Functional Analysis in Three-Dimensional Category Theory

Dissertation

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Abstract

Operator algebras first arose through the mathematical foundations of quantum mechanics. We have since seen a movement throughout mathematics of categorifying structures and their results–motivated by the fact that n-categories naturally describe n-dimensional structures. This leads us to develop tools for describing a theory of functional analysis in three-dimensional category theory.

The main content of this thesis is divided into four parts.

In Chapter §1, we introduce the operator algebraic and category theoretic preliminaries needed for our results. In particular, we discuss operator 1-categories, operator 2-categories, algebraic 3-categories and their semi-strict version, Gray-categories.

In Chapter §2, we show the localization of the category of Gray-categories at the weak equivalences is equivalent to the category of algebraic 3-categories and equivalence classes of weak 3-functors. This result finishes establishing the homotopy hypothesis for homotopy 3-types.

In Chapter §3, we provide a definition of operator 3-categories and prove many of their desiderata. In particular, we show every operator algebraic tricategory is equivalent to an operator Gray-category, categorify the Gelfand-Naimark theorem for operator algebras, and provide several examples of interest. Of particular importance is the Morita operator 3-category $\mathsf{AbC^*Alg}$ of abelian C*-algebras, C*-algebras with central maps, C*-correspondences, and adjointable bimodule maps.

In Chapter §4, we compute the homotopy groups of the homotopy 3-type associated to $\mathsf{AbC^*Alg}$. We describe these groups purely in terms of the topological and cohomological data and compute the actions of the first homotopy group on the second and third homotopy groups in terms of these topological invariants.

This is dedicated to my parents, Nell and Tito

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—sea lo que sea que hagas en la vida, hazlo con pasión y cariño.

Vita

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Publications

Research Publications

Manifestly unitary higher Hilbert spaces (with Chen, Hungar, Penneys, Sanford), accepted by J. Lond. Math. Soc., `arXiv:2410.05120`

Gray-categories model algebraic tricategories, *Theor. Appl. Categ.* **38** (2022), no. 29, pp 1136-1155, `arXiv:2203.03748`

Classifying module categories for generalized Temperley-Lieb-Jones $*$-2-Categories (with Hernández-Palomares), *Internat. J. Math.* **31** (2020), no. 04, 2050027, `MR4098904`, `arXiv:1905.00471`

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Fields of Study

Major Field: Mathematics

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Chapter 1: Preliminaries

1.1 Background on operator algebras

1.1.1 Operator theory

We first recall some facts about Hilbert spaces in order to introduce our notations and conventions.

Definition 1.1.1.1. A Hilbert space $(H, \langle\cdot|\cdot\rangle)$ consists of a vector space H together with a complete inner product $\langle\cdot|\cdot\rangle\colon \overline{H}\otimes H \to \mathbb{C}$ satisfying:

- $\langle\xi|\eta\rangle = \overline{\langle\eta|\xi\rangle}$;
- $\langle\xi|\xi\rangle \geq 0$ with equality only when $\xi = 0$;
- H is Cauchy complete with respect to the norm $\|\xi\| := \langle\xi|\xi\rangle^{1/2}$.

We will also use the convention $\langle\cdot,\cdot\rangle\colon H\otimes\overline{H}\to\mathbb{C}$ for inner products which are linear on the left and conjugate linear on the right. These are of course interchangeable by setting $\langle\cdot,\cdot\rangle = \overline{\langle\cdot|\cdot\rangle}$.

Example 1.1.1.2. The prototypical example of a Hilbert space is $\mathbb{C}$ with the inner product $\langle z_1|z_2\rangle := \overline{z_1}z_2$.

Definition 1.1.1.3. Let H, K be Hilbert spaces and $x\colon H \to K$ be a linear map. We define the operator norm $\|x\| \in [0, \infty]$ by

$$\|x\| \coloneqq \sup_{\substack{\xi \in H \\ \|\xi\|=1}} \|x\xi\|.$$

We say that x is a (bounded) operator whenever $\|x\| < \infty$ and denote the space of such operators by $B(H \to K)$. Moreover, we use the shorthand $B(H) \coloneqq B(H \to H)$ and $H^* \coloneqq B(H \to \mathbb{C})$.

Fact 1.1.1.4. *A map $x\colon H \to K$ is bounded if and only if x admits an adjoint $x^\dagger\colon K \to H$ satisfying*

$$\langle x\xi | \eta \rangle = \langle \xi | x^\dagger \eta \rangle.$$

1.1.2 Operator algebras

Unless otherwise stated, we will assume all C*-algebras to be unital.

Definition 1.1.2.1. A C*-algebra A consists of the following data.

- A (unital) algebra A. We will use lowercase $a, b, \ldots$ for elements of A and $1_A \in A$ for its unit. When it is unambiguous, we will write $z = z \cdot 1_A \in A$ for $z \in \mathbb{C}$.
- A (conjugate-linear) involution $\dagger\colon \overline{A} \to A$ satisfying $(ab)^\dagger = b^\dagger a^\dagger$ and $a^{\dagger\dagger} = a$.
- A complete submultiplicative norm $\|\cdot\|$ on A.

We require that this data satisfy the C*-identity:

$$\|a\| = \|a^\dagger a\|^{1/2} \qquad \forall a \in A.$$

An element $a \in A$ is *positive* ($a \geq 0$) if there exists $b \in A$ with $b^\dagger b = a$. We use A^+ to denote the (positive) cone of positive elements. Moreover, we denote the group of invertible elements of A by $A^\times$.

