Research Statement

Giovanni Ferrer 2026

1. INTRODUCTION

Functional analysis. The mathematical foundations for *quantum mechanics* were first established during the early 20^{th} century by pioneering mathematicians like Hilbert and von Neumann. This lead to the development of *functional analysis*, where one can encode the state space of a system with a Hilbert space H and observables with a von Neumann algebra $A \subset B(H)$ acting on H.

Categorification. Functional analysis has since flourished in its own right as a field of mathematics. One particular approach in studying a given operator algebra A is to examine its *representations* $A \xrightarrow{\pi} B(H_\pi)$, i.e. the possible ways A can *act* on different Hilbert spaces H_π. The mathematical structure $\mathsf{Rep}(A)$ consisting of all such representations of A has many properties analogous to that of Hilbert spaces, while being much more mathematically rich.

Functional Analysis		Representation Theory	
Hilbert spaces	$H = L^2(X, \mu)$	Representation categories	$\mathcal{H} = \mathsf{Rep}(A)$
Vectors	$\eta \in H$	Representations	$H_A \in \mathcal{H}$
Scalars	$z \in \mathbb{C}$	Hilbert spaces	$H \in \mathsf{Hilb}$
Scaling	$\times \colon \mathbb{C} \times H \to H$	Tensoring	$\otimes \colon \mathsf{Hilb} \times \mathcal{H} \to \mathcal{H}$
Addition	$+ \colon H \times H \to H$	Direct sum	$\oplus \colon \mathcal{H} \times \mathcal{H} \to \mathcal{H}$
$\mathbb{C}$-valued inner product	$\langle \cdot \vert \cdot \rangle \colon \overline{H} \times H \to \mathbb{C}$	Hilb-valued inner product	$L^2\,\mathrm{Hom}(\cdot \to \cdot) \colon \overline{\mathcal{H}} \times \mathcal{H} \to \mathsf{Hilb}$
Dual Hilbert space	$H^* := \mathrm{Hom}(H \to \mathbb{C})$	Presheaf category	$\mathcal{H}^* := \mathrm{Hom}(\mathcal{H} \to \mathsf{Hilb})$
Riesz Representation Theorem	$\overline{H} \cong H^*$	Yoneda Embedding Theorem	$\overline{\mathcal{H}} \cong \mathcal{H}^*$

In this sense, these structures are thus *higher* Hilbert spaces, or *2-Hilbert spaces*:

$$\underbrace{A}_{\text{Operator algebra}} \xmapsto{\text{Represent}} \underbrace{\mathsf{Rep}(A)}_{\text{2-Hilbert space}}$$

Bicommutant categories. Just as before, consider operators (i.e. endofunctors) that act on a 2-Hilbert space $\mathcal{H} = \mathsf{Rep}(A)$. These are again higher quantum symmetries forming the mathematical structure of a *higher* von Neumann algebra.

Algebras	Tensor categories
Finite dimensional algebra	Fusion category
$*$-algebra	Bi-involutive category
Operators on a Hilbert space H $B(H) = \mathrm{End}(H) = \mathrm{Hom}(H \to H)$	Bimodules over an operator algebra A $\mathsf{Bim}(A) = \mathrm{End}(\mathcal{H}) = \mathrm{Hom}(\mathcal{H} \to \mathcal{H})$
Center $Z(A)$ of an algebra A Commutant $A' = Z_{B(H)}(A)$ of $A \subset B(H)$	Drinfeld center $\mathcal{Z}(\mathcal{A})$ of a tensor category $\mathcal{A}$ Commutant $\mathcal{A}' = \mathcal{Z}_{\mathrm{Bim}(A)}(\mathcal{A})$ of $\mathcal{A} \hookrightarrow \mathrm{Bim}(A)$
von Neumann algebras $A \subset B(H)$ with $A = A''$	Bicommutant categories $\mathcal{A} \hookrightarrow \mathrm{Bim}(A)$ with $\mathcal{A} \cong \mathcal{A}''$

This fits into the following research program, which has been worked out in the *finite dimensional* case. Indeed, the representations $\mathsf{Rep}(\mathcal{A})$ of a 2-operator algebra $\mathcal{A}$, i.e. the ways $\mathcal{A}$ can *act* on 2-Hilbert spaces $\mathcal{A} \xrightarrow{F} B(\mathcal{H}_F)$, then form an *even higher* Hilbert space known as a *3-Hilbert space*. One may thus continue constructing a staircase of higher and higher Hilbert spaces and operator algebras, that is, a theory of *higher functional analysis*.

