Algebraic geometry for ~~dummies~~ quantum symmetry folk

[[work in progress]]

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Contents

1

Steps towards higher spaces: What is a site?

> The wise man looks into space and does not regard the small as too little, nor the great as too big, for he knows that there is no limit to dimensions.
>
> Zhuang Zhou

Recall that, for a category $\mathcal{X}$, a presheaf of a given flavor of mathematical structure on $\mathcal{X}$ is simply a contravariant functor:

$$\mathcal{X}^{\mathrm{op}} \to \mathsf{Structures}$$

where $\mathsf{Structures}$ can be taken to be category of Sets, Groups, Ab, Vec, Hilb, $\mathsf{C^*Alg}$, $\mathsf{W^*Alg}$, etc. depending on our interests. But where does the term “pre-sheaf” come from, and what would make a “pre”-sheaf not just “pre”?

Idea. *If presheaves are like functions on a space, sheaves are to be thought of as the* ***continuous*** *functions.*

But in order to talk about “continuity” of a functor on a category, we need the notion of a *topology* on a category: thus turning it into a *higher space* known as a *site*.

Algebraic geometry cheat sheet

Category level 0	Category level 1
Set	Category
Topology	Grothendieck Topology
Space	Site
Function $X \to \mathbb{C}$	Presheaf $\mathcal{X}^{\mathrm{op}} \to \mathsf{Hilb}$
Continuous function $f\colon X \to \mathbb{C}$	Sheaf $F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Hilb}$
Abelian group $C(X)$	Topos $Sh(\mathcal{X})$
$(f+g)(x) \coloneqq f(x)+g(x)$	$(F \oplus G)(U) \coloneqq F(U) \oplus G(U)$
Vector space $C(X)$	2-vector space $Sh(\mathcal{X})$
Scalars $\lambda \in \mathbb{C}$	Hilbert spaces $\Lambda \in \mathsf{Hilb}$
$(\lambda \triangleright f)(x) \coloneqq \lambda \cdot f(x)$	$(\Lambda \triangleright F)(U) \coloneqq \Lambda \otimes F(U)$
Algebra $C(X)$	Monoidal category $Sh(\mathcal{X})$
$(f \cdot g)(x) \coloneqq f(x) \cdot g(x)$	$(F \otimes G)(U) \coloneqq F(U) \otimes G(U)$
Commutative C*-algebra $C(X)$	Symmetric *C*-2-algebra* $Sh(\mathcal{X})$
$\overline{f}(x) \coloneqq \overline{f(x)}$	$\overline{F}(U) \coloneqq \overline{F(U)}$

Idea. *The category $\mathcal{O}(X)$ of open sets is the same data as a topological space X, and is hence a space.*[1]

Albeit unorthodox, we may describe the topology on X as follows:

For every $x \in X$, there is a collection of so-called open neighborhoods $T_x \subseteq \mathcal{O}(X)$ of x where each $U \in T_x$ contains $x \in U$. These are required to satisfy the following three axioms:

(T1) If $V \in T_y$ is an open neighborhood of $y \in X$, then $V \cap T_x \subseteq T_y$.[2]

(T2) For every collection of points $\{x_j\} \subset X$, and every collection of open neighborhoods $\{U_{ij}\} \subseteq T_{x_j}$, we have $\bigcup_{ij} U_{ij} \in T_{x_j}$ for every x_j.

(T3) $X \in T_x$ is an open neighborhood for every $x \in X$

Notice (T1) captures the fact that the topology on X is closed under finite intersections, (T2) captures closure under arbitrary unions, and (T3) implies X is open.[3]

One thing to note is that the points $x \in X$ generally live "outside" of $\mathcal{O}(X)$, as the singletons $\{x\}$ are seldom open. Hence, one should *morally* modify these conditions in terms of coverings for open sets U, instead of neighborhoods for a point x.[4] This leads us to the following notion of a Grothendieck topology:

Definition 1.0.1 — A *Grothendieck topology* τ on a category $\mathcal{X}$ consists of the following data:

For every $U \in \mathcal{X}$, there is a collection of so-called covering sieves τ_U where each $S \in \tau_U$ is a subfunctor of $\mathcal{X}(- \to U)$.[a] These are required to satisfy the following three axioms:

(T1) For $f \in \mathcal{X}(V \to U)$ and $\mathcal{S} \in \tau_U$, we have $f^*\mathcal{S} \in \tau_V$ where $f^*\mathcal{S}$ is the pullback[b] sieve:

$$\begin{tikzcd} V \times_U W \arrow[r, dashed] \arrow[d, dashed, "f^*\mathcal{S}(V \times_U W) \ni f^*(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "g \in \mathcal{S}(W)"] \\ V \arrow[r, "f"'] & U \end{tikzcd}$$

(T2) For every covering sieve $\mathcal{S} \in \tau_U$, and every collection of covering sieves $\{\mathcal{S}_{U_j} \in \tau_{U_j}\}$, we have $\bigcup_j \mathcal{S}_{U_j} \circ \mathcal{S} \in \tau_U$ where

$$\left(\bigcup_j \mathcal{S}_{U_j} \circ \mathcal{S}\right)(V) = \bigcup_j \mathcal{S}_{U_j}(V) \circ \mathcal{S}(U_j) := \left\{ V \xrightarrow{g} U_j \xrightarrow{f} U \,\middle|\, g \in \mathcal{S}_{U_j}(V) \text{ and } f \in \mathcal{S}(U_j) \right\}$$

(T3) $\mathcal{X}(- \to U) \in \tau_U$ is a covering sieve for every $U \in \mathcal{X}$.