Fact 1.1.2.2. *Given a C*-algebra A, its norm is uniquely determined by spectral theory. In particular, for $a \in A$, we define its spectrum* $\mathsf{spec}(a)$ *and spectral radius* $\rho(a)$ *by*

$$\mathsf{spec}(a) = \{z \in \mathbb{C} \mid a - z \notin A^\times\}$$

$$\rho(a) = \sup_{z \in \mathsf{spec}(a)} |z|$$

It is a well-known fact that $\|a\| = \|a^\dagger a\|^{1/2} = \rho(a^* a)^{1/2}$, *where the latter quantity only depends on the* $*$*-algebra structure on* A. *In particular, being a C*-algebra is a property of a* $*$*-algebra and not extra structure.*

Example 1.1.2.3. The prototypical example of a C*-algebra is the complex numbers $\mathbb{C}$ with complex conjugation $\overline{\cdot}$ and the norm $|z| := (\overline{z}z)^{1/2}$. Notice $\mathbb{C}^+ = [0, \infty)$.

Example 1.1.2.4. For a Hilbert space H, the algebra of (bounded) operators $B(H)$ forms a C*-algebra. Moreover, any norm closed subalgebra of $B(H)$ is again a C*-algebra. We will refer to these as the *concrete* C*-algebras.

Example 1.1.2.5. For a compact Hausdorff space T, the algebra of (continuous) complex-valued functions $C(T) := \mathrm{Hom}(T \to \mathbb{C})$ is a commutative C*-algebra.

Example 1.1.2.6. Given a C*-algebra A, its center $Z(A) := \{a \in A \mid ab = ba \text{ for all } b \in A\}$ is again a C*-algebra.

Definition 1.1.2.7. A C*-algebra A is called a *W*-algebra* if A admits a predual A_*, i.e. a Banach space with $A_*^* \cong A$. The weak*-topology on A inherited from A_* is also known as the ultraweak topology.

Theorem 1.1.2.8 (von Neumann Bicommutant Theorem)**.** *Let* $A \subseteq B(H)$ *be a concrete C*-algebra of operators on* H. *The following are equivalent:*

- *A is WOT closed, i.e. if $(x_\lambda) \subseteq A$ and $x \in B(H)$ satisfy $\langle x_\lambda \xi | \eta \rangle \to \langle x\xi | \eta \rangle$ for all $\xi, \eta \in H$, then $x \in A$;*
- *A is SOT closed, i.e. if $(x_\lambda) \subseteq A$ and $x \in B(H)$ satisfy $x_\lambda \xi \to x\xi$ for all $\xi \in H$, then $x \in A$;*
- *$A'' = A$ where the commutant S' of a set $S \subseteq B(H)$ is given by*

$$S' \coloneqq \{x \in B(H) \mid xs = sx \text{ for all } s \in S\};$$

- *A is a W*-algebra.*

When these conditions hold, A is also said to be a von Neumann algebra.

Definition 1.1.2.9. A map of C*-algebras $\phi\colon A \to B$ consists of a (unital) $*$-homomorphism, i.e. a linear map such that

- $\phi(ab) = \phi(a)\phi(b)$;
- $\phi(1_A) = 1_B$;
- $\phi(a^\dagger) = \phi(a)^\dagger$.

We say that ϕ is a $*$-isomorphism whenever it is bijective. When A and B are W*, we say ϕ is *normal* whenever it is weak*-continuous.

Fact 1.1.2.10. *A $*$-homomorphism $\phi\colon A \to B$ of C*-algebras is always contractive, i.e. $\|\phi(a)\| \leq \|a\|$ for all $a \in A$. Moreover, φ injective if and only if it is isometric, i.e. $\|\phi(a)\| = \|a\|$.*

Fact 1.1.2.11. *If $\phi\colon A \to B$ is a $*$-isomorphism between W*-algebras, then it is automatically normal.*

1.1.3 Representation Theory

We now provide some background on the representation theory of operator algebras.

Definition 1.1.3.1. A *representation* of a C*-algebra A consists of a Hilbert space H_λ equipped with a $*$-homomorphism $\lambda\colon A \to B(H_\lambda)$. We say H_λ is faithful or full whenever λ is injective or surjective respectively. When A is W*, we further require λ to be normal.

Example 1.1.3.2. All Hilbert spaces are representations of $\mathbb{C}$.

Example 1.1.3.3. Each Hilbert space H is canonically a representation of $B(H)$.

Example 1.1.3.4. For a compact Hausdorff space T, each point $t \in T$ induces a representation of $C(T)$ on $\mathbb{C}$ by

$$C(T) \xrightarrow{\mathrm{ev}_t} \mathbb{C} = B(\mathbb{C})$$

$$f \longmapsto f(t)$$

Definition 1.1.3.5. Let H_λ and K_ρ be representations of A. A map of representations $x\colon H_\lambda \to K_\rho$ consists of an operator $x \in B(H \to K)$ such that $\lambda(a)x = x\rho(a)$ for all $a \in A$. Two representations λ and ρ of A are unitarily equivalent if there is a unitary $(u^\dagger = u^{-1})$ operator $u\colon H_\rho \to H_\lambda$ so that

$$\lambda(a) = u\rho(a)u^\dagger \qquad \forall a \in A.$$

We will denote the category of representations of A and their maps by $\mathsf{Rep}(A)$.

1.1.4 GNS construction

Definition 1.1.4.1. A functional $\varphi \in A^*$ is called positive when $\varphi(a^*a) \geq 0$ for every $a \in A$. It is further called a state when $\|\varphi\| = 1$ and definite when $\varphi(a^*a) = 0$ if and only if $a = 0$. When A is W*, we say φ is normal whenever it is weak*-continuous.

We will denote the space of (normal) states on A by $\mathbb{P}(A)$.

