Research Statement

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1. INTRODUCTION

Functional analysis. The mathematical foundations for *quantum mechanics* lead to the development of *functional analysis*, where one can encode the state space of a system with a *Hilbert space* H and observables with a *von Neumann algebra* $A \subset B(H)$ of operators acting on H.

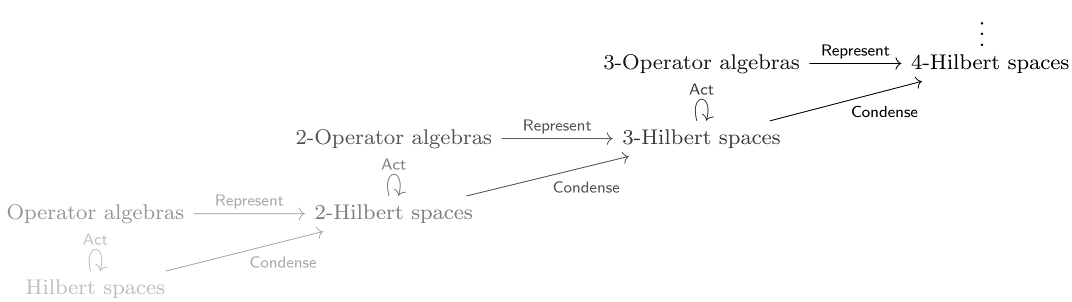
Categorifying Hilbert spaces. Functional analysis has since flourished in its own right as a field of mathematics. One particular approach in studying a given operator algebra A is to examine its *representations* $A \xrightarrow{\pi} B(H_\pi)$, i.e. the possible ways A can *act* on different Hilbert spaces H_π. The mathematical structure $\mathsf{Rep}(A)$ consisting of all such representations of A has many properties analogous to that of Hilbert spaces, while being much more mathematically rich. In this sense, these structures are *higher* Hilbert spaces, or *2-Hilbert spaces.*

Functional Analysis		**Representation Theory**	
Hilbert spaces	$H = L^2(X, \mu)$	Representation categories	$\mathcal{H} = \mathsf{Rep}(A)$
Vectors	$\eta \in H$	Representations	$H_A \in \mathcal{H}$
Scalars	$z \in \mathbb{C}$	Hilbert spaces	$H \in \mathsf{Hilb}$
Scaling	$\times \colon \mathbb{C} \times H \to H$	Tensoring	$\otimes \colon \mathsf{Hilb} \times \mathcal{H} \to \mathcal{H}$
Addition	$+ \colon H \times H \to H$	Direct sum	$\oplus \colon \mathcal{H} \times \mathcal{H} \to \mathcal{H}$
$\mathbb{C}$-valued inner product	$\langle \cdot \vert \cdot \rangle \colon \overline{H} \times H \to \mathbb{C}$	Hilb-valued inner product	$L^2 \operatorname{Hom}(\cdot \to \cdot) \colon \overline{\mathcal{H}} \times \mathcal{H} \to \mathsf{Hilb}$
Dual Hilbert space	$H^* := \operatorname{Hom}(H \to \mathbb{C})$	Presheaf category	$\mathcal{H}^* := \operatorname{Hom}(\mathcal{H} \to \mathsf{Hilb})$
Riesz Representation Theorem	$\overline{H} \cong H^*$	Yoneda Embedding Theorem	$\overline{\mathcal{H}} \cong \mathcal{H}^*$

Categorifying von Neumann algebras. Just as before, consider operators that act on a 2-Hilbert space $\mathcal{H} = \mathsf{Rep}(A)$. These are higher quantum symmetries forming the structure of a *higher* von Neumann algebra $\operatorname{Bim}(A)$. More generally, Henriques defines a *bicommutant category* as a bi-involutive $\otimes$-category $\mathcal{A} \hookrightarrow \operatorname{Bim}(A)$ with $\mathcal{A} \cong \mathcal{A}''$.