[1] In this note we will use this idea to motivate the notion of a *site*. In particular, we will focus on abstracting "open covers" to arbitrary categories. However, there is another, equally valid direction in formalizing the very same idea. Indeed, one can instead consider lattices which behave like the lattice $\mathcal{O}(X)$ by admitting:

- finite limits (meets, greatest lower bounds, intersections) $\wedge = \cap$,
- arbitrary colimits (joins, least upper bounds, unions) $\bigvee = \bigcup$
- initial (minimal, smallest, False) and terminal (maximal, greatest, True) objects $0 = \varnothing$ and $1 = X$.

These lattices are known as *frames/locales* in the field of *pointless geometry* (pun intended). These also play a role in intuitionistic logic, where they are known as *complete Heyting algebras*. Of course these structures will turn out to be closely related to sites. In fact, the representations of a site $\mathcal{X}$ will form a so-called *topos* $Sh(\mathcal{X})$, where a topos is the categorification (homotopification) of a locale.

[2] Here we're not worrying about $\varnothing$

[3] Either include $\varnothing$ in each T_x (which is admittedly not great conceptually), or include it at the end

[4] We note that one should also be able to interpret this discussion in terms of filters and ultrafilters.

A site $(\mathcal{X}, \mathcal{T})$ is then a category $\mathcal{X}$ equipped with a Grothendieck topology τ.

[a]More concretely, a covering sieve $\mathcal{S}$ picks out a collection of morphisms $\mathcal{S}(V) \subseteq \mathcal{X}(V \to U)$ for every $V \in \mathcal{X}$, which we say *cover* U. In the case of $\mathcal{X} = \mathcal{O}(X)$, note that $\mathcal{X}(V \to U)$ is either $\varnothing$ or $\{\subseteq\}$. Hence, when defining a covering sieve $\mathcal{S}$ for U, we have a choice of whether or not to include V as part of our cover for U whenever $V \subset U$.

[b]In the case when $\mathcal{X} = \mathcal{O}(X)$, pulling back along an inclusion $V \subseteq U$ is the same as taking an intersection with V.

Remark 1.0.2. As one can see, the notation for sieves gets a bit heavy and obfuscates this relatively easy concept: *A site is a category equipped with a notion of "covering", where*:

(T1) The preimage of a cover is a cover

(T2) Covering a cover is a cover

(T3) The whole space[5] covers everything

Alternatively, we can state these axioms in parallel to those of a topology as long as we're willing to squint at the meaning of "intersections", "unions", and "trivial":

(T1) Finite *intersections* of covers are covers

(T2) Arbitrary *unions* of covers are covers[6]

(T3) *Trivial* covers are covers

Of course, by construction, we recover our guiding example:

Example 1.0.3 (Spaces are spaces) — The category $\mathcal{O}(X)$ of open sets with inclusions for a topological space X admits a natural Grothendieck topology τ, where $\mathcal{S} \in \tau_U$ is a covering sieve on an open set $U \subseteq X$ if and only if

$$U = \bigcup_{\mathcal{S}(V) \neq \varnothing} V$$

When working in categorification, one notices the following common motif:

Idea. *Structures of a certain mathematical flavor assemble into* ***higher*** *mathematical structures with a resembling taste.*

We see this for example as abelian groups themselves form an abelian category Ab, vector spaces form a 2-vector space Vec, Hilbert spaces form a 2-Hilbert space Hilb, and so on. Thus, in the spirit of (vertical) categorification, we should expect that topological spaces themselves form a higher space, i.e. a *site*.

Vertical categorification

Category level 0	Category level 1
Abelian groups	Abelian category Ab
Vector spaces	2-vector space Vec
Hilbert spaces	2-Hilbert space Hilb
Topological spaces	Site CHaus

[5]Following the yoga of Yoneda, we identify a space $\mathcal{X}$ with its Yoneda embedding.

[6]The ambiguity of this statement is particularly egregious, as we will see later on.

Example 1.0.4 (Spaces are a space) — The category CHaus of compact Hausdorff spaces forms a site, where each non-trivial covering sieve $\mathcal{S}$ on a compact Hausdorff space X corresponds to a finite[a] cover $\{U_i\}_{i=1}^n$ of X with compact Hausdorff spaces U_i, i.e., $\mathcal{S}$ is determined by a surjective map:

$$\coprod_{i=1}^{n} U_i \twoheadrightarrow X.$$

[a] Here one might be worried about satisfying (T2). However it is true that the arbitrary "union" of finite covers $\{\mathcal{S}_{X_j}\}$ is again finite... not great, but the reader was warned. Indeed, in the formal statement of (T2), the union $\bigcup_j \mathcal{S}_{U_j}$ gets post-composed with a chosen finite cover $\mathcal{S}$, and now $\bigcup_j \mathcal{S}_{U_j} \circ \mathcal{S}$ is again finite.

