Research Statement

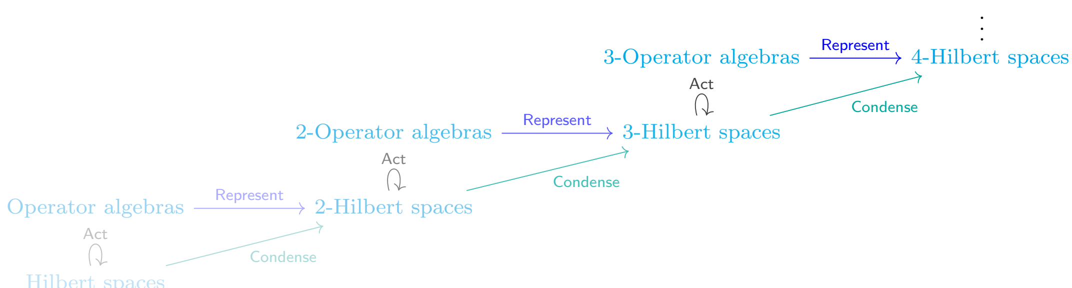
Giovanni Ferrer 2025

1. INTRODUCTION

Functional analysis. The mathematical foundations for *quantum mechanics* lead to the development of *functional analysis*, where one can encode the state space of a system with a *Hilbert space* H and observables with an *operator algebra* $A \subset B(H)$ of operators acting on H.

Higher Hilbert spaces. Functional analysis has since flourished in its own right as a field of mathematics. One particular approach in studying a given operator algebra A is to examine its *representations* $A \xrightarrow{\pi} B(H_\pi)$, i.e. the possible ways A can *act* on different Hilbert spaces H_π. The mathematical structure $\mathsf{Rep}(A)$ consisting of all such representations of A has many properties analogous to that of Hilbert spaces, while being much more mathematically rich. In this sense, these structures are *higher* Hilbert spaces, or *2-Hilbert spaces*.

Higher operator algebras. Just as before, consider operators that act on a 2-Hilbert space $\mathcal{H} = \mathsf{Rep}(A)$. These are higher quantum symmetries forming the structure of a *higher* operator algebra $\mathrm{Bim}(A)$. These motifs fit into the following research program, which has been worked out in the *finite dimensional* case. Indeed, the representations $\mathsf{Rep}(\mathcal{A})$ of a 2-operator algebra $\mathcal{A}$, i.e. the ways $\mathcal{A}$ can *act* on 2-Hilbert spaces $\mathcal{A} \xrightarrow{F} B(\mathcal{H}_F)$, then form an *even higher* Hilbert space known as a *3-Hilbert space*. One may thus continue constructing a staircase of higher and higher Hilbert spaces and operator algebras, that is, a theory of *higher functional analysis*.



The focus of my research is to develop the theory of higher functional analysis.

2. IMPORTANCE

The framework of higher functional analysis provides a powerful lens to systematically study quantum systems, using abstract mathematical concepts to obtain physical applications. From advancing our understanding of topological phenomena to supporting robust quantum computation, these structures illuminate the interplay between mathematics and physics.

Quantum symmetry. The classical symmetries of a set S are captured by the corresponding group of symmetries G, which act on S through a map $G \to \mathrm{End}(S)$. In the same way, tensor categories $\mathcal{A}$ act on 2-Hilbert spaces as so-called *quantum symmetries*. In particular for an operator algebra A, its quantum symmetries are captured by a $\otimes$-functor $\mathcal{A} \to \mathsf{Bim}(A)$. One important example central to subfactor theory is the *standard invariant* of a finite index II_1-subfactor $A \subseteq B$, which is generated by the action of ${}_A L^2 B_B \in \mathrm{Bim}(A \oplus B)$.

Topological order. Another motivating force driving us up this mathematical staircase is the connection between higher operator algebras and *quantum computation*. Higher operator algebras provide an algebraic structure for excitations in so-called *topologically ordered* systems, capturing the fusion, braiding, and commutation relations of the quasi-particle excitations (or anyons) in your material. In particular, the excitations or defects of a $(2+1)D$ topological order gives rise to a 3-category which is expected to form a 4-Hilbert space.