Example 1.1.4.2 (Riesz-Markov-Kakutani Representation Thm.)**.** States on $C(T)$ are precisely the Borel probability measures on T via the map

$$\begin{aligned} \mathbb{P}(T) &\xrightarrow{\sim} \mathbb{P}(C(T)) \\ \mu &\longmapsto \int_T - \, d\mu \end{aligned}$$

Theorem 1.1.4.3 (Hahn-Banach)**.** *For every $a \in A$ in a C*-algebra, there exists $\varphi \in \mathbb{P}(A)$ with $\varphi(a) := \|a\|$. In particular, $a = 0$ if and only if $\varphi(a) = 0$ for every $\varphi \in \mathbb{P}(A)$. When A is W*, one may take φ to be normal.*

Definition 1.1.4.4. Any positive $\varphi \in A^*$ defines a sesquilinear form on A by

$$\langle a|b\rangle_\varphi := \varphi(a^*b).$$

We denote the seminorm induced from $\langle\cdot|\cdot\rangle_\varphi$ by $\|\cdot\|_\varphi$. Notice the subspace of negligibles $N_\varphi := \{a \in A \mid \|a\|_\varphi = 0\}$ is the obstruction to φ being definite. The GNS construction associated to φ is the Hilbert space obtained by completing the following quotient

$$L^2(A, \varphi) := \overline{A/N_\varphi}$$

equipped with the inner product induced from $\langle\cdot|\cdot\rangle_\varphi$. We denote the representative of $1 \in A$ in $L^2(A,\varphi)$ by Ω_φ, for which $\|\Omega_\varphi\|_\varphi = 1$ if and only if φ is a state. Notice $L^2(A,\varphi)$ admits a (left) action of A by left multiplication $a \triangleright b\Omega := ab\Omega$. When A is W* and φ is normal, this action is also normal.

Operator algebras *are* algebras of operators on Hilbert spaces. This is made precise by the Gelfand-Naimark theorem [GN43] and, in particular, the Gelfand-Naimark-Segal (GNS) construction [Seg47].

Theorem 1.1.4.5 (Gelfand-Naimark-Segal). *The universal GNS construction*

$$H_\Upsilon := \bigoplus_{\varphi\in\mathbb{P}(A)} L^2(A,\varphi)$$

inherits a faithful A-action $\Upsilon\colon A\to B(H_\Upsilon)$ such that for $\varphi\in A^*$, there exist $\xi,\eta\in H_\Upsilon$ with

$$\varphi(a) = \langle\Upsilon(a)\xi,\eta\rangle \quad \textit{for all } a\in A.$$

In this case we write $\varphi = \langle\Upsilon\cdot\xi,\eta\rangle$. When A is W, this representation is normal when we interpret $\mathbb{P}(A)$ as the space of* normal *states.*

Remark 1.1.4.6. The previous statement can be interpreted as saying every abstract C*-algebra (W*-algebra resp.) A can be realized as a concrete C*-algebra (W*-algebra resp.) of operators on a Hilbert space, namely A is isomorphic to its image $\mathsf{GNS}(A) := \operatorname{Im}\Upsilon$ in $B(H_\Upsilon)$.

1.1.5 W*-completion of a C*-algebra

Recall that, for a C*-algebra A, there exists an organic inclusion $\operatorname{ev}\colon A\hookrightarrow A^{**}$ given by

$$\operatorname{ev}_a(\varphi) := \varphi(a) \quad \text{for } a\in\mathcal{A} \text{ and } \varphi\in\mathcal{A}^*.$$

Clearly A^{**} has a predual, namely A^*, and the Goldstine theorem tells us that A^{**} is "small" in the following sense:

Fact 1.1.5.1 (Goldstine). *For a C*-algebra*[1] *A, $\operatorname{ev}(A)$ is weak*-dense in A^{**}.*

Thus, if one equips A with the structure of a C*-algebra such that ev is a $*$-homomorphism, A^{**} would then serve as the universal W*-completion of A. This is precisely the content of Arens' results from [Are51b] and [Are51a]. We will state the main result here and refer the reader to [Pal74] for an exposition on the topic.

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[1] This theorem holds more generally for any Banach space.

Fact 1.1.5.2 (Arens)**.** *Let A be a C*-algebra. There is a unique W*-algebra structure on A^{**} such that:*

- *the map* $\mathrm{ev}\colon A \hookrightarrow A^{**}$ *is a $*$-homomorphism;*
- *for every $*$-homomorphism $f\colon A \to B$ into a W*-algebra B, there exists a unique normal $*$-homomorphism $\widetilde{f}\colon A^{**} \to B$ making the following triangle commute.*

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, hook, "\mathrm{ev}"'] & B \\
A^{**} \arrow[ur, dashed, "\exists!\widetilde{f}"'] &
\end{tikzcd}$$

On the other hand, the GNS construction realizes A as a concrete C*-algebra $\mathsf{GNS}(A) \subseteq B(H_\Upsilon)$, which one can complete to obtain a W*-algebra $\mathsf{GNS}(A)''$. The following result relates these two constructions.

Fact 1.1.5.3 (Sherman-Takeda for C*-algebras)**.** *Let A be a C*-algebra. There is a unique normal $*$-homomorphism $\widetilde{\Upsilon}\colon A^{**} \to \mathsf{GNS}(A)''$ making the following square commute:*

$$\begin{tikzcd}
A \arrow[r, "\Upsilon"] \arrow[d, hook, "\mathrm{ev}"'] & \mathsf{GNS}(A) \arrow[d, hook] \\
A^{**} \arrow[r, "\widetilde{\Upsilon}"'] & \mathsf{GNS}(A)''
\end{tikzcd}$$

1.1.6 Tensor products

Definition 1.1.6.1. For C*-algebras A and B, their spatial tensor product is defined to be the norm completion

$$A \otimes_{\min} B \coloneqq \overline{\mathsf{GNS}(A) \otimes \mathsf{GNS}(B)}^{\|\cdot\|_{\min}} \subseteq B(H_{\Upsilon_A} \otimes K_{\Upsilon_B})$$

where $\|\cdot\|_{\min}$ is the standard operator norm on $B(H_{\Upsilon_A} \otimes K_{\Upsilon_B})$.