We note that these covering sieves in CHaus are quite tame. In general, Grothendieck topologies can be much more fine or coarse. Indeed, just as in point-set topology, there are maximal and minimal Grothendieck topologies:

Example 1.0.5 — For a category $\mathcal{X}$,

- The discrete topology is obtained by declaring every subfunctor $\mathcal{X}(- \to U)$ to be a covering sieve for every $U \in \mathcal{X}$;
- The trivial topology is obtained by declaring that the only covering sieves are $\mathcal{X}(- \to U)$.
- The canonical topology is obtained by declaring that the only covering sieves are the representable presheaves, i.e. those isomorphic to $\mathcal{X}(- \to U)$ for some $U \in \mathcal{X}$.

2

Being hungry, they carry the sheaves: What is a sheaf?

> Those who go out weeping, carrying seed to sow, will return with songs of joy, carrying sheaves with them.
>
> Psalms 126:6

Now that we have fleshed out the notion of a topology on a category $\mathcal{X}$, we may talk about presheaves of a certain flavor

$$\mathcal{X}^{\mathrm{op}} \to \mathsf{Structures}$$

which preserve the topology we've chosen on our site.

Idea. If we think of a covering sieve $\mathcal{S}$ for $U \in \mathcal{X}$ as an honest covering of U, "preserving" $\mathcal{S}$ corresponds to satisfying a certain "gluing condition"[1] with respect to this covering. In the case when $\mathcal{X} = \mathcal{O}(X)$ for a topological space X, we think of a presheaf F as assigning a whole structure's-worth of functions over each $U \subset X$. Indeed, consider the prototypical example where $\mathsf{Structures} = \mathsf{Groups}$ and $F = C(- \to G)$

$$C(- \to G) \colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Groups}$$

assigns to each open $U \subset X$ the group of G-valued continuous functions for a topological group G:[2]

$$C(U \to G) \coloneqq \{f \colon U \to G \text{ continuous}\}.$$

Now for an open covering $\{V_i\} \subseteq X$ of U, consider the commutative diagram

$$\begin{tikzcd} C(U \to G) \arrow[r] \arrow[d] & C(V_j \to G) \arrow[d] \\ C(V_i \to G) \arrow[r] & C(V_i \cap V_j \to G) \end{tikzcd} \qquad \begin{tikzcd} (f : U \to G) \arrow[r, mapsto] \arrow[d, mapsto] & f|_{V_j} \arrow[d, mapsto] \\ f|_{V_i} \arrow[r, mapsto] & f|_{V_i \cap V_j} \end{tikzcd}$$

Observe that each $f \colon U \to G$ is uniquely determined by the collection of functions $(f_i \coloneqq f|_{V_i})$ where $f_i|_{V_j} = f_j|_{V_i}$. Conversely, given such a collection $(f_i \colon V_i \to G)$ of continuous functions, we may uniquely *glue* these to obtain a continuous $f \colon U \to G$. The slick way to express this categorically is that the following

[1] You'll also hear of algebraic geometers talking about "descent conditions", which are synonymous.

[2] This is actually closely related to how one thinks of a topological group as a *condensed* group in Condensed Mathematics.

diagram is a pullback square[3]:

$$\begin{tikzcd}
C(U \to G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_j C(V_j \to G) \arrow[d] \\
\coprod_i C(V_i \to G) \arrow[r] & \coprod_{ij} C(V_i \cap V_j \to G)
\end{tikzcd}$$

In any case, what the category theory is trying to express is that the space $C(U \to G)$ is built from each $C(V_i \to G)$ by gluing along their intersections $C(V_i \cap V_j \to G)$.

> **Definition 2.0.1** (Sheaf) — A sheaf of groups $F\colon \mathcal{X} \to \mathsf{Groups}$ on a site $(\mathcal{X}, \tau)$ is a presheaf satisfying:
>
> - For every covering sieve $\mathcal{S}$ on $U \in \mathcal{X}$, we have a pullback square:
>
> $$\begin{tikzcd}
F(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_j F(V_j) \arrow[d] \\
\coprod_i F(V_i) \arrow[r] & \coprod_{ij} F(V_i \times_U V_j)
\end{tikzcd}$$
>
> Of course, we may equivalently describe this gluing condition more concretely:
>
> - For every covering sieve $\mathcal{S}$ on $U \in \mathcal{X}$, we have that every collection $(f_i \in F(V_i))$ with $f_i|_{V_j} = f_j|_{V_i}$ can be uniquely glued into $f \in F(U)$ such that each $f|_{V_i} = f_i$.[a]
>
> [a] Here one actually needs to consider collections $(f_{i,s} \in F(V_i))$ indexed not only by objects $V_i \in \mathcal{X}$, but also by morphisms $s \in \mathcal{S}(V_i) \subseteq \mathcal{X}(V_i \to U)$. One then needs to reinterpret our restriction notation $f|_{V_i}$ as $F(s)(f)$ where $F(s)\colon F(U) \to F(V_i)$. As the notation is already quite cumbersome, we will omit such ennui.