We recall that in classical systems, "order" refers to patterns like the alignment of spins in a ferromagnet, where the spins break rotational symmetry and align in a particular direction. This type of order can be captured by simple order parameters, such as magnetization. In many-body quantum systems, however, order can be more complex due to non-local quantum correlations, such as entanglement, where distant parts of the system become strongly correlated. These correlations are not visible at the level of an individual particle yet become evident at the global scale, distinguishing quantum order from classical order. Further key features of these quantum systems are the presence of exotic excitations, such as anyons in two dimensional materials.

Quantum order is remarkably robust, often persisting despite local perturbations. This is due to the fact that in systems with topological order, quantum states are determined by global properties, such as the system's shape or topology, rather than local details. This stability makes topological order especially valuable for quantum computing, as it offers protection against errors.

Topological Quantum field theory. Higher Hilbert spaces are also expected to serve as the appropriate receptacles for fully-extended unitary topological quantum field theories (TQFTs). Since Lurie's work on *cobordism hypothesis*, it is known that these TQFTs are uniquely determined by what they assign to the point $*$. Henriques showed that Reshetikhin-Turaev theories, e.g. Chern-Simons theories, are fully extended. Moreover, he then showed that the mathematical structure of what such a TQFT assigns to $*$ forms a *higher operator algebra*.

We note that TQFTs not only hold physical significance but also yield invariants with deep mathematical applications in knot theory, differential geometry, and algebraic topology. The importance of such invariants is underscored by the recognition of mathematicians like Vaughan Jones (1990), Edward Witten (1990), and Maxim Kontsevich (1998), who have been awarded Fields Medals for their groundbreaking contributions to knot invariants and related fields.

3. PAST ACCOMPLISHMENTS

In my past work, I have focused on the finite dimensional theory of *higher functional analysis*.

Manifestly unitary higher Hilbert spaces. In joint work with Chen, Hungar, Penneys, and Sanford, we define *finite dimensional* 2-operator algebras and 3-Hilbert spaces. We then describe a formal process of constructing 3-Hilbert spaces from 2-Hilbert spaces from Hilbert spaces, known as *unitary condensation*. In particular, this is a *manifestly unitary* formulation of the work of Gaiotto and Johnson-Freyd for higher vector spaces.

Foundations for operator algebraic tricategories. In single author work, we provide foundational results for the theory of 3-operator algebras and 4-Hilbert spaces. More technically, we define and prove both coherence and concreteness results for so-called *operator algebraic tricategories*. We then show there is an operator algebraic tricategory $E_2^\dagger(\mathsf{Hilb})$ consisting of commutative operator algebras and their 3-categorical quantum symmetries.

Dagger n-categories. In joint work with Hungar, Johnson-Freyd, Krulewski, Muller, Nivedita, Penneys, Reutter, Scheimbauer, Stehouwer, and Vuppulury, we establish a theory of *dagger n-categories*, one of the key ingredients in the future formulation of n-Hilbert spaces and n-operator algebras for arbitrary $n = 1, 2, 3, \ldots$

Classifying module categories for generalized Temperley-Lieb-Jones $*$-2-categories. In joint work with Hernandez-Palomares, we provide a universal construction for a family of finite dimensional 3-Hilbert spaces related to the Jones polynomial and classify their *representations*.

Gray-categories model algebraic tricategories. In single author work, we prove the Grothendieck's *homotopy hypothesis*, as popularized by Baez, for *algebraic trigroupoids*.

4. CURRENT WORK

My current research is focused on furthering these results in the direction of homotopy theory and higher category theory.

Bases for 3-Hilbert spaces and higher unitary duality. Building on work from *Manifestly unitary higher Hilbert spaces*, joint with Hungar, Penneys, and Wesley. We define and characterize unitary dual 2-functors on finite dimensional 3-operator algebras.

The homotopy 3-type of commutative operator algebras. Building on work from *Foundations for operator algebraic tricategories*, joint with Faurot. We describe an equivalent topological model to the Morita operator algebraic tricategory $E_2^\dagger(\mathsf{Hilb})$ of commutative operator algebras and their quantum symmetries, and provide a tricategorical Gelfand duality, simultaneously extending classical Gelfand duality, the Serre-Swan theorem, and the Dauns-Hoffman theorem. We then describe the underlying homotopy 3-type of this 3-category in terms of its Postnikov data.

