TANNAKA-KREIN DUALITY FOR FINITE GROUPS

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Abstract. The category of representations of a group forms a symmetric fusion category with a fiber functor, which is a strict faithful symmetric functor into the category of vector spaces. We show that we may recover a group from its representations in a process known as Tannaka reconstruction. Conversely, we prove Krein's theorem which states that every symmetric fusion category equipped with a fiber functor arises in this way.

Contents

The standard reference for tensor (fusion) categories is [2]. Our treatment of the Tannaka reconstruction theorem is based on [3] and the David Green's treatment of classical Tannaka reconstruction in an upcoming pre-print. The proof of the converse theorem is based on many useful discussions with David Green. However, a generalized version of this result can be found in [1].

1. Introduction

In what follows, we will only consider vector spaces and representations over $\mathbb{C}$. Similarly, we consider categories where each hom-set is equipped with the structure of a $\mathbb{C}$-vector space such that composition is bilinear. Motivated by representation theory, we provide the following definition.

Definition 1.1 — An object V in a ($\mathbb{C}$-linear) category $\mathcal{C}$ is said to be *simple* when

$$\mathrm{End}(V) := \mathrm{Hom}_{\mathcal{C}}(V, V) = \mathbb{C}\,\mathrm{id}_V.$$

A category $\mathcal{C}$ is *semisimple* when every object can be decomposed as a finite direct sum of simples. We further say that $\mathcal{C}$ is *finitely* semisimple when there are only finitely many simple objects (up to isomorphism). We denote the set of simple objects (up to isomorphism) by $P(\mathcal{C})$.

Remark 1.1. One can show that every semisimple category is an abelian category such that every short exact sequence splits. In particular, any functor from a semisimple category is exact, i.e. preserves short exact sequences.

The following definition categorifies the notion of a commutative semisimple algebra.

Definition 1.2 (SFC) — A symmetric fusion category $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, \alpha, \lambda, \rho, \beta)$ consists of:

($\mathcal{C}$) A finitely semisimple category $\mathcal{C}$;

($\otimes$) A bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;

($1_{\mathcal{C}}$) A simple object $1_{\mathcal{C}} \in \mathcal{C}$ called the *unit*;

(α) An *associator* natural isomorphism

$$\alpha_{UVW}\colon U \otimes (V \otimes W) \to (U \otimes V) \otimes W \quad \text{for } U, V, W \in \mathcal{C};$$

(λ) A *left unitor* natural isomorphism

$$\lambda_V\colon 1_{\mathcal{C}} \otimes V \to V \quad \text{for } V \in \mathcal{C};$$

(ρ) A *right unitor* natural isomorphism

$$\rho_V\colon V \otimes 1_{\mathcal{C}} \to V \quad \text{for } V \in \mathcal{C};$$

(β) An *interchanger* natural isomorphism

$$\beta_{V,W}\colon V \otimes W \to W \otimes V \quad \text{for } V, W \in \mathcal{C}.$$

We require that:

- Any two ways of reparenthesization, adding or removing the unit, swapping objects with α, λ, ρ, β, and identities agree.
- For every object $V \in \mathcal{C}$, there exists a *dual* object $V^* \in \mathcal{C}$ with maps

  $$\mathsf{ev}\colon V^* \otimes V \to 1_{\mathcal{C}} \quad \text{and} \quad \mathsf{coev}\colon 1_{\mathcal{C}} \to V \otimes V^*,$$

  such that any two ways of annihilating and producing V,V^* with ev, coev, and identities agree.

Example 1.3 — The category Vec of vector spaces and linear maps admits a canonical structure of a symmetric fusion category. In particular,

- $\otimes$ is given by the ordinary tensor product,
- 1_{Vec} is given by the one-dimensional vector space $\mathbb{C}$,
- the coherence natural isomorphisms α, λ, ρ, and β are given on simple tensors in the obvious way,
- V^* is given by the dual vector space of V, ev is given by function evaluation on simple tensors, and coev is determined on $1 \in \mathbb{C}$ by the canonical tensor $\sum e^i \otimes e_i$ in $V \otimes V^*$ where $\{e_i\}$ is any basis on V and $\{e^i\}$ is the induced dual basis on V^*.

Example 1.4 — For a finite group G, consider the category $\mathsf{Rep}(G)$ of finite dimensional representations of G and G-intertwiners.

- In class, we showed that $\mathsf{Rep}(G)$ is finitely semisimple.
- Furthermore, recall that $\mathsf{Rep}(G)$ inherits tensors and duals from Vec, hence turning $\mathsf{Rep}(G)$ into a symmetric fusion category.