Definition 1.1.6.2. For C*-algebras A and B, their maximal tensor product is defined to be the norm completion

$$A \otimes_{\max} B \coloneqq \overline{A \otimes B}^{\|\cdot\|_{\max}}$$

where $\|\cdot\|_{\max}$ is defined on $x \in A \otimes B$ by

$$\|x\| \coloneqq \sup\{\|(\rho \otimes \sigma)x\| \mid H_\rho \in \mathsf{Rep}(A), H_\sigma \in \mathsf{Rep}(B) \text{ with commuting images in } B(H)\}.$$

In general, there are many possible norms one can equip the algebraic tensor product $A \otimes B$ of C*-algebras such that completion yields a C*-algebra. It is a well-known fact that for any such norm $\|\cdot\|$, we have

$$\|\cdot\|_{\min} \leq \|\cdot\| \leq \|\cdot\|_{\max}.$$

We refer the reader to [WO93, Appendix T] for an exposition on this topic.

Remark 1.1.6.3. Out of all the C*-tensor products, it is the maximal tensor product that enjoys a universal property in the category theoretic sense. Since maps between C*-algebras do not themselves form C*-algebras, we will defer till §1.2.5 to express this fact more naturally as a hom-tensor adjunction for the more general C*-categories.

Remark 1.1.6.4. When A and B are W*-algebras, there is an analogous universal tensor product $A \underset{\max}{\overline{\otimes}} B$ constructed as a quotient of the W*-completion $(A \underset{\max}{\otimes} B)^{**}$ of their maximal tensor product as C*-algebras [Dau72]. We state these results for operator categories in §1.2.7 and build the necessary machinery for operator 2-categories in §3.2.4.

1.1.7 Spectra & Dauns–Hofmann

In this section, we will take a deeper look at representations of a C*-algebra. Of particular importance are the irreducible representations as these form the building blocks of larger representations. A closed subspace $X \subseteq H_\lambda$ is *invariant* for λ if $\lambda(a)x \in X$ for all $a \in A$ and $x \in X$.

Definition 1.1.7.1. A representation λ is said to be *irreducible* if the only closed invariant subspaces are $\{0\}$ and H_λ. Equivalently, the only operators in $B(H_\lambda)$ commuting with $\lambda(A)$ are scalar multiples of the identity by [RW98, Lem. A.1].

Remark 1.1.7.2. For an irreducible representation λ of A, observe that $\lambda(Z(A)) \subseteq \mathbb{C}1_{H_\lambda}$ because $\lambda(Z(A))$ commutes with $\lambda(A)$. As a result, when we write $\lambda|_{Z(A)}$, we will also implicitly restrict the codomain to be $\mathbb{C} \cong \mathbb{C}1_{H_\lambda}$.

Definition 1.1.7.3. The *spectrum* $\hat{A}$ of a C*-algebra A is defined to be the set of unitary equivalence classes of irreducible representations of A.

Currently, the spectrum is simply a set; it does not carry a topology. A topology on $\hat{A}$ is defined using the ideals that arise as kernels of these irreducible representations.

Definition 1.1.7.4. A closed, two-sided ideal P of A is *primitive* if P is the kernel of an irreducible representation of A. Let $\operatorname{Prim} A$ denote the set of primitive ideals of A.

The topology on $\operatorname{Prim} A$ is defined in the following manner.

Definition 1.1.7.5. Given a subset $F \subseteq \operatorname{Prim} A$, define its closure $\overline{F}$ to be the set

$$\overline{F} := \{P \in \operatorname{Prim} A : \bigcap_{I \in F} I \subseteq P\}$$

One verifies, using the Kuratowski closure axioms, that this closure operation defines a topology on $\operatorname{Prim} A$ called the *hull-kernel topology.* Since two unitarily equivalent representations have the same kernel, the map $[\lambda] \mapsto \ker \lambda$ is a well-defined surjection $\ker\colon \hat{A} \twoheadrightarrow \operatorname{Prim} A$. The spectrum is thus equipped with a topology by pulling back the topology on $\operatorname{Prim} A$ via the kernel map. In general, neither of these topologies is Hausdorff. However, as we will be interested in unital C*-algebras, the spectrum and primitive ideal space will at least be compact.

Proposition 1.1.7.6 ([RW98, Lem. A.30])**.** *Let* A *be a unital C*-algebra. Then* $\hat{A}$ *and* $\operatorname{Prim} A$ *are compact.*

We now discuss the Dauns–Hofmann Theorem. Since we are only interested in the unital case, we can avoid discussing multiplier algebras.

Theorem 1.1.7.7 ([DH68, Lem. 8.15], Dauns–Hofmann)**.** *Let* A *be a unital C*-algebra. Then there is a* $*$*-isomorphism* $\mathcal{F}\colon C(\operatorname{Prim} A) \to Z(A)$ *so that*

$$q_P(\mathcal{F}(f)a) = f(P)q_P(a)$$

for all $P \in \operatorname{Prim} A$ *and* $a \in A$*, where* $q_P\colon A \to A/P$ *is the quotient map.*

As a consequence, we obtain a diagram of the following form.

Corollary 1.1.7.8 (c.f. Lem. 4.2.1.5)**.** *Let* A *be a unital C*-algebra. Then there exist continuous surjections:*

$$\begin{tikzcd} \hat{A} \arrow[r, two heads, "\ker"] & \operatorname{Prim} A \arrow[r, two heads, "D\text{–}H"] & \widehat{Z(A)}. \end{tikzcd}$$

Moreover, these map witness $\widehat{Z(A)}$ *as the Stone–Čech compactifications*[2] *of* $\operatorname{Prim} A$ *and* $\hat{A}$ *[Nil95, Lem. 1.1].*

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[2] Usually, this construction is applied to a Hausdorff space X and is commonly referred to as the compactification βX. In this case, however, $\hat{A}$ is already compact, but not Hausdorff. Thus, this construction would be aptly named the "Hausdorffication" $\beta\hat{A}$ of $\hat{A}$. We will further discuss this construction in the later section §4.2.1.