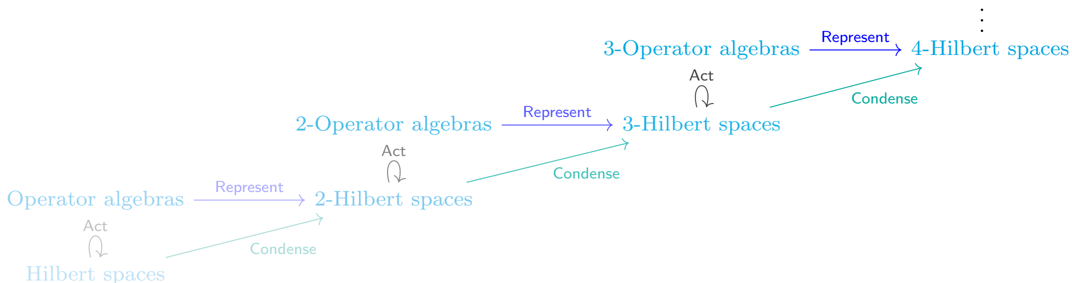

Figure 1: Higher Functional Analysis Program

The focus of my research is to develop the foundational theory of higher functional analysis.

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2. IMPORTANCE

The framework of higher functional analysis provides a powerful lens to systematically study quantum systems, using abstract mathematical concepts to obtain profound physical applications. From advancing our understanding of topological phenomena to supporting robust quantum computation, these structures illuminate the intricate age-old interplay between mathematics and physics.

Quantum symmetry. The classical symmetries of a set S are captured by the corresponding group of symmetries G, which act on S through a map $G \to \mathrm{End}(S)$. In the same way, tensor categories $\mathcal{A}$ act on 2-Hilbert spaces as so-called *quantum symmetries*. In particular for an operator algebra A, its quantum symmetries are captured by a $\otimes$-functor $\mathcal{A} \to \mathsf{Bim}(A)$. One important example central to subfactor theory is the *standard invariant* of a finite index II_1-subfactor $A \subseteq B$, which is generated by the action of ${}_AL^2B_B \in \mathrm{Bim}(A \oplus B)$.

Topological order. Another motivating force driving us up this mathematical staircase is the connection between higher operator algebras and *quantum computation*. Higher operator algebras provide an algebraic structure for excitations in *topologically ordered* systems, capturing the fusion, braiding, and commutation relations of the quasi-particle excitations (or anyons) in your material. In particular, the excitations or defects of a $(3+1)D$ topological order gives rise to a 3-category which is expected to form a 4-Hilbert space.

In classical systems, "order" refers to patterns like the alignment of spins in a ferromagnet, where the spins break rotational symmetry and align in a particular direction. This type of order can be captured by simple order parameters, such as magnetization. In many-body quantum systems, however, order can be more complex due to non-local quantum correlations, such as entanglement, where distant parts of the system become strongly correlated. These correlations are not visible at the level of an individual particle yet become evident at the global scale, distinguishing quantum order from classical order. Further key features of these quantum systems are the presence of exotic excitations, such as anyons in two dimensional materials.

Quantum order is remarkably robust, often persisting despite local perturbations. This is due to the fact that in systems with topological order, quantum states are determined by global properties, such as the system's shape or topology, rather than local details. This stability makes topological order especially valuable for quantum computing, as it offers protection against errors. As a testament to the significance of this field, David Thouless, Duncan Haldane, and Michael Kosterlitz were awarded the 2016 Nobel Prize in Physics for their pioneering work on topological order.

TODO: Add more people!