In particular, for G-representation (V, ρ_V) and (W, ρ_W), we have

$$(V, \rho_V) \otimes (W, \rho_W) \coloneqq (V \otimes W, \rho_V \otimes \rho_W),$$

that is, $\rho_{V\otimes W} \coloneqq \rho_V \otimes \rho_W$. Given (V, ρ_V) one also defines the action of G on V^* by

$$(g \cdot \varphi)(v) = \varphi(g^{-1} \cdot v) \quad \text{for } \varphi \in V^* \text{ and } v \in V.$$

Remark 1.2. For a finite group G, notice that $\mathsf{Rep}(G)$ admits a forgetful functor

$$F\colon \mathsf{Rep}(G) \to \mathsf{Vec}$$
$$(V, \rho_V) \mapsto V.$$

By construction, F is faithful and preserves the tensor, unit, and coherence natural isomorphisms on $\mathsf{Rep}(G)$. In this note, we will call such strict faithful symmetric functors from a general symmetric fusion category into Vec a *fiber functor*.

Definition 1.5 — Given any ($\mathbb{C}$-linear) functor $F\colon \mathcal{C} \to \mathcal{D}$ between ($\mathbb{C}$-linear) categories, there is an algebra $\mathrm{End}(F)$ of natural transformations from F to itself. The vector space structure on $\mathrm{End}(F)$ is determined component-wise and the multiplication is given by composition of natural transformations.

Moreover, when $\mathcal{C}$ and $\mathcal{D}$ are symmetric fusion categories with F perserving the tensor, unit, and coherence natural isomorphisms, we may define a group $\mathrm{Aut}_\otimes(F)$ of natural isomorphisms from F to itself such that $\eta_{V\otimes W} = \eta_V \otimes \eta_W$ for $V, W \in \mathcal{C}$.

2. Tannaka Reconstruction

Given a finite group G, we may produce its category of representations $\mathsf{Rep}(G)$ together with its fiber functor $F\colon \mathsf{Rep}(G) \to \mathsf{Vec}$. Conversely, given this symmetric fusion category $\mathsf{Rep}(G)$ with its fiber functor F, we may construct a group $\mathrm{Aut}_\otimes(F)$ of $\otimes$-natural isomorphisms from F to itself. In this section, we will show that this reconstructs the original group G, i.e.

$$G \cong \mathrm{Aut}_\otimes(\mathsf{Rep}(G) \xrightarrow{F} \mathsf{Vec}).$$

We first recall some facts about the group algebra $\mathbb{C}[G]$ of a finite group G, which is given by

$$\mathbb{C}[G] \coloneqq \left\{ \sum_{g\in G} z_g \delta_g \;\middle|\; z_g \in \mathbb{C} \right\}.$$

First notice that $\mathbb{C}[G]$ admits a G-action determined by left multiplication

$$g_0 \cdot \delta_g \coloneqq \delta_{g_0 g} \quad \text{for } g_0 \in G \text{ and } \delta_g \in \mathbb{C}[G].$$

In this sense, we obtain the following fact.

Fact 2.1 — For $(V, \rho_V) \in \mathsf{Rep}(G)$, we have $F(V) = \mathrm{Hom}_{\mathsf{Rep}(G)}(\mathbb{C}[G], V)$.

Proof of Fact. Given $v \in F(V)$, we define $T\colon \mathbb{C}[G] \to V$ by

$$T \sum z_g \delta_g = \sum z_g \rho_V(g)(v).$$

Conversely, we identify $T\colon \mathbb{C}[G] \to V$ with $T\delta_e \in V$. ■

In particular, since $\mathbb{C}[G]$ acts on $\mathrm{Hom}(\mathbb{C}[G], V)$, each G-representation can be uniquely extended to a $\mathbb{C}[G]$-representation. In what follows, we will identity $\mathsf{Rep}(G)$ with $\mathsf{Rep}(\mathbb{C}[G])$.