1.1.8 Imprimitivity Bimodules

As before, we will use $\langle\cdot|\cdot\rangle$ to denote (A-valued) inner products that are conjugate linear in the first coordinate and linear in the second. On the other hand, we will use $\langle\cdot,\cdot\rangle$ for inner products that are linear in the first coordinate and conjugate linear in the second.

Definition 1.1.8.1. A right *Hilbert C*-module* $(X, \langle\cdot|\cdot\rangle_A)$ for a C*-algebra A consists of a (complex) vector space X equipped with a right action $\triangleleft\colon X \otimes A \to X$ and an A-valued inner product $\langle\cdot|\cdot\rangle_A\colon \overline{X} \otimes X \to A$ satisfying:

- $\langle x|y \triangleleft a\rangle_A = \langle x|y\rangle_A \cdot a$
- $\langle y|x\rangle_A = \langle x|y\rangle_A^\dagger$;
- $\langle x|x\rangle_A \geq 0$ with equality only when $x = 0$;
- X is Cauchy complete with respect to the norm $\|x\|_X = \|\langle x|x\rangle_A\|^{1/2}$.

When A is W*, we say that $(X, \langle\cdot|\cdot\rangle_A)$ is a *Hilbert W*-module* if we further have that:

- the Banach space $(X, \|\cdot\|_X)$ admits a predual X_*;
- each map $\langle x|\colon X \to A$ given by $y \mapsto \langle x|y\rangle_A$ is weak*-continuous.

A left Hilbert C*/W*-module is defined similarly with a left module action of A on X and an A-valued inner product ${}_A\langle\cdot,\cdot\rangle$. We will often omit the subscript A whenever it is clear.

Observe that $\langle X|X\rangle_A \coloneqq \operatorname{span}\{\langle x|y\rangle_A : x, y \in X\}$ is a two-sided ideal of A. A Hilbert A-module X is called *full* if $\langle X|X\rangle_A$ is a dense ideal of A. The following remark about full Hilbert modules will be used numerous times.

Remark 1.1.8.2. We note that fullness implies that the action is faithful. Indeed, for X_A a full Hilbert module, if $x \lhd a = 0$ for every $x \in X$,

$$\langle y|x\rangle_A \cdot a = \langle y|x \lhd a\rangle_A = 0.$$

As this holds for all $x, y \in X$, and since $\langle X|X\rangle_A \subseteq A$ is dense, we have $a = 1_A \cdot a = 0$.

Definition 1.1.8.3. Given two right Hilbert C*-modules X_A and Y_A over A, a function $f\colon X \to Y$ is *adjointable* if there is another function $f^\dagger\colon Y \to X$ satisfying

$$\langle fx|y\rangle_A = \langle x|f^\dagger y\rangle_A \text{ for all } x \in X,\ y \in Y.$$

We denote the set of adjointable A-module maps from X_A to Y_A by $\mathcal{L}_A(X, Y)$, or simply $\mathcal{L}_A(X)$ if $X_A = Y_A$.

Remark 1.1.8.4. Adjointable functions are automatically continuous A-module maps. When A is W* and X, Y are right Hilbert W*-modules, it is further true that adjointable functions are automatically weak*-continuous.

Definition 1.1.8.5. An A–B *C*-correspondence* ${}_AX_B$ is a right Hilbert B-module X equipped with a compatible left action, i.e. a $*$-homomorphism $A \to \mathcal{L}_B(X)$. When A and B are W*, we say that ${}_AX_B$ is a W*-correspondence whenever the maps $a \mapsto \langle x|a \rhd y\rangle_B$ are normal.

The following definition describes the invertible correspondences.

Definition 1.1.8.6. An A–B correspondence ${}_AX_B$ is called an *imprimitivity bimodule* if:

1. ${}_AX$ is a full left Hilbert A-module and a full right Hilbert B-module X_B.

2. For all $x, y \in X, a \in A$, and $b \in B$, we have

$$\langle a \triangleright x|y\rangle_B = \langle x|a^\dagger \triangleright y\rangle_B \quad \text{and} \quad {}_A\langle x \triangleleft b, y\rangle = {}_A\langle x, y \triangleleft b^\dagger\rangle$$

   so that A and B act as adjointable operators relative to the other's inner product.

3. For all $x, y, z \in X$, we have

$$_A\langle x, y\rangle \triangleright z = x \triangleleft \langle y|z\rangle_B.$$

Imprimitivity bimodules induce an equivalence relation on operator algebras, analogous to Morita equivalence in algebra.

Definition 1.1.8.7. Two operator algebras A and B are said to be *Morita equivalent* if there exists an A-B imprimitivity bimodule.

1.1.9 Continuous-Trace C*-Algebras

Before discussing continuous-trace C*-algebras, we need to address the interaction between Morita equivalence and spectra of C*-algebras.

Theorem 1.1.9.1 ([RW98, Thm. 3.29]). *Let A and B be Morita equivalent C*-algebras. Then $\hat{A} \cong \hat{B}$.*

Theorem 1.1.9.2 ([RW98, Cor. 3.33]). *Suppose $_AX_B$ is an A–B imprimitivity bimodule. Then X induces a homeomorphism $h_X\colon \operatorname{Prim} B \to \operatorname{Prim} A$ called the Rieffel homeomorphism. The map h_X is given on $P \in \operatorname{Prim} B$ by*

$$h_X(P) = \overline{\operatorname{span}}\{_A\langle x \triangleleft b, y\rangle : x, y \in X, b \in P\}.$$

Since imprimitivity bimodules induce homeomorphisms of the spectra and primitive ideal spaces, we may expect that the Dauns–Hofmann Theorem can be used to understand the actions of $Z(A)$ and $Z(B)$ on $_AX_B$. This is made precise in the following proposition.