Again, by construction, we obtain our first example:

> **Example 2.0.2** (Continuous functions form sheaves) — For a topological group G, the sheaf $C(- \to G)$ on $\mathcal{O}(X)$ is known as the *sheaf of germs of continuous G-valued functions on X.*

In the following section, we will discuss the meaning of this curious term "germ". However, prior to such a digression, we present more examples of sheaves:

> **Example 2.0.3** (Manifolds form sheaves) — For a smooth manifold X, its *structure sheaf*
>
> $$C^\infty\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Vec}$$
>
> assigns to an open set $U \subset X$ the vector space
>
> $$C^\infty(U) \coloneqq \{f\colon U \to \mathbb{R} \text{ or } \mathbb{C} \text{ smooth}\},$$
>
> where the target depends on which flavor of manifolds one desires.

> **Non-example 2.0.4** (C*-algebras) — With the previous example in mind, one might wish to construct a C*-algebraic analogue of C^∞ as follows: For a compact Hausdorff space, we define
>
> $$C\colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{C^*Alg}$$

[3] or equalizer diagram, pick your poison.

which assigns to an open set $U \subset X$ the C*-algebra

$$C(U) := \{f : U \to \mathbb{C} \text{ continuous}\}.$$

Of course there's a problem in that functions in $C(U)$ need not be bounded because U is seldom compact, i.e. this obviously doesn't form a C*-algebra. But we may try to modify this construction by considering:

$$C_0 : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{C^*Alg} \text{ (not necessarily unital)}$$

on a locally compact Hausdorff space X, which assigns to an open set $U \subset X$ the C*-algebra

$$C_0(U) := \{f : U \to \mathbb{C} \text{ continuous and vanishing at infinity}\}.$$

Albeit a more promising candidate, as this does form a presheaf, we note that C_0 does *not* satisfy the desired gluing condition for sheaves.

Exercise 2.0.5. Find a locally compact Hausdorff space X so that $C_0 : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{C^*Alg}$ is not a sheaf on X. In particular, build a family of continuous functions vanishing at infinity $(f_i : U_i \to \mathbb{C})$ on an open cover $\{U_i\}$ of X which does *not* glue to a global function $f : X \to \mathbb{C}$ which vanishes at infinity.
Hint: Consider $X = \mathbb{R}$ covered by $U_i = (i-1, i+1)$ for $i \in \mathbb{Z}$.

Example 2.0.6 (C*-algebras v2.0) — A way to fix the previous non-example is to instead consider

$$C : \mathcal{O}(X)^{\mathrm{op}} \to \dagger\mathrm{Alg}$$

as a sheaf of †-algebras on our chosen compact Hausdorff space X. It just so happens that $C(X)$ is a C*-algebra[a], which is a *property* and not a *structure* on a †-algebra.

[a] More generally $C(\widetilde{X})$ is a C*-algebra for every component $\widetilde{X} \subseteq X$ as closed sets in a Hausdorff space are compact.

Sheafs of smooth functions? Continuous functions? Imaginably one can cook up many more sheaves of this form. Indeed, one may consider sheaves of holomorphic functions on compact Riemann surfaces, meromorphic functions on such surfaces equipped with divisors, you name it. In fact, one might be willing to naively conjecture: *the nicer the flavor of functions, the easier it is to form a sheaf out of them.* However, there is a certain element of robustness these families of functions must satisfy. We provide the following counterexample as a warning.

Non-example 2.0.7 (Constant function presheaf) — Let G be a group. We define to *constant function* presheaf on a space X, denoted by:

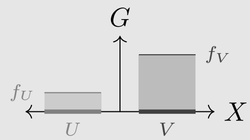
$$C_c(- \to G) : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Groups},$$

by assigning to each open $U \subseteq X$, the group of constant functions on U:

$$C_c(U \to G) := \{f : U \to G \text{ constant}\}.$$

This presheaf in general fails to satisfy the require gluing condition. Indeed, as long as G is non-trivial and there exist disjoint open sets $U, V \subset X$, we can always construct constant functions f_U and f_V

which do not glue to a constant function:



There is a way to fix this example, which is to consider a slightly larger, more robust class of functions.

Example 2.0.8 (Locally constant function sheaf) — Let G be a group. We define to *locally constant function* sheaf on a space X, denoted by:

$$C_{lc}(- \to G) \colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Groups},$$

by assigning to each open $U \subseteq X$, the group of locally[a] constant functions on U:

$$C_{lc}(U \to G) \coloneqq \{f \colon U \to G \text{ locally constant}\}.$$

[a]We say that a function f is locally *flavored* when every $x \in X$ admits an open neighborhood $x \in U \subseteq X$ such that f has said *flavor* on U.

Exercise 2.0.9. Let G be a group equipped with its discrete topology. Show that

$$C(U \to G) = C_{lc}(U \to G) \text{ for any space } U.$$

We will see that there is a more systematic way of enhancing a presheaf into a sheaf, i.e. a free construction:

$$\begin{tikzcd} \mathsf{Sheaves} \arrow[r, bend right, "\mathsf{forget}"'] \arrow[r, phantom, "\perp"] & \mathsf{Presheaves} \arrow[l, bend right, "\mathsf{free}"'] \end{tikzcd}$$

For this we will introduce the "stalk picture" and finally discuss "germs" of functions. Before this, we include a brief digression on spaces of sheaves.