Algebras	**Tensor categories**
Finite dimensional algebra	Fusion category
$*$-algebra	Bi-involutive $\otimes$-category
Operators on a Hilbert space H $B(H)$	**Bimodules** over an operator algebra A $\mathsf{Bim}(A)$
Center $Z(A)$ of an algebra A Commutant $A' = Z_{B(H)}(A)$ of $A \subset B(H)$	Drinfeld center $\mathcal{Z}(\mathcal{A})$ of a tensor category $\mathcal{A}$ Commutant $\mathcal{A}' = \mathcal{Z}_{\operatorname{Bim}(A)}(\mathcal{A})$ of $\mathcal{A} \hookrightarrow \operatorname{Bim}(A)$
von Neumann algebras $A \subset B(H)$ with $A = A''$	**Bicommutant categories** $\mathcal{A} \hookrightarrow \operatorname{Bim}(A)$ with $\mathcal{A} \cong \mathcal{A}''$

Higher Functional Analysis. These motifs fit into the following research program, which has been worked out in the *finite dimensional* case. Indeed, the representations $\mathsf{Rep}(\mathcal{A})$ of a 2-operator algebra $\mathcal{A}$, i.e. the ways $\mathcal{A}$ can *act* on 2-Hilbert spaces $\mathcal{A} \xrightarrow{F} B(\mathcal{H}_F)$, then form an *even higher* Hilbert space known as a *3-Hilbert space.* One may thus continue constructing a staircase of higher and higher Hilbert spaces and operator algebras, that is, a theory of *higher functional analysis.*



The focus of my research is to develop the theory of higher functional analysis.

2. IMPORTANCE

The framework of higher functional analysis provides a powerful lens to systematically study quantum systems, using abstract mathematical concepts to obtain physical applications. From advancing our understanding of topological phenomena to supporting robust quantum computation, these structures illuminate the interplay between mathematics and physics.

Quantum symmetry. The classical symmetries of a set S are captured by the corresponding group of symmetries G, which act on S through a map $G \to \mathrm{End}(S)$. In the same way, tensor categories $\mathcal{A}$ act on 2-Hilbert spaces as so-called *quantum symmetries*. In particular for an operator algebra A, its quantum symmetries are captured by a $\otimes$-functor $\mathcal{A} \to \mathsf{Bim}(A)$. One important example central to subfactor theory is the *standard invariant* of a finite index II_1-subfactor $A \subseteq B$, which is generated by the action of ${}_AL^2B_B \in \mathrm{Bim}(A \oplus B)$.

Topological order. Another motivating force driving us up this mathematical staircase is the connection between higher operator algebras and *quantum computation*. Higher operator algebras provide an algebraic structure for excitations in so-called *topologically ordered* systems, capturing the fusion, braiding, and commutation relations of the quasi-particle excitations (or anyons) in your material. In particular, the excitations or defects of a $(2+1)D$ topological order gives rise to a 3-category which is expected to form a 4-Hilbert space.

We recall that in classical systems, "order" refers to patterns like the alignment of spins in a ferromagnet, where the spins break rotational symmetry and align in a particular direction. This type of order can be captured by simple order parameters, such as magnetization. In many-body quantum systems, however, order can be more complex due to non-local quantum correlations, such as entanglement, where distant parts of the system become strongly correlated. These correlations are not visible at the level of an individual particle yet become evident at the global scale, distinguishing quantum order from classical order. Further key features of these quantum systems are the presence of exotic excitations, such as anyons in two dimensional materials.

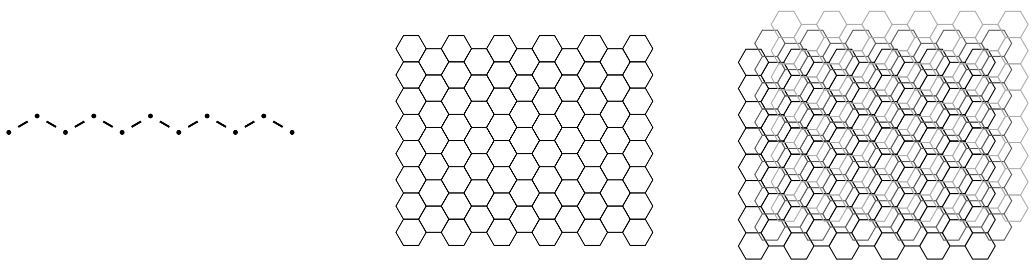


Figure 1: Depictions of 1D (spin chain), 2D (lattice), and 3D phases of matter

Quantum order is remarkably robust, often persisting despite local perturbations. This is due to the fact that in systems with topological order, quantum states are determined by global properties, such as the system's shape or topology, rather than local details. This stability makes topological order especially valuable for quantum computing, as it offers protection against errors.

Topological Quantum field theory. Higher Hilbert spaces are also expected to serve as the appropriate receptacles for fully-extended unitary topological quantum field theories (TQFTs). Since Lurie's work on *cobordism hypothesis*, it is known that these TQFTs are uniquely determined by what they assign to the point $*$. Henriques showed that Reshetikhin-Turaev theories, e.g. Chern-Simons, are fully extended. Moreover, he then showed that the mathematical structure of what such a TQFT assigns to $*$ forms a *higher von Neumann algebra*, i.e. a bicommutant category.