Proposition 1.1.9.3 ([RW98, Prop. 5.7]). *Suppose $_AX_B$ is an A–B imprimitivity bimodule, and A and B are unital C*-algebras with Hausdorff spectrum. With $h_X\colon \operatorname{Prim} B \to \operatorname{Prim} A$ the Rieffel homeomorphism, we have*

$$\mathcal{F}(f) \triangleright x = x \triangleleft \mathcal{F}(f \circ h_X) \qquad \text{for } f \in C(\operatorname{Prim} A) \text{ and } x \in X.$$

Because the Rieffel homeomorphism intertwines the central actions, we can also consider equivariant bimodules where this homeomorphism is the identity.

Definition 1.1.9.4. Let A and B be C*-algebras with $*$-isomorphisms $\phi\colon C(T) \to Z(A)$ and $\psi\colon C(T) \to Z(B)$. Furthermore, assume $\hat{A}$ and $\hat{B}$ are Hausdorff. We say an imprimitivity A-B bimodule X is an imprimitivity bimodule *over* T if, for all $f \in C(T)$ and $x \in X$,

$$\phi(f) \triangleright x = x \triangleleft \psi(f).$$

When such ${}_AX_B$ exists, we say A and B are *Morita equivalent over* T.

Proposition 1.1.9.5 ([RW98, Prop. 5.7])**.** *With the hypotheses of Definition 1.1.9.4,* ${}_AX_B$ *is an* A-B *imprimitivity bimodule over* T *if and only if the following triangle commutes:*

$$\begin{tikzcd}
\operatorname{Prim} B \arrow[r, "h_X"] \arrow[d, "\hat{\psi}"'] & \operatorname{Prim} A \arrow[dl, "\hat{\phi}"] \\
T &
\end{tikzcd}$$

We are now ready to work towards the definition of continuous-trace C*-algebras. We first need to describe how traces on irreducible representations interact with elements of A.

Definition 1.1.9.6. Let A be a C*-algebra and $a \in A^+$. We define the function

$$\operatorname{Tr}_a\colon \hat{A} \to \mathbb{R}^+ \cup \{\infty\}$$

$$[\lambda] \mapsto \operatorname{Tr}(\lambda(a))$$

where Tr is the (unnormalized) trace on $B(H_\lambda)$. In the special case when $a = 1_A$, we write $\dim_A$ instead of Tr_{1_A}, as the trace of the unit detects the dimension of the representation.

In general, these functions are not continuous on $\hat{A}$. They are, however, always lower semi-continuous.

Proposition 1.1.9.7 ([Dix77, Prop. 3.5.9]). *For any C*-algebra A, the functions* $\{\mathrm{Tr}_a : a \in A^+\}$ *are lower semi-continuous on* $\hat{A}$.

There are three equivalent definitions of continuous-trace C*-algebras. We will use the definition involving continuous-trace elements and Tr_a.

Definition 1.1.9.8. Let A be a C*-algebra with Hausdorff spectrum. A positive element $a \in A^+$ is said to be a *continuous-trace* element if Tr_a is a continuous function on $\hat{A}$.

The span of the continuous-trace elements of a C*-algebra forms a two-sided (not necessarily closed) ideal J. The closure of this ideal then determines when an algebra has continuous-trace.

Definition 1.1.9.9. A (generally non-unital) C*-algebra A with Hausdorff spectrum is said to have *continuous-trace* if the ideal J, generated by continuous-trace elements, is dense in A.

Remark 1.1.9.10. Observe that, given a unital C*-algebra A, A has continuous-trace if and only if 1_A is a continuous-trace element. This is because the set of invertible elements $A^\times$ is an open subset of A, and so there is an invertible element in J when $\overline{J} = A$. Furthermore, notice $\dim_A$ takes values in $\mathbb{N} \cup \{\infty\}$. Therefore, 1_A is a continuous-trace element if and only if $\dim_A$ is constant on each component of $\hat{A}$.

A special class of continuous-trace C*-algebras is the class of homogeneous C*-algebras.

Definition 1.1.9.11. A C*-algebra A is *homogeneous* if the dimension of each irreducible representation is the same natural number $n < \infty$.

Homogeneous C*-algebras are automatically continuous-trace. When a homogeneous C*-algebra has compact spectrum, it is automatically unital ([Fel61, Thm. 3.2]). These will be important because unital continuous-trace C*-algebras are homogeneous on each connected component of their spectrum (see Remark 1.1.9.10). We now present the Dixmier–Douady classification of continuous-trace C*-algebras.

Theorem 1.1.9.12 ([RW98, Thm. 5.29])**.** *To each continuous-trace C*-algebra A with Hausdorff spectrum T, there is an associated element $\delta(A) \in H^3(T;\mathbb{Z})$ called the Dixmier–Douady class of A. Two continuous-trace C*-algebras with spectrum T are Morita equivalent over T if and only if $\delta(A) = \delta(B)$.*

Equivalence classes of continuous-trace algebras form a group, where the group operation is given by the following relative tensor product. However, the assumption of continuous-trace is not necessary for the construction of this tensor product.