3

Where the sea advances insensibly in silence: What is a topos?

> It is better to have a good category with bad objects than a bad category with good objects.
>
> Attributed to A. Grothendieck

Consider the following example of a sheaf, which is the "dirac-delta" function's higher analogue. Of course this can be stated in terms of any sort of mathematical structure. But, as we are not interested in *centipede mathematics*[1], we will state it here for sheaves of groups.

Example 3.0.1 (Skyscraper sheaves) — Let G be a group and $p \in X$ a point in a space X. The *Skyscraper sheaf* over a point $p \in X$, denoted by

$$G\delta_p \colon \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Groups}$$

is given on open sets by:

$$G\delta_p(U) = \begin{cases} G & p \in U \\ 0 & \text{else} \end{cases}$$

and has "restriction" maps for $U \subseteq V$:

$$G\delta_p(V) \to G\delta_p(U) = \begin{cases} \mathrm{id}_G & p \in U \subseteq V \\ 0 & \text{else} \end{cases}$$

One can more generally extend this construction to $G\chi_P$ for characteristic functions χ_P where $P \subset X$, "add" these $G\chi_P \oplus_X H\chi_Q$ to assign different groups G, H to different points in $P, Q \subseteq X$, etc.

Exercise 3.0.2. What conditions on $P \subset X$, if any, does one need to impose so that one can define a sheaf $G\chi_P$ on X?

The idea of sheaves being "continuous[2] functions" on a site leads us to the following insight:

Idea. *The space of sheaves $Sh(\mathcal{X})$ on a site $\mathcal{X}$ plays the role of $C(X)$ on a space X.*

[1] This is the yoga of removing as many hypothesis from a theorem as possible while retaining its form. *How many legs can you remove from a centipede until it is no longer a centipede?* One? Fifty? Fifty-one? Ninety-nine? A hundred? Moreover, how much discussion can one include in a footnote until it is no longer a footnote? We leave this as an exercise to the reader.

[2] This last example might be a bit counterintuitive, since characteristic functions are generally not continuous. But what we normally think of as continuity will arise as a local-triviality condition later on.

From practice, we know that the C*-algebra $C(X)$ is quite a nice mathematical object, due to the fact that $\mathbb{C}$ is quite rich. More generally, the yoga of the Yoneda lemma teaches us that algebraic structures on an object T correspond to structures on $\mathrm{Hom}(- \to T)$, which in turn descend onto structure for each $\mathrm{Hom}(X \to T)$. So the fact that $\mathbb{C}$ is a C*-algebra is what endows $C(X)$ is said structure.

Now the same can be said for higher structures: When $\mathcal{T}$ is, for example, an abelian category, it follows that the category $Sh(\mathcal{X} \to \mathcal{T})$ of sheaves on a site $\mathcal{X}$ valued in $\mathcal{T}$ will also form an abelian category. This is the guiding principle of *condensed mathematics*[3], where topological abelian groups are replaced by sheaves of abelian groups on a suitable site CHaus of topological spaces.

Indeed, albeit the category of topological abelian groups is not an abelian category, we may identify a topological abelian group G with its associated sheaf:

$$C(- \to G)\colon\ \mathsf{CHaus}^{op} \to \underbrace{\mathsf{Ab}}_{\text{abelian category}}$$

which does live in an abelian category of condensed abelian groups $\mathsf{cAb} := Sh(\mathsf{CHaus} \to \mathsf{Ab})$.[4]

Definition 3.0.3 — A category of the form $Sh(\mathcal{X})$ for a site $\mathcal{X}$ is known as a *(Grothendieck) topos.*

Idea. *Condensed structures are like structures that need not have enough points.*

For a condensed structure $F\colon\ \mathsf{CHaus}^{\mathrm{op}} \to \mathsf{Structure}$, we may consider the *underlying structure*

$$|F| := F(*) = F(\{p\}),$$

which are to be thought of as the space of "points" of F.[5] Notice how $|F|$ could be trivial yet $\Omega_F := F(S^1)$, the loop space of F, could be non-trivial. For example, consider the condensed abelian group

$$H^1(-;\mathbb{Z}) := \mathsf{CHaus}^{\mathrm{op}} \to \mathsf{Ab}$$

which only has one point $H^1(*;\mathbb{Z}) = \{0\}$ and an infinite loop space $H^1(S^1;\mathbb{Z}) = \mathbb{Z}$. One way to think about this is that H^1 is the condensed abelian group representing the *infinitely small circle*.

Moreover, since we may view finite sets Δ as discrete compact Hausdorff spaces in CHaus, there is also an *underlying simplicial structure* $F|_{\Delta^{op}} : \Delta^{op} \to \mathsf{Structure}$.

Indeed, for condensed sets, we obtain some nice adjunctions:

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend right, "\mathsf{Kan}"'] \arrow[r, phantom, "\top"] & \mathsf{cSet} \arrow[l, bend right, "|_{\Delta^{op}}"'] \arrow[r, bend right, "|\cdot|"'] \arrow[r, phantom, "\top"] & \mathsf{Top} \arrow[l, bend right, "\text{よ}"'] \end{tikzcd}$$

Note that here, instead of $|F|$ landing in Set, we equip $F(*)$ with an organic topology.

