in \Upsilon(B)$ with $\|\langle \Upsilon \cdot \xi, \eta\rangle\| = 1$ such that

$$|\langle \widetilde{\Upsilon}(\Phi)\xi, \eta\rangle| = |\Phi(\langle \Upsilon \cdot \xi, \eta\rangle)| \geq \|\Phi\| - \varepsilon.$$

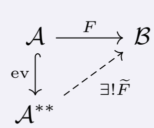
Since $\widetilde{\Upsilon}$ is weak*-WOT continuous, $\widetilde{\Upsilon}(\Phi) \in \overline{\operatorname{Im}\Upsilon}^{\text{WOT}} = \overline{\operatorname{Im}\Upsilon}^{\text{SOT}} \subseteq \mathsf{Hilb}(\Upsilon A \to \Upsilon B)$. By the Kaplansky density theorem, there exist $(x_\lambda) \subset \mathcal{A}(A \to B)$ with $\|x_\lambda\| = \|\Upsilon(x_\lambda)\| \leq \|\widetilde{\Upsilon}(\Phi)\|$ such that $\Upsilon(x_\lambda) \to \widetilde{\Upsilon}(\Phi)$ SOT. Observe

$$\|\widetilde{\Upsilon}(\Phi)\| \geq \|x_\lambda\| \geq |\langle \Upsilon(x_\lambda)\xi, \eta\rangle| \to |\langle \widetilde{\Upsilon}(\Phi)\xi, \eta\rangle| \geq \|\Phi\| - \varepsilon.$$

Since $\varepsilon \geq 0$ was arbitrary, $\|\widetilde{\Upsilon}(\Phi)\| \geq \|\Phi\|$.

We conclude that $\mathcal{A}^{**}$ is a W*-category. □

Universal Property 1.4.2 — For every †-functor $F\colon \mathcal{A} \to \mathcal{B}$ into a W*-category $\mathcal{B}$, there exists a unique normal extension $\widetilde{F}\colon \mathcal{A}^{**} \to \mathcal{B}$ making the following diagram commute.



[2] We equip $\mathcal{A}^{**}$ with the standard operator norm for functionals, which is complete on each hom-space. By showing this norm satisfies the C*-identity, this implies the spectral norm agrees with the operator norm, and is hence complete.

Verification. Since each hom set $\mathcal{A}(A \to B)$ is weak* dense in $\mathcal{A}^{**}(A \to B)$, we may try to extend F to $\mathcal{A}^{**}$ by continuity.

Indeed, suppose $x_\lambda \to x$ weakly in $\mathcal{A}(A \to B)$, which is equivalent to $\mathrm{ev}_{x_\lambda} \to \mathrm{ev}_x$ weak* in $\mathcal{A}^{**}(A \to B)$. Let $\psi \in \mathcal{B}(FA \to FB)_*$ be a weak*-continuous functional and consider the functional $F^*\psi \in \mathcal{A}(A \to B)^*$ given by $F^*\psi(a) := \psi(Fa)$. Then

$$\psi(Fx_\lambda) = F^*\psi(x_\lambda) \to F^*\psi(x) = \psi(Fx),$$

from which we conclude $Fx_\lambda \to Fx$ weak*. Therefore, we may extend F by continuity to a map $\widetilde{F}\colon \mathcal{A}^{**} \to \mathcal{B}$, which is normal by construction. We also have $\widetilde{F}(\mathrm{ev}_x) = F(x)$, and this implies that $\widetilde{F}$ preserves identities. To see that F preserves compositions, consider $\Phi \in \mathcal{A}^{**}(A \to B)$ and $\Psi \in \mathcal{A}^{**}(B \to C)$. Choose $(x_\lambda) \subset \mathcal{A}(A \to B)$ and $(y_\mu) \subset \mathcal{A}(B \to C)$ such that $x_\lambda \to \Phi$ and $y_\mu \to \Phi$ weak*. Using the fact that composition in W*-categories is separately normal, we obtain

$$\begin{aligned}\widetilde{F}(\Psi) \circ \widetilde{F}(\Phi) &= \lim_\mu F(y_\mu) \circ \widetilde{F}(\Phi) = \lim_\mu \lim_\lambda F(y_\mu) \circ F(x_\lambda) \\ &= \lim_\mu \lim_\lambda F(y_\mu \circ x_\lambda) = \lim_\mu \widetilde{F}(y_\mu \circ \Phi) = \widetilde{F}(\Psi \circ \Phi).\end{aligned}$$