Definition 1.1.9.13. Given unital C*-algebras A and B with central $*$-homomorphisms $\phi\colon C(T) \to Z(A)$ and $\psi\colon C(T) \to Z(B)$, we write $A \otimes_T B$ for the C*-algebra $(A \otimes_{\max} B)/I_T$, where I_T is the balancing ideal generated by elements of the form

$$a\phi(f) \otimes b - a \otimes \psi(f)b \qquad \text{for } a \in A, b \in B, f \in C(T).$$

Continuous-trace C*-algebras implicitly have such central $*$-homomorphisms due to the identifications of their spectra with T. This allows for the group of Morita equivalence classes of continuous-trace algebras to be defined as follows:

Definition 1.1.9.14 ([RW98, Thm. 6.3])**.** Given a compact Hausdorff space T, define the *Brauer group* of T, denoted $\mathrm{Br}(T)$, to be the group whose elements are Morita equivalence classes (over T) of (non-unital) continuous-trace C*-algebras with spectrum T. The group operation is given by $[A][B] = [A \otimes_T B]$. The identity element is $[C(T)]$, and $[A]^{-1} = [A^{\mathrm{op}}]$.

Considering non-unital continuous-trace C*-algebras is important for the Brauer group, as it allows for the Dixmier–Douady class to be a surjective map onto $H^3(T;\mathbb{Z})$. When restricting to unital continuous-trace algebras, the (unital) Brauer group surjects onto the torsion subgroup $H^3_{\mathrm{tor}}(T;\mathbb{Z})$ (see Lem. 4.2.1.10).

Theorem 1.1.9.15 ([RW98, Thm. 6.3])**.** *The map* $\delta\colon Br(T) \to H^3(T;\mathbb{Z})$ *given by* $[A] \mapsto \delta(A)$ *is a group isomorphism.*

We refer the reader to [RW98] for a thorough exposition on continuous-trace algebras.

1.2 Background on operator categories

In this section, we provide the requisite background for operator 1-categories, which we will directly categorify in Section §1.3.2 in order to provide novel results in Section §3.2.

1.2.1 Algebraic categories

We briefly recall some facts about categories in order to introduce our notation and some diagrammatics.

Definition 1.2.1.1. A category $\mathcal{A}$ consists of the following data:

(0) A collection of objects, which we denote by $A, B, C, \ldots \in \mathcal{A}$;

(1) For every pair of objects $A, B \in \mathcal{A}$, a collection $\mathcal{A}(A \to B)$ of morphisms;

($\circ$) For every triple of objects $A, B, C \in \mathcal{A}$, a composition law

$$\mathcal{A}(A \to B) \times \mathcal{A}(B \to C) \xrightarrow{\circ} \mathcal{A}(A \to C)$$

$$(f, g) \mapsto g \circ f$$

We require that $\circ$ is associative and unital, where we denote the identity on $A \in \mathcal{A}$ by id_A. We say that $\mathcal{A}$ is ($\mathbb{C}$-)linear when each $\mathcal{A}(A \to B)$ is equipped with the structure of a (complex) vector space and $\circ$ is bilinear.

Remark 1.2.1.2. We can express the data of a category diagrammatically as follows:

(∘) For $A, B, C \in \mathcal{A}$, the map $\circ\colon \mathcal{A}(A \to B) \times \mathcal{A}(B \to C) \to \mathcal{A}(A \to C)$ is represented in Figure 1.1.

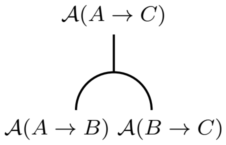

Figure 1.1: 2D string diagram for composition in categories

(I) For $A \in \mathcal{A}$, the identity $\mathrm{id}_A \in \mathcal{A}(A \to A)$ is represented in Figure 1.2.

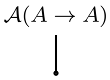

Figure 1.2: 2D string diagram for identities in categories

(a) For $A, B, C, D \in \mathcal{A}$, associativity is represented in Figure 1.3.

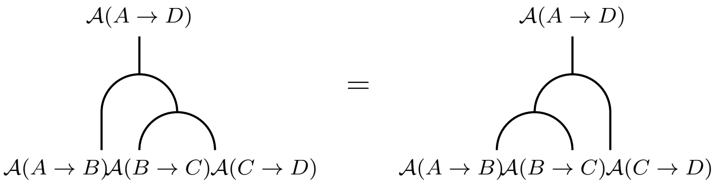

Figure 1.3: Associativity condition for composition in categories

(u) For $A, B \in \mathcal{A}$, unitality is represented in Figure 1.4.

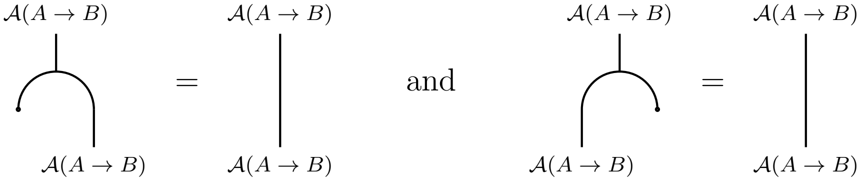

Figure 1.4: Unitality condition for composition in categories

Example 1.2.1.3. We denote the category of C*-algebras and $*$-homomorphisms by $\mathsf{AbC^*Alg_1}$. Similarly, the category of W*-algebras and normal $*$-homomorphisms by $\mathsf{AbW^*Alg_1}$.

Example 1.2.1.4. We denote the category of compact Hausdorff spaces and continuous maps by CHaus.

Example 1.2.1.5. Given a category $\mathcal{A}$, the *opposite category* $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$ with morphisms $\mathcal{A}^{\mathrm{op}}(A \to B) := \mathcal{A}(B \to A)$ and composition $g \circ^{\mathrm{op}} f = f \circ g$.

Definition 1.2.1.6. A functor $F\colon \mathcal{A} \to \mathcal{A}'$ between categories consists of the following data:

(0) A map of objects $\mathcal{A} \to \mathcal{A}'$;

(1) For every pair of objects $A, B \in \mathcal{A}$, a map $\mathcal{A}(A \to B) \to \mathcal{A}'(FA \to FB)$.