Moreover, the group algebra $\mathbb{C}[G]$ admits the structure of a co-commutative Hopf algebra with comultiplication $\Delta\colon \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$, counit $\epsilon\colon \mathbb{C}[G] \to \mathbb{C}$, and antipode $S\colon \mathbb{C}[G] \to \mathbb{C}[G]$ determined by:

$$\begin{aligned}\Delta(\delta_g) &\coloneqq \delta_g \otimes \delta_g \\ \epsilon(\delta_g) &\coloneqq 1 \\ S(\delta_g) &\coloneqq \delta_{g^{-1}}.\end{aligned}$$

Fact 2.2 — The grouplike elements of $\mathbb{C}[G]$ are precisely those $\delta_g \in \mathbb{C}[G]$ for $g \in G$, i.e.

$$a \neq 0 \text{ with } \Delta(a) = a \otimes a \quad \text{if and only if} \quad a = \delta_g \text{ for some } g \in G.$$

Proof of Fact. Indeed, if $a \neq 0$ such that $\Delta(a) = a \otimes a$, then

$$\epsilon(a) = (\epsilon \otimes \epsilon)(\Delta(a)) = \epsilon(a)^2,$$

implying $\epsilon(a) = 0$ or 1. So if $a = \sum z_g \delta_g$, then $\epsilon(a) = \sum z_g = 0$ or 1. Now observe

$$\sum_{g \in G} z_g \, \delta_g \otimes \delta_g = \Delta(a) = a \otimes a = \sum_{g,h \in G} z_g z_h \, \delta_g \otimes \delta_h.$$

Hence $z_g = z_g^2$ which occurs only when $z_g = 0$ or 1 for every $g \in G$. Since $a \neq 0$, we conclude that $a = \delta_g$ for some $g \in G$. The converse is immediate by definition of Δ. ∎

We are now ready to prove the main result of this section.

Theorem 2.3 (Tannaka Reconstruction)**.** For a finite group G, we have

$$G \cong \mathrm{Aut}_{\otimes}(\mathsf{Rep}(G) \xrightarrow{F} \mathsf{Vec}).$$

Proof. By the Yoneda Lemma, we have

$$\mathrm{Hom}_{\mathsf{Rep}(G)}(\mathbb{C}[G], \mathbb{C}[G]) \cong \mathrm{Hom}(\mathrm{Hom}(\mathbb{C}[G], -), \mathrm{Hom}(\mathbb{C}[G], -)).$$

So from Fact 2, we have $\mathbb{C}[G] \cong \mathrm{End}(F)$ as vector spaces. In particular, the Yoneda map[1] $\rho\colon \mathbb{C}[G] \to \mathrm{End}(F)$ is determined on $\delta_g \in \mathbb{C}[G]$ by the natural transformation

$$\rho(\delta_g) = (\rho(\delta_g)_V \colon F(V) \to F(V)),$$

whose component on $(V, \rho_V) \in \mathsf{Rep}(G)$ is given by

$$\rho(\delta_g)_V \coloneqq \rho_V(g).$$

From this definition, it is clear that ρ is also an algebra isomorphism, i.e.

$$\rho(\delta_g)\rho(\delta_h) = \rho(\delta_{gh}) \quad \text{for all } g, h \in G.$$

Furthermore, each $\rho(\delta_g) \in \mathrm{Aut}_{\otimes}(F)$ since every $\rho_V(g)$ is invertible with

$$\rho_{V \otimes W}(g) \coloneqq \rho_V(g) \otimes \rho_W(g).$$

[1]Recall that the Yoneda map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathrm{Hom}(X, -) \to \mathrm{Hom}(Y, -))$ is given on $f\colon X \to Y$ by precomposition f^* by f. Through our identifications, for $(V, \rho_V) \in \mathsf{Rep}(G)$ and $v \in F(V)$, we have $v = T\delta_e$ for some $T\colon \mathbb{C}[G] \to V$ and δ_g^* is precisely the map which sends $v = T\delta_e$ to $\rho_V(g)(v) = T\delta_g\delta_e$.

Now suppose $\rho(a) \in \mathrm{Aut}_\otimes(F)$. Then $a \neq 0$ and $\rho_{\mathbb{C}[G]\otimes\mathbb{C}[G]} = \rho_{\mathbb{C}[G]} \otimes \rho_{\mathbb{C}[G]}$. Hence

$$\Delta(a) = \rho_{\mathbb{C}[G]\otimes\mathbb{C}[G]}(a)(\delta_e \otimes \delta_e) = \rho_{\mathbb{C}[G]}(a)(\delta_e) \otimes \rho_{\mathbb{C}[G]}(a)(\delta_e) = a \otimes a.$$