Exercise 3.0.4. What topology do we need to equip $F(*)$ with in order to obtain the desired adjunction?

[3] One will find different formalisms, all based on this principle, which have their own ways of dealing with size issues:

- *Condensed mathematics* only considers spaces smaller than an uncountable inaccessible cardinal κ, taking a (large) colimit on κ whenever needed. In fact, they tend to restrict themselves to so-called *pro-finite sets*, which form a site with more-or-less the same sheaves.
- *Pyknotic mathematics* only considers spaces smaller than the first strongly inaccessible cardinal κ.
- *Quasi-mathematics* completely disregards size issues. A *quasi-topological space* in the sense of Spanier is precisely a sheaf $\mathsf{CHaus}^{\mathrm{op}} \to \mathsf{Ab}$ on the large category CHaus. This is the philosophy we will follow, noting that "quasi-mathematics" is not a standard term.

[4] Here it is curious that $\mathsf{CHaus}^{\mathrm{op}} \cong \mathsf{C}^*\mathsf{Alg}_{\mathrm{comm.}}$ appears. Is this just a coincidence?

[5] Again practicing the yoga of Yoneda, the points of F are $\mathrm{Hom}(* \Rightarrow F) = F(*)$ where $* := \text{よ}_* = \mathsf{Top}(- \to *)$.

4

Love in the Time of Cholera: What is a germ?

> Today, when I saw you, I realized that what is between us is nothing more than an illusion.
>
> Gabriel García Márquez

We have seen how sheaves on a space X serve to encode classes of partially defined functions on X together with the way they glue together. In this section, our aim is to present the *"Stalk picture"* for sheaves, which is motivated by *fiber bundles*.

Definition 4.0.1 (Bundles) — A (locally trivial) fiber bundle $E \xrightarrow{p} B$ with fiber F consists of:

(E) A space E called the *total space*;

(B) A space B called the *base space*

(F) A space F called the *fiber*, which might be equipped with $\mathsf{Structure}$.

(p) A continuous map $p\colon E \to B$ satisfying two conditions:

- (locally trivial) Each $b \in B$ admits an open neighborhood $U \subseteq B$ such that:

$$\begin{tikzcd} U \times F \arrow[r, equal] \arrow[d, "p_U"'] & E_U \arrow[r, hook] \arrow[d, "p|^U"'] & E \arrow[d, "p"] \\ U \arrow[r, equal] & U \arrow[r, hook] & B \end{tikzcd}$$

  where p_U is the projection $(u, f) \mapsto u$ and $p|^U : E_U := p^{-1}(U) \to U$ is the co-restriction of p.

- (transition maps) For two such $U, V \subseteq X$, there is a transition map $t_{U,V} : U \cap V \to \mathrm{Aut}(F)$. Indeed, consider a basepoint $b \in U \cap V$. For each $e \in E_b := p^{-1}(b)$, we have two equivalent expressions (or coordinates): $(b, f) \in U \times F$ and $(b, f') \in V \times F$. We then define $t_{U,V}(b)$ by $f \mapsto f'$. When the fiber F has $\mathsf{Structure}$, we require these transition maps to be structure preserving[a].

[a] That is, we compile $\mathrm{Aut}(F)$ in the category $\mathsf{Structure}$.

We will quickly talk only of bundles $E \xrightarrow{p} B$, suppressing the fibers from our notation when possible. In order to clarify this talk of $\mathsf{Structure}$ and structure preserving maps, let us instantiate our cases of interest:

- In the case when $\mathsf{Structure} = \mathsf{Vec}$, such a fiber bundle is known as a *vector bundle* and we require the transition maps $t_{U\cap V} : U \cap V \to \mathrm{Aut}(V)$ to have their image in *linear* automorphisms $\mathsf{Vec}(V \to V)$.
- When $\mathsf{Structure} = \mathsf{Hilb}$, these are known as *Hilbert bundles* and we require the transition maps to land in bounded maps (really, the "correct" choice is unitary maps). These are closely related to so-called *Riemannian* manifolds.
- One can also consider $\mathsf{Structure} = \mathsf{sHilb}$, which are then related to the *semi-Riemannian* manifolds appearing in general relativity.
- Finally, for *C*-bundles* with $\mathsf{Structure} = \mathsf{C^*Alg}$, we require that $t_{U\cap V}$ land in $*$-algebra automorphisms.

More generally, one speaks of so-called *structure groups*:

Definition 4.0.2 — We say that a bundle has structure group $G \leq \mathrm{Aut}(F)$ when every $\mathrm{Im}\, t_{U\cap V} \subseteq G$.

These structure groups are more refined than just equipping the fibers F of a bundle $E \xrightarrow{p} B$ with Structure[1]. For example, we may talk of n-dimensional vector bundles with structure group $O(n)$. Similarly, we can consider Hilbert bundles with fiber H having structure group $U(H)$.

Before we move on to relating these bundles to our story about sheaves, we present some examples.