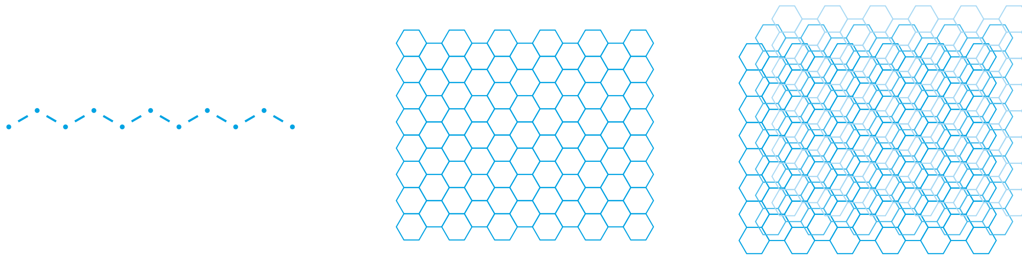

Figure 2: Depictions of 1D (spin chain), 2D (lattice), and 3D phases of matter

Quantum field theory. Higher Hilbert spaces are also expected to serve as the appropriate receptacles for fully-extended unitary topological quantum field theories (TQFTs). Since Lurie's work on *cobordism hypothesis*, it is known that these TQFTs are uniquely determined by what they assign to the point $*$. Henriques showed that Reshetikhin-Turaev theories, e.g. Chern-Simons theories, are fully extended. Moreover, he then shows that the mathematical structure of what these assign to $*$ must be a *bicommutant category.*

We note that TQFTs not only hold profound physical significance but also yield invariants with deep mathematical applications in knot theory, differential geometry, and algebraic topology. The importance of such invariants is underscored by the recognition of mathematicians like Vaughan Jones (1990), Edward Witten (1990), and Maxim Kontsevich (1998), who have been awarded Fields Medals for their groundbreaking contributions to knot invariants and related fields.

3. PAST ACCOMPLISHMENTS

In my past work, I have focused on the finite dimensional theory of *higher linear algebra.* Moreover, I have established foundational category-theoretic results which will play a role in the development of higher functional analysis.

Manifestly unitary higher Hilbert spaces. In joint work with Chen, Hungar, Penneys, and Sanford, we define *finite dimensional* 2-operator algebras and 3-Hilbert spaces. We then describe a formal process of constructing 3-Hilbert spaces from 2-Hilbert spaces from Hilbert spaces, known as *unitary condensation.*

Foundations for operator algebraic tricategories. In single author work, we provide foundational results for the theory of 3-operator algebras and 4-Hilbert spaces. More technically, we define and prove both coherence and concreteness results for so-called *operator algebraic tricategories.* We then show there is an operator algebraic tricategory $E_2^\dagger(\mathsf{Hilb})$ consisting of commutative operator algebras and their 3-categorical quantum symmetries.

Dagger n-categories. In joint work with Hungar, Johnson-Freyd, Krulewski, Muller, Nivedita, Penneys, Reutter, Scheimbauer, Stehouwer, and Vuppulury, we establish a theory of *dagger n-categories*, one of the key ingredients in the future formulation of n-Hilbert spaces and n-operator algebras for arbitrary $n = 1, 2, 3, \ldots$

Classifying module categories for generalized Temperley-Lieb-Jones $*$-2-categories. In joint work with Hernandez-Palomares, we provide a universal construction for a family of finite dimensional 3-Hilbert spaces related to the Jones polynomial and classify their *representations.*

Gray-categories model algebraic tricategories. In single author work, we prove the Grothendieck's *homotopy hypothesis*, as popularized by Baez, for *algebraic trigroupoids.*

4. CURRENT WORK

My current doctoral research is focused on furthering these results, particularly in the direction of homotopy theory and higher category theory.

Bases for 3-Hilbert spaces and higher unitary duality. Building on work from *Manifestly unitary higher Hilbert spaces*, joint with Di, Hungar, Penneys, and Wesley. We define and characterize unitary dual 2-functors on finite dimensional 3-operator algebras.

The homotopy 3-type of abelian operator algebras. Building on work from *Foundations for operator algebraic tricategories*, joint with Faurot. We describe an equivalent topological model to the Morita operator algebraic tricategory $E_2^\dagger(\mathsf{Hilb})$ of commutative operator algebras and their quantum symmetries, and provide a tricategorical Gelfand duality construction, simultaneously extending the Spectral theorem, Serre-Swan duality, and the Dauns-Hoffman theorem. We then describe the underlying homotopy 3-type of this 3-category in terms of its Postnikov data.

Models for higher Hilbert spaces as homotopy fixed points. Building on work from *Dagger n-categories*, joint with Müller, Penneys, and Stehouwer. We systematically describe the different theories of operator algebras (C*-algebras, W*-algebras, H*-algebras) in terms of homotopy fixed points for different subgroups of $O(2)$.

5. FUTURE RESEARCH OBJECTIVES

TODO: Cut down detail on this?

My future work will focus on the infinite-dimensional side of this higher functional analysis program. In particular, my immediate **research objectives** are developing the theory of bicommutant categories, the higher 2-analogues of von Neumann algebras.

(1) Establish foundational results for the theory of bicommutant categories.

(2) Construct expected examples of bicommutant categories coming from operator algebras, representation theory, and conformal nets.