We note that TQFTs not only hold physical significance but also yield invariants with deep mathematical applications in knot theory, differential geometry, and algebraic topology. The importance of such invariants is underscored by the recognition of mathematicians like Vaughan Jones (1990), Edward Witten (1990), and Maxim Kontsevich (1998), who have been awarded Fields Medals for their groundbreaking contributions to knot invariants and related fields.

3. RESEARCH PLAN

Our goal is to develop the *infinite-dimensional* side of this higher functional analysis program. In particular, one of my main research objectives is to develop the theory of bicommutant categories, which are higher analogues of von Neumann algebras introduced by Prof. André Henriques.

(1) Establish foundational results for the theory of bicommutant categories.

- **Concreteness theorem.** Identify analytic and categorical conditions under which abstract (universal) and concrete (realization-based) definitions of bicommutant categories coincide, i.e. so that every abstract bicommutant category admits a faithful module category.

- **Representation theory.** Prove that every module category for a bicommutant category $\mathcal{A}$ decomposes as a direct sum of a faithful and a kernel part. Deduce that every module category admits non-trivial maps into a faithful module category, and hence any faithful module for $\mathcal{A}$ generates $\mathsf{Rep}(\mathcal{A})$.

- **Higher bicommutant theorem.** Show that for any faithful module $\mathcal{H}$ of a bicommutant category $\mathcal{A}$, the commutant $\mathcal{A}' \subset B(\mathcal{H}) = \mathrm{End}(\mathcal{H})$ contains absorbing objects, and $\mathcal{A} \cong \mathcal{A}''$, mirroring the classical bicommutant theorem for von Neumann algebras.

(2) Construct expected examples of bicommutant categories coming from operator algebras, representation theory, and conformal nets.

(3) Construct the Morita operator algebraic tricategory $E_1^\dagger(\mathsf{2Hilb})$ of bicommutant categories and their quantum symmetries. It was proven by Bartels-Douglas-Henriques that conformal nets form a 3-category. However this 3-category is not Cauchy complete. We propose:

- **Completion of conformal nets.**
  - $E_1^\dagger(\mathsf{2Hilb})$ is a Cauchy complete operator algebraic tricategory;
  - The category of conformal nets embeds into $E_1^\dagger(\mathsf{2Hilb})$; and
  - $E_1^\dagger(\mathsf{2Hilb})$ is the Cauchy completion of the 3-category of conformal nets.

(4) Describe a higher analogue of Gelfand duality, spectral theory, and the functional calculus.

Indeed, classical Gelfand duality provides a foundational correspondence between commutative von Neumann and compact measure spaces. This result can be aimed at a particular operator $x \in B(H)$ on a Hilbert space H. Namely, if x is normal, the W^*-algebra $W^*(x)$ generated by x corresponds to functions $L^\infty(\mathrm{Spec}(x))$ on the spectrum $\mathrm{Spec}(x)$ of x. This correspondence then allows us to perform a measurable *functional calculus* for x. On the other hand, there are various instantiations of Tannaka-reconstruction which afford correspondences between symmetric monoidal categories and (super)groupoids under suitable conditions. In particular, we expect a higher Gelfand duality theorem unifying these two motifs.

- **Higher Gelfand duality.** Describe a correspondence between symmetric bicommutant categories and compact measurable (super)groupoids.

  Aiming such a result at a bimodule ${}_AX_A \in \mathrm{Bim}(A)$ over a von Neumann algebra A would then yield a higher spectral theorem when viewing ${}_AX_A$ as an operator on the 2-Hilbert space $\mathsf{Rep}(A)$. Indeed, such a bimodule ${}_AX_A$ equipped with a symmetric braiding for ${}_AX_A$ and ${}_A\overline{X}_A$ generates a symmetric bicommutant category $W^*({}_AX_A)$, which would then correspond to representations $\mathsf{Rep}(\mathcal{G})$ on a compact measurable (super)groupoid $\mathcal{G} = \mathrm{Spec}({}_AX_A)$.

- **Higher spectral theory.** Determine what information about ${}_AX_A$ this higher spectrum $\mathrm{Spec}({}_AX_A)$ encodes. Moreover, determine how this construction depends on the choice of symmetric braiding.

- **Higher functional calculus.** Employ this representation-theoretic functional calculus to solve problems in higher functional analysis arising from quantum symmetries, topological order, and topological quantum field theory.