Example 4.0.3 (Möbius) — [[todo]]

Example 4.0.4 — For an n-dimensional (smooth) manifold M, its *tangent bundle* TM has fibers $\mathbb{R}^n$ where, more concretely, the fiber over a basepoint $b \in M$ is its *tangent space* T_bM. Viewing this as the space of derivations at b,

$$T_bM \coloneqq \{\partial \colon C^\infty(M) \to \mathbb{R} \text{ or } \mathbb{C} \mid \partial(fg) = f(b)\partial(g) + \partial(f)g(b) \text{ for all } f, g \in C^\infty(M)\}.$$

Exercise 4.0.5. Figure out how to equip TM with a topology so that $TM \to M$ is a vector bundle.

Non-example 4.0.6 (Unitary algebras) — [[todo]]

We now discuss how to view bundles as sheaves.

Example 4.0.7 (Bundles as sheaves) — Let $E \xrightarrow{p} B$ be a fiber bundle with fiber F equipped with some Structure. We define its *sheaf of sections*, denoted by:

$$\Gamma(- \to E) \colon \mathcal{O}(B)^{\mathrm{op}} \to \mathsf{Structure},$$

by assigning to an open set $U \subseteq B$ the space of *sections*[a] on U:

$$\Gamma(U \to E) \coloneqq \{s \colon U \to E \text{ continuous or smooth} \mid U \xrightarrow{s} E \xrightarrow{p} U = \mathrm{id}_U\}.$$

[a] In general, the sections of a morphism $f \colon A \to B$ are its right-inverses, i.e. the $g \colon B \to A$ such that $fg = \mathrm{id}_B$.

A particular instance of this example to keep in mind is the sheaf of vector fields on a manifold:

$$\mathfrak{X}_M \coloneqq \Gamma(- \to TM) \colon \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{Vec}.$$

[1] Equivalently, one could restrict the morphisms in the category Structure, so that $\mathrm{Aut}(F)$ is our desired group in this subcategory. For example, one could pass from Hilb to the subcategory $\mathsf{Hilb}_{\mathrm{isom}}$ of Hilbert spaces with isometric maps in order to obtain bundles with structure group $U(H)$. This, however, would be a notational nightmare.

Idea. *Sheafs are like bundles where we allow the "fibers", called* stalks*, to be different.*

In order to recover the fiber over a basepoint $b \in B$ from the section sheaf $\Gamma(- \to E)$ of a bundle E, we somehow need to "shrink" the $\Gamma(U \to E)$ by taking smaller and smaller open neighborhoods of $b \in U \in \mathcal{O}(B)$. Using the language of ultrafilters on $\mathcal{O}(X)$, or more generally, of directed limits in $\mathcal{X}$, we obtain the notion of a *stalk*:

Definition 4.0.8 — For a sheaf $F\colon \mathcal{O}(X)^{\mathrm{op}}$, its *stalk* over the point $x \in X$ is given by the colimit

$$F_x \coloneqq \lim_{\mathcal{O}(X) \ni U \ni x} F(U).$$

This stalk is determined by a universal property where each $F(U)$ admits a map F_x compatible with restrictions.

In the case where F is a sheaf of functions, for example when $F = C(- \to G)$ for a topological group G, each $f \in C(U \to G)$ determines a *germ* at $x \in U \in \mathcal{O}(X)$ i.e. its image under the map $C(U \to G) \to F_x$. Thus, the stalk F_x is known as the space of germs at x, and $F = C(- \to G)$ the sheaf of germs of G-valued functions.
To summarize our discussion so far:

$$\begin{tikzcd}
\text{Presheaves} \arrow[r, hookleftarrow] \arrow[d, rightsquigarrow] & \text{Sheaves} \arrow[r, hookleftarrow] \arrow[d, rightsquigarrow] & \text{Fiber bundles} \arrow[d, rightsquigarrow] \\
\text{Functions} \arrow[r, hookleftarrow] & \text{Continuous functions} \arrow[r, hookleftarrow] & \text{Constant functions}
\end{tikzcd}$$

Let us now view operator algebras as sheaves through this stalk picture:

Definition 4.0.9 (unitary algebras) — [[To do]]

Theorem 4.0.10 (Dauns-Hofmann). [[To do]]

Theorem 4.0.11 (Factor decomposition). [[To do]]

We now provide a method of *sheafifying* any presheaf F, by first viewing it as a "generalized bundle" and then taking its sheaf of sections.

Definition 4.0.12 (Sheafification) —

[[Figure out where to move this]]

Example 4.0.13 (2-functionals) — For a finite dimensional 2-Hilbert space $\mathcal{X}$, recall that the Yoneda embedding $\mathcal{X} \to \mathsf{Hilb}^{\mathcal{X}^{\mathrm{op}}}$ is a unitary equivalence. Hence, all Grothendieck topologies on $\mathcal{X}$ agree with the trivial topology and every presheaf on $\mathcal{X}$ is a sheaf.

Remark 4.0.14. The analogous statement one category level down is that every functional on a Hilbert space is bounded/continuous.

5

Failure as an option: What is sheaf cohomology?