(3) Build the Morita operator algebraic tricategory $E_1^\dagger(2\mathsf{Hilb})$ of bicommutant categories and their quantum symmetries.

(4) [[Higher spectral theory]]

5.1 Foundations

Our first goal is to establish core structural results for bicommutant categories, drawing parallels with the theory of von Neumann algebras.

Concreteness theorem. Identify analytic and categorical conditions under which abstract (universal) and concrete (realization-based) definitions of bicommutant categories coincide, i.e. so that every abstract bicommutant category admits a faithful module category.

Representation theory. Prove that every module category for a bicommutant category $\mathcal{A}$ decomposes as a direct sum of a faithful and a kernel part. Deduce that every module category admits non-trivial maps into a faithful module category, and hence any faithful module for $\mathcal{A}$ generates $\mathsf{Rep}(\mathcal{A})$.

Higher bicommutant theorem. Show that for any faithful module $\mathcal{H}$ of a bicommutant category $\mathcal{A}$, the commutant $\mathcal{A}' \subset B(\mathcal{H}) = \mathrm{End}(\mathcal{H})$ contains absorbing objects, and $\mathcal{A} = \mathcal{A}''$, mirroring classical bicommutant theorems in operator algebras.

Preduals and absorbing objects. Define a predual category $\mathcal{A}_*$, potentially as the subcategory of absorbing objects, and explore the extent to which $\mathcal{A} \cong \mathcal{A}_*^* = \mathrm{Hom}(\mathcal{A}_* \to \mathsf{Hilb})$. Characterize bicommutant categories with $\mathcal{A} = \mathcal{A}_{\mathsf{abs}}$.

5.2 Expected Examples

Consolidate these examples

After setting up this groundwork, we will then exhibit many families of expected examples.

Proposed bicommutant categories	Objects
$\mathsf{Bim}(A)$	bimodules over a von Neumann algebra A
$\mathsf{Hilb}(X, \mu)$	measurable Hilbert bundles over a measure space (X, μ)
$C(\mathcal{G})$	representations of a measurable groupoid $\mathcal{G}$
$\mathsf{Rep}(G)$	unitary representations of a discrete/(locally) compact group G
$\mathsf{Rep}(H)$	representations of a (locally) compact quantum group H
$\mathsf{Hilb}(G)$	G-graded Hilbert spaces for a discrete/(locally) compact group
$\mathcal{C}^* = \mathrm{Hom}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Hilb})$	unitary ind-objects of a rigid C*/W* tensor category $\mathcal{C}$
$\mathsf{Rep}(\mathcal{A})$	representations of a conformal net $\mathcal{A}$
?	topological line defects of a unitary quantum field theory

5.3 The Morita operator algebraic tricategory of Bicommutant Categories

Just as operator algebras and their quantum symmetries form an operator algebraic bicategory, we expect bicommutant categories to assemble into an operator algebraic tricategory.

Morita category. Construct the operator algebraic tricategory $E_1^\dagger(\mathsf{2Hilb})$ of bicommutant categories and their higher quantum symmetries.

Completion of conformal nets. It was proven by Bartels-Douglas-Henriques that conformal nets form a 3-category. However this 3-category is not Cauchy complete. We propose:

- $E_1^\dagger(\mathsf{2Hilb})$ is Cauchy complete;
- The category of conformal nets embeds into $E_1^\dagger(\mathsf{2Hilb})$; and
- $E_1^\dagger(\mathsf{2Hilb})$ is the Cauchy completion of the category of conformal nets.

5.4 Higher spectral theory

TODO: Wrap things up with a nice bow!

[[Add some cool stuff here!]]

6. BROADER IMPACTS

My broader impact activities will center on mentoring, education, and dissemination.

Mentoring. I have extensive experience supervising undergraduate research, mentoring graduate students, and organizing seminars and workshops. As a part of my post-doctoral work, I will continue these efforts through outreach and mentoring within [[Place I'm applying to]] mathematical community.

Book. A key educational component will be the coauthored textbook *Higher Quantum Symmetries* (with Kawagoe and Penneys), which connects categorical operator theory and topological order. Building on this foundation, I plan to develop a series of lectures translating its contents to the infinite-dimensional setting of higher functional analysis, introducing W*-complete and bicommutant categories to a broad mathematical audience.

Applications. This research contributes to the foundational mathematics underpinning quantum computation and information theory, helping build the conceptual and analytic infrastructure that supports emerging quantum technologies.