Models for higher Hilbert spaces as homotopy fixed points. Building on work from *Dagger n-categories*, joint with Müller, Penneys, and Stehouwer. We systematically describe the different theories of operator algebras (C*-algebras, W*-algebras, H*-algebras) in terms of homotopy fixed points for different subgroups of $O(2)$.

5. FUTURE RESEARCH

One of my goals is to develop the infinite-dimensional side of this higher functional analysis programme, e.g. by describing a higher analogue of Gelfand duality, spectral theory, and the functional calculus.

Recall that classical Gelfand duality provides a foundational correspondence between commutative C*-algebras and compact Hausdorff spaces. This result can be aimed at a particular operator $x \in B(H)$ on a Hilbert space H. Namely, if x is normal, the commutative C*-algebra $C^*(x)$ generated by x corresponds to functions $C(\text{Spec}(x))$ on the spectrum $\text{Spec}(x)$ of x. This correspondence then allows us to perform a *functional calculus* for x. On the other hand, there are various instantiations of Tannaka-reconstruction which afford correspondences between symmetric tensor categories and (super)groupoids under suitable conditions. In particular, we expect a higher Gelfand duality theorem unifying these two motifs.

Higher Gelfand duality. Under suitable conditions, describe a concrete correspondence between symmetric C*-tensor categories and compact (super)groupoids.

Aiming such a result at a bimodule ${}_AX_A \in \text{Bim}(A)$ over an operator algebra A would then yield a higher spectral theorem when viewing ${}_AX_A$ as an operator on the 2-Hilbert space $\mathsf{Rep}(A)$. Indeed, such a bimodule ${}_AX_A$ equipped with a symmetric braiding with ${}_AX_A$ and ${}_A\overline{X}_A$ would generate a symmetric C*-tensor category $C^*({}_AX_A)$, which would then correspond to representations $\mathsf{Rep}(\text{Spec}({}_AX_A))$ on a compact (super)groupoid $\text{Spec}({}_AX_A)$.

Higher spectral theory. Determine what information about ${}_AX_A$ this higher spectrum $\text{Spec}({}_AX_A)$ encodes. Moreover, determine whether this construction depends on the choice of symmetric braiding and, if so, in what way.

Higher functional calculus. Employ this representation-theoretic calculus to solve higher functional analysis problems.

6. BROADER IMPACTS AND OUTREACH

Having extensive experience supervising undergraduate research, mentoring graduate students, and organizing seminars, I will continue these efforts as a part of my post-doctoral work.

REU Research. Joint work with Poudel, mentored undergraduate student Lu during our quantum symmetries REU summer program. We established a skein-theoretic immersed curve model for $\mathsf{Rep}(\mathcal{U}_q(\mathfrak{sl}_4))$, the representation theory of the quantum group associated to SL_4. In particular, this project relates $\mathsf{Rep}(\mathcal{U}_q(\mathfrak{sl}_4))$ to work of Kauffmand and Vogel on the HOMFLY-PT knot invariant.

Undergraduate Research. Mentoring undergraduate student Circele as part of the CYCLE program at OSU. Our project is focused on establishing a duality theory in the spirit of functional analysis. In particular, we generalize the notion of rigidity in a monoidal category to account for infinite dimensional spaces and provide a formal calculus for Hilbert spaces internal to a monoidal category under suitable conditions.

REU Expositional Work. Mentored undergraduate student Kolt, producing an expositional video on the *Synoptic Chart of Tensor Categories* with the goal of disseminating the geometric aspects of tensor categories to a broader audience.

Textbook. Writing a coauthored book *Unitary Quantum Symmetries Lite* (UQSL) with Kawagoe and Penneys. Our primary goal is to make the tools of higher category theory accessible to a large audience, in order to bridge the gap between mathematicians and physicists working on unitary tensor categories and topological phases of matter. Our main theme is the aforementioned higher functional analysis program in *finite dimensions.*

Building on this foundation, I plan to develop a series of expositional lectures translating its contents to the *infinite-dimensional* setting of higher functional analysis with a broader math and physics audience in mind.