By Fact 2.2, we conclude that $a = \delta_g$ for some $g \in G$. □

3. Krein's Theorem

Given a symmetric fusion category $\mathcal{C}$ with a fiber functor $F\colon \mathcal{C} \to \mathsf{Vec}$, we may construct the group $\mathrm{Aut}_\otimes(F)$ of $\otimes$-natural automorphisms of F. Conversely, given the group $\mathrm{Aut}_\otimes(F)$, we may construct a symmetric fusion category of its representations $\mathsf{Rep}(\mathrm{Aut}_\otimes(F))$ together with its fiber functor. In this section, we will show that this reconstructs the original symmetric fusion category $\mathcal{C}$ with fiber functor $F\colon \mathcal{C} \to \mathsf{Rep}$, i.e.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathsf{Rep}(\mathrm{Aut}_\otimes(\mathcal{C} \xrightarrow{F} \mathsf{Vec})) \\ & {\scriptstyle F}\searrow \quad \swarrow & \\ & \mathsf{Vec} & \end{array}$$

Lemma 3.1 — For a SFC $\mathcal{C}$ with fiber functor $F\colon \mathcal{C} \to \mathsf{Vec}$, there exists an equivalence of categories $\widehat{F}\colon \mathcal{C} \to \mathsf{Rep}(\mathrm{End}(\mathcal{C} \xrightarrow{F} \mathsf{Vec}))$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\widehat{F}} & \mathsf{Rep}(\mathrm{End}(F)) \\ & {\scriptstyle F}\searrow \quad \swarrow & \\ & \mathsf{Vec} & \end{array}$$

Proof of Lemma. We will first construct an equivalence of (linear) categories

$$\widehat{F}\colon \mathcal{C} \to \mathsf{Rep}(\mathrm{End}(F)).$$

For $C \in \mathcal{C}$, we define

$$\widehat{F}(C) \coloneqq (F(C), \rho_{F(C)})$$

where $\rho_{F(C)}(\alpha) \coloneqq \alpha_C$ for $\alpha\colon F \Rightarrow F$. For $f\colon C \to C'$ in $\mathcal{C}$, we then set $\widehat{F}(f) = F(f)$. Since F is faithful, $\widehat{F}$ is a faithful. It is also clear by construction that the desired triangle commutes.

To show $\widehat{F}$ is essentially surjective and full, we first introduce a tensoring for $\mathcal{C}$ compatible with F. By choosing a basis $V \xrightarrow{\sim} \mathbb{C}^n$ for every vector space V, we may define

$$\mathcal{C} \times \mathsf{Vec} \to \mathcal{C}$$
$$(C, V) \mapsto C \otimes V$$

together with natural isomorphisms

$$\tau_{C,V}\colon F(C \otimes V) \to F(C) \otimes V$$

and

$$\mu_{C,V,W}\colon C \otimes (V \otimes W) \to (C \otimes V) \otimes W$$

as follows. Set $C \otimes V \coloneqq V^{\oplus n}$ and choose the maps

$$\tau_{C,V}\colon F(C^{\oplus n}) = F(C)^{\oplus n} = F(C) \otimes \mathbb{C}^n \xrightarrow{\sim} F(C) \otimes V$$

determined by the basis $V \xrightarrow{\sim} \mathbb{C}^n$, and

$$\mu_{C,V,W} \colon C^{\oplus(n+m)} \to C^{\oplus(n+m)}$$

to be the obvious morphism in $\mathcal{C}$ determined by the change of basis matrix

$$\mathbb{C}^{n+m} \to V \otimes W \to \mathbb{C}^n \otimes \mathbb{C}^m = \mathbb{C}^{n+m}.$$

One extends $\otimes$ to morphisms similarly. We now set

$$X_F \coloneqq \bigoplus_{X_i \in P(\mathcal{C})} X_i \otimes F(X_i)^*.$$

Now notice that each $\alpha \in \mathrm{End}(F)$ is uniquely determined by $\alpha_{X_i} \in \mathrm{End}(F(X_i))$ for $X_i \in P(\mathcal{C})$ since $\mathcal{C}$ is semisimple and $\alpha_{C \oplus C'} = \alpha_C \oplus \alpha_{C'}$ for $C, C' \in \mathcal{C}$. Conversely, one can easily show that choosing $\beta_{X_i} \in \mathrm{End}(F(X_i))$ for every $X_i \in P(\mathcal{C})$ determines a natural transformation $\beta \in \mathrm{End}(F)$. In summary,

$$\mathrm{End}(F) = \bigoplus_{X_i \in P(\mathcal{C})} \mathrm{End}(F(X_i)).$$

Now notice

$$\mathrm{End}(F) = \bigoplus_{X_i \in P(\mathcal{C})} F(X_i) \otimes F(X_i)^* = \bigoplus_{X_i \in P(\mathcal{C})} F(X_i \otimes F(X_i)^*) = F(X_F).$$