Similar arguments using the linearity and †-preservation of F together with the continuity of addition and scalar action, and Exercise 1.3.9 reveal that $\widetilde{F}$ is also linear and †-preserving. We conclude that $\widetilde{F}\colon \mathcal{A}^{**} \to \mathcal{B}$ is a normal †-functor, whose uniqueness is immediate by the local weak* density of $\mathcal{A}$ in $\mathcal{A}^{**}$. ■

1.5 Corollaries

Corollary 1.5.1 (Sherman-Takeda for C*-categories) — For a small C*-category $\mathcal{A}$, the †-functor

$$\widetilde{\Upsilon}\colon \mathcal{A}^{**} \to \mathsf{GNS}(\mathcal{A})''$$

constructed in Theorem 1.4.1 is an isomorphism of W*-categories extending $\Upsilon\colon \mathcal{A} \to \mathsf{GNS}(A)$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \Upsilon\ } & \mathsf{GNS}(\mathcal{A}) \\ {\scriptstyle \mathrm{ev}}\big\downarrow & & \big\downarrow \\ \mathcal{A}^{**} & \xrightarrow[\ \widetilde{\Upsilon}\ ]{} & \mathsf{GNS}(\mathcal{A})'' \end{array}$$

Proof of Corollary. We already know $\widetilde{\Upsilon}\colon \mathcal{A}^{**} \to \mathsf{Hilb}$ is weak*-WOT continuous, so that $\operatorname{Im}\widetilde{\Upsilon} \subseteq \mathsf{GNS}(\mathcal{A})''$. To show that $\widetilde{\Upsilon}\colon \mathcal{A}^{**} \to \mathsf{GNS}(\mathcal{A})''$ is full, consider some $T\colon \mathsf{Hilb}(\Upsilon(A) \to \Upsilon(B))$. Since every functional $\varphi \in \mathcal{A}(A \to B)^*$ is of the form

$$\varphi = \langle \Upsilon \cdot \xi, \eta \rangle, \quad \text{for some } \xi \in \Upsilon(A) \text{ and } \eta \in \Upsilon(B),$$

we may define $\Phi_T\colon \mathcal{A}^{**}(A \to B)$ by

$$\Phi_T(\langle \Upsilon \cdot \xi, \eta \rangle) = \langle T\xi, \eta \rangle \quad \text{for } \xi \in \Upsilon(A) \text{ and } \eta \in \Upsilon(B).$$

To see that Φ_T is well-defined, suppose $\langle \Upsilon \cdot \xi, \eta \rangle = \langle \Upsilon \cdot \xi', \eta' \rangle$. Then choose $(x_\lambda) \subset \mathcal{A}(A \to B)$ such that $\Upsilon(x_\lambda) \to T$ WOT and observe

$$\Phi_T(\langle \Upsilon \cdot \xi, \eta \rangle) = \langle T\xi, \eta \rangle = \lim \langle \Upsilon(x_\lambda)\xi, \eta \rangle = \lim \langle \Upsilon(x_\lambda)\xi', \eta' \rangle = \Phi_T(\langle \Upsilon \cdot \xi', \eta' \rangle).$$

Hence Φ_T is well-defined. Now note that $\widetilde{\Upsilon}(\Phi_T) = T$ as

$$\langle \widetilde{\Upsilon}(\Phi_T)\xi, \eta \rangle = \Phi_T(\langle \Upsilon \cdot \xi, \eta \rangle) = \langle T\xi, \eta \rangle \quad \text{for all } \xi \in \Upsilon(A) \text{ and } \eta \in \Upsilon(B).$$

Thus $\widetilde{\Upsilon}$ is full. Since $\widetilde{\Upsilon}$ is bijective on objects and faithful by Theorem 1.4.1, we conclude that $\widetilde{\Upsilon}$ is an isomorphism of W*-categories. ■

1.6 Future work

(1) We have actually already used the W*-completion of a C*-category to construct the W*-tensor product $\underset{\max}{\overline{\otimes}}$ of W*-categories.

(2) In the not-so-distant future, we will be upgrading these constructions for C*-2-categories and W*-2-categories, in order to obtain a symmetric closed monoidal category $(\mathrm{W}^*\mathsf{2Cat}, \underset{\max}{\overline{\boxtimes}})$ of W*-2-categories.

(3) By enriching over $\mathrm{W}^*\mathsf{2Cat}$, we end up with the correct notion of a W*-Gray-category, the semi-strict version of a W*-3-category. The main goal of our future work is to show this coherence theorem.

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