> We don't make mistakes, just happy little accidents.
>
> Bob Ross

Consider a morphism[1] $\phi\colon F \Rightarrow G$ of sheaves $F, G\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Structure}$:

$$\begin{tikzcd} \mathcal{X}^{\mathrm{op}} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathsf{Structure} \arrow[Rightarrow, from=U, to=D, "\phi"'] \end{tikzcd}$$

Recall the following idea:

Idea. *Maps into an algebraic structure tend to absorb this structure and reflect its properties.*

In particular, when $\mathsf{Structure}$ forms an abelian category, we expect $Sh(\mathcal{X})$ to form a category which is also abelian.

So, for $\phi\colon F \Rightarrow G$, we should be able to construct sheaves $\operatorname{Ker}\phi$ and $\operatorname{Im}\phi$ that fit into a short exact sequence:

$$0 \to \operatorname{Ker}\phi \to F \xrightarrow{\phi} \operatorname{Im}\phi \to 0$$

The first construction one would guess is to define

$$\operatorname{Ker}\phi\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Structure} \qquad \text{and} \qquad \operatorname{Im}\phi\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Structure}$$

pointwise, i.e. on $U \in \mathcal{X}$ by

$$(\operatorname{Ker}\phi)(U) \coloneqq \operatorname{Ker}(\phi_U\colon F(U) \to G(U)) \qquad \text{and} \qquad (\operatorname{Im}\phi)(U) \coloneqq \operatorname{Im}(\phi_U\colon F(U) \to G(U))$$

Unfortunately, while $\operatorname{Ker}\phi$ is indeed a sheaf, this naive construction for $\operatorname{Im}\phi$ *fails* to be more than a presheaf.

[1] By this, we just mean a natural transformation as functors $\mathcal{X}^{\mathrm{op}} \to \mathsf{Structure}$

Non-example 5.0.1 (exp) — The continuous map $e^{i\pi(-)} \colon \mathbb{R} \to \mathbb{T}$ induces a sheaf map

$$C(- \to \mathbb{R}) \overset{\phi}{\Longrightarrow} C(- \to \mathbb{T})$$

given by post-composition $\phi \coloneqq \fatsemi_{e^{i\pi(-)}}$, i.e. for $U \in \mathcal{O}(X)$ we define the corresponding component by:

$$\begin{aligned} C(U \to \mathbb{R}) &\xrightarrow{\phi_U} C(U \to \mathbb{T}) \\ f(x) &\mapsto e^{i\pi f(x)} \end{aligned}$$

Consider the open cover of $\mathbb{T} = U \cup V$ where $U = \mathbb{T} - \{1\}$ and $V = \mathbb{T} - \{-1\}$ are contractible. We claim that $\mathrm{id}_U \in (\operatorname{Im}\phi)(U)$ and $\mathrm{id}_V \in (\operatorname{Im}\phi)(V)$ yet

$$\mathrm{id}_U \cup \mathrm{id}_V = \mathrm{id}_{\mathbb{T}} \notin (\operatorname{Im}\phi)(\mathbb{T}).$$

Exercise 5.0.2. Show the previous claim by proving:

- There exists a continuous section of $e^{i\pi(-)}$ on U, suggestively named $\frac{1}{i\pi}\ln$. Convince yourself that this is equivalent to $\mathrm{id}_U \in (\operatorname{Im}\phi)(U)$.
- Convince yourself the same holds true for V.
- Show there exists no continuous split monomorphism of $\mathbb{T} \to \mathbb{R}$ by homotopical[2] considerations:

$$\underbrace{\pi_1(\mathbb{R})}_{0} \leftarrow \underbrace{\pi_1(\mathbb{T})}_{\mathbb{Z}}$$

Convince yourself this means that $e^{i\pi(-)}$ has no continuous section on $\mathbb{T}$, and hence $\mathrm{id}_{\mathbb{T}} \notin (\operatorname{Im}\phi)(\mathbb{T})$.

Okay, so just sheafify this construction to obtain the desired $\operatorname{Im}\phi$ sheaf, big whoop. *Well actually...*

Idea. *The failure of our naive construction is a **feature**, not a **bug**.*

Indeed, notice the obstruction we constructed was homotopical in nature: $\mathbb{T}$ has nontrivial holes in dimension 1 whereas $\mathbb{R}$ does not.

The idea behind *sheaf* cohomology is to exploit this failure in order to detect holes.

To recap, given a short exact sequence of sheaves on $\mathcal{O}(X)^{\mathrm{op}}$:

$$0 \to K \to F \to I \to 0$$

we only have exact sequences:

$$0 \to K(U) \to F(U) \to I(U)$$

which we will extend into long exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K(U) \arrow[r] & F(U) \arrow[r] & I(U) \arrow[dll, "\alpha^0"] \\
& H^1(U;K) \arrow[r] & H^1(U;F) \arrow[r] & H^1(U;I) \arrow[dll, "\alpha^1"] \\
& H^2(U;K) \arrow[r] & H^2(U;F) \arrow[r] & H^2(U;I) \arrow[dddll, dashed] \\
& & & \\
& & & \\
& H^n(U;K) \arrow[r] & H^n(U;F) \arrow[r] & H^n(U;I)
\end{tikzcd}$$

[2]Recall that $\pi_1(Y) \coloneqq C(\mathbb{T} \to Y)/\sim$ up to homotopy for a (pointed) space Y.