One verifies that through this identification, the action of $\mathrm{End}(F)$ on $F(X_F)$ agrees with left multiplication (postcomposition) on $\mathrm{End}(F)$. We are now ready to show $\widehat{F}$ is essentially surjective. Consider some $(V, \rho_V) \in \mathsf{Rep}(\mathrm{End}(F))$. Let $C \in \mathcal{C}$ be the coequalizer

$$C \coloneqq \mathsf{coeq}\left(X_F \otimes \mathrm{End}(F) \otimes V \underset{\mathrm{ev} \otimes \mathrm{id}_V}{\overset{\mathrm{id}_F \otimes \rho_V}{\rightrightarrows}} X_F \otimes V \right)$$

where the bottom arrow is given by

$$X_F \otimes \mathrm{End}(F) \otimes V = X_F \otimes \mathrm{End}(X_F) \otimes V \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_V} X_F \otimes V.$$

Here, we have used the map $\mathrm{ev} \colon C \otimes \mathrm{Hom}(C, C') \to C'$ given on the summand indexed by $f \colon C \to C'$ by f itself. Now notice that

$$\begin{aligned} F(C) &= \mathsf{coeq}\left(F(X_F) \otimes \mathrm{End}(F) \otimes V \rightrightarrows F(X_F) \otimes V\right) \\ &= \mathsf{coeq}\left(\mathrm{End}(F) \otimes \mathrm{End}(F) \otimes V \rightrightarrows \mathrm{End}(F) \otimes V\right) \\ &= \mathrm{End}(F) \otimes_{\mathrm{End}(F)} V \\ &= V. \end{aligned}$$

By our previous remarks about the action of $\mathrm{End}(F)$ on $F(X_F)$, we see that

$$\widehat{F}(C) = (V, \rho).$$

Finally, to see that $\widehat{F}$ is full, notice that an $\mathrm{End}(F)$-intertwiner $\varphi\colon (V,\rho_V) \to (W,\rho_W)$ induces a morphism of diagrams:

$$\begin{array}{ccc} X_F \otimes \mathrm{End}(F) \otimes V & \underset{\mathrm{ev}\otimes\mathrm{id}_V}{\overset{\mathrm{id}_F\otimes\rho_V}{\rightrightarrows}} & X_F \otimes V \\ {\scriptstyle \mathrm{id}\otimes\mathrm{id}\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi} \\ X_F \otimes \mathrm{End}(F) \otimes W & \underset{\mathrm{ev}\otimes\mathrm{id}_W}{\overset{\mathrm{id}_F\otimes\rho_W}{\rightrightarrows}} & X_F \otimes W \end{array}$$

We then obtain a morphism f between the coequalizers of both diagrams. One then see that $\widehat{F}(f) = F(f) = \mathrm{id}_{\mathrm{End}(F)} \otimes_{\mathrm{End}(F)} \varphi = \varphi$. We conclude that $\widehat{F}\colon \mathcal{C} \to \mathsf{Rep}(\mathrm{End}(F))$. ■

Fact 3.2 — For a Hopf algebra H, the grouplike elements G of H form a group.

Proof of Fact. The fact that G is closed under multiplication follows from the condition that comultiplication is an algebra map. As in Fact 2.2, one verifies that $g \in G$ satisfies $\epsilon(g) = 1$. It then follows that $S(g) = g^{-1}$ since

$$S(a)a = m(S \otimes \mathrm{id})(a \otimes a) = m(S \otimes \mathrm{id})\Delta a = 1\epsilon(a) = 1,$$

and similarly $aS(a) = 1$. ■

Fact 3.3 — Let H be a finite dimensional (complex) co-commutative Hopf algebra. Then H is the group algebra $\mathbb{C}[G]$ of the group-like elements G of H.

Proof of Fact. This seems to be corollary of the Cartier-Konstant-Milnor-Moore classification theorem. Unfortunately, I could not find a simpler proof of this fact. ■

Lemma 3.4 — We may equip $\mathrm{End}(F)$ with the structure of a Hopf algebra such that the grouplike elements of $\mathrm{End}(F)$ are $\mathrm{Aut}_\otimes(F)$.

Proof of Lemma. Similar to the tensor product of algebras, there is a Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ of semisimple categories $\mathcal{C}, \mathcal{D}$ which admits a separately linear functor

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$$

such that for linear functors $G\colon \mathcal{D} \to \mathcal{E}$ and $H\colon \mathcal{D} \to \mathcal{E}$, there exists a unique linear functor $G \boxtimes H$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} \boxtimes \mathcal{D} & & \\ \uparrow & \searrow^{G\boxtimes H} & \\ \mathcal{C} \times \mathcal{D} & \xrightarrow[G\times H]{} & \mathcal{E} \end{array}$$

The Deligne tensor product satisfies

$$\mathrm{End}(G) \otimes \mathrm{End}(H) = \mathrm{End}(G \boxtimes H).$$

In particular, $\mathrm{End}(F) \otimes \mathrm{End}(F) = \mathrm{End}(F \boxtimes F)$. We then define the coproduct

$$\Delta\colon\ \mathrm{End}(F) \to \mathrm{End}(F) \otimes \mathrm{End}(F)$$

by $\Delta\alpha \in \mathrm{End}(F \boxtimes F)$ for $\alpha \in F$ by

$$(\Delta\alpha)_{(C,C')} \coloneqq \alpha_{C\otimes C'}.$$

From this definition together with Fact 3.2, it is clear that $\Delta\alpha = \alpha \otimes \alpha$ if and only if $\alpha \in \mathrm{Aut}_{\otimes}(F)$. We then define the counit

$$\epsilon \colon \mathrm{End}(F) \to \mathbb{C}$$

by $\epsilon(\alpha) \coloneqq \alpha_1$ where $\alpha_1 \in \mathrm{End}(F(1)) = \mathrm{End}(1) = \mathbb{C}$. Finally, we define the antipode

$$S \colon \mathrm{End}(F) \to \mathrm{End}(F)$$

by $S(\alpha)_C \coloneqq \alpha^*_{C^*}$ for $C \in \mathcal{C}$. One checks that the associator, unitors, and interchanger in $\mathcal{C}$ are absorbed by the canonical associator, unitors, and interchangers of algebras and the identification of $\mathrm{End}(G) \otimes \mathrm{End}(H)$ with $\mathrm{End}(G \boxtimes H)$. In particular, Δ is co-associative, ϵ is co-unital, and Δ is co-commutative. ■

Theorem 3.5 (Krein)**.** For a SFC $\mathcal{C}$ with fiber functor $F \colon \mathcal{C} \to \mathsf{Vec}$,

$$\mathcal{C} \cong \mathsf{Rep}(\mathrm{Aut}_{\otimes}(\mathcal{C} \xrightarrow{F} \mathsf{Vec}))$$

as symmetric fusion categories, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathsf{Rep}(\mathrm{Aut}_{\otimes}(F)) \\ & {\scriptstyle F}\searrow \quad \swarrow & \\ & \mathsf{Vec} & \end{array}$$

Proof. By Lemma 3.4, $\mathrm{End}(F) = \bigoplus_{X_i \in P(\mathcal{C})} \mathrm{End}(F(X_i))$ is a finite dimensional co-commutative Hopf algebra, with grouplike elements $\mathrm{Aut}_{\otimes}(F)$. By Fact 3.3,

$$\mathrm{End}(F) = \mathbb{C}[\mathrm{Aut}_{\otimes}(F)].$$

Therefore, Lemma 3.1 yields that

$$\mathcal{C} \cong \mathsf{Rep}(\mathbb{C}[\mathrm{Aut}_{\otimes}(F)]) = \mathsf{Rep}(\mathrm{Aut}_{\otimes}(F))$$

as categories. Finally, for $\alpha \in \mathrm{Aut}_{\otimes}(F)$ and $C, C' \in \mathcal{C}$, observe

$$\rho_{F(C\otimes C')}(\alpha) = \alpha_{F(C\otimes C')} = \alpha_{F(C)} \otimes \alpha_{F(C')} = \rho_{F(C)} \otimes \rho_{F(C')}(\alpha).$$

So $\widehat{F}(C) \otimes \widehat{F}(C') = \widehat{F}(C \otimes C')$ when we restrict ourselves to the action of $\mathrm{Aut}_{\otimes}(F) \subset \mathrm{End}(F)$. Therefore, $\mathcal{C} \cong \mathsf{Rep}(\mathrm{Aut}_{\otimes}(F))$ as symmetric fusion categories. □

References

[1] P. Deligne and J. S. Milne. Tannakian categories.

[2] Pavel I. Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*. American Mathematical Society, 2017.

[3] Pavel I. Etingof and Olivier Schiffmann. *Lectures on quantum groups*. International Press, 2012.