We require that $F(g \circ f) = Fg \circ Ff$ and $F(\mathrm{id}_A) = \mathrm{id}_{FA}$. When $\mathcal{A}$ and $\mathcal{A}'$ are linear, we say F is linear when every $\mathcal{A}(A \to B) \to \mathcal{A}'(FA \to FB)$ is linear. We denote functor composition by $G \circ F\colon \mathcal{A} \to \mathcal{A}'$ for $F\colon \mathcal{A} \to \mathcal{A}'$ and $G\colon \mathcal{A}' \to \mathcal{A}''$.

Remark 1.2.1.7. We can express the data of a functor diagrammatically as follows:

(hom) For $A, B \in \mathcal{A}$, the map $F\colon \mathcal{A}(A,B) \to \mathcal{A}'(FA,FB)$ is represented in Figure 1.5.

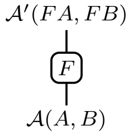

Figure 1.5: 2D string diagram for the action of functors on morphisms

(∘) For $A, B, C \in \mathcal{A}$, monoidality/composibility is represented in Figure 1.6.

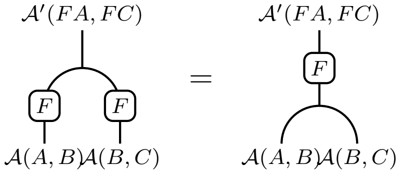

Figure 1.6: Monoidality condition for functors

(I) For $A \in \mathcal{A}$, unitality is represented in Figure 1.7.

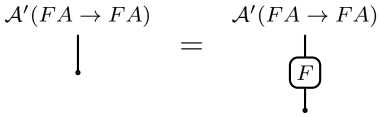

Figure 1.7: Unitality condition for functors

Example 1.2.1.8. There is a functor $C\colon \mathsf{CHaus}^{\mathrm{op}} \to \mathsf{AbC^*Alg_1}$ given on objects by $T \mapsto C(T)$ where we note that $C(T) = \mathsf{CHaus}(T \to \mathbb{C})$, and on morphisms $\Phi\colon T \to S$ by

$$C(S) \xrightarrow{\Phi^*} C(T)$$

$$f \mapsto f \circ \Phi$$

Example 1.2.1.9. There is a functor $\mathsf{spec}\colon {\mathsf{AbC^*Alg_1}}^{\mathrm{op}} \to \mathsf{CHaus}$ given on objects by $A \mapsto \mathsf{spec}(A)$ where $\mathsf{spec}(A) \coloneqq \mathsf{AbC^*Alg}(A \to \mathbb{C})$ is equipped with the pointwise topology, and on morphisms $\phi\colon A \to B$ by

$$\mathsf{spec}(B) \xrightarrow{\phi^*} \mathsf{spec}(A)$$

$$\varphi \mapsto \varphi \circ \phi.$$

Definition 1.2.1.10. A natural transformation $\alpha\colon F \to G$ between functors $F, G\colon \mathcal{A} \to \mathcal{A}'$ consists of:

(0) A collection of morphisms $(\alpha_A)_{A\in\mathcal{A}}$ indexed by objects of $\mathcal{A}$ where $\alpha_A \in \mathcal{A}'(FA \to GA)$.

We require that this collection is natural in $A \in \mathcal{A}$, i.e. for every $f \in \mathcal{A}(A \to B)$, the following square commutes:

$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\alpha_B"'] & GB
\end{tikzcd}$$

Remark 1.2.1.11. We can express the data of a natural transformation diagrammatically as follows:

(0) For $A \in \mathcal{A}$, the morphism $\theta_A \in \mathcal{A}'(FA \to F'A)$ is represented in Figure 1.8.

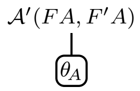

Figure 1.8: 2D string diagram for the components of natural transformations

(hom) For $A, B \in \mathcal{A}$, naturality is represented in Figure 1.9.

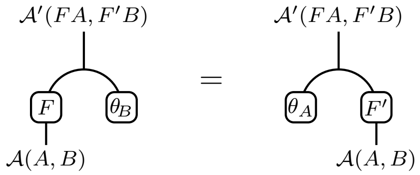

Figure 1.9: Naturality condition for natural transformations

Definition 1.2.1.12. There are several operations one may apply on natural transformations:

- For natural transformations $\theta\colon F \Rightarrow F'$ and $\sigma\colon G \Rightarrow G'$ between functors $F, F'\colon \mathcal{A} \to \mathcal{B}$ and $G, G'\colon \mathcal{B} \to \mathcal{C}$, we denote their 1-composition by

  $$\sigma \odot \theta\colon G \circ F \Rightarrow G' \circ F' \qquad (\sigma \odot \theta)_A \coloneqq \sigma_{F'A} \circ G(\theta_A) = G'(\theta_A) \circ \sigma_{FA} \qquad \text{for } A \in \mathcal{A}.$$

- For natural transformations $\theta\colon F \Rightarrow F'$ and $\theta'\colon F' \Rightarrow F''$ between functors $F, F', F''\colon \mathcal{A} \to \mathcal{B}$, we denote their 2-composition by

  $$\theta' \circ \theta\colon F \Rightarrow F'' \qquad (\theta' \circ \theta)_A \coloneqq \theta'_A \circ \theta_A \qquad \text{for } A \in \mathcal{A}.$$

- For natural transformations $\theta_1, \theta_2 \colon F \Rightarrow G$ between functors $F, G \colon \mathcal{A} \to \mathcal{B}$, if $\mathcal{A}, \mathcal{B}$ and F, G are linear, we may also take linear combinations by

$$z\theta_1 + \theta_2 \colon F \Rightarrow G \qquad (z\theta_1 + \theta_2)_A \coloneqq z(\theta_1)_A + (\theta_2)_A \qquad \text{for } z \in \mathbb{C}, A \in \mathcal{A}.$$

Using these operations, we say:

- $\theta \colon F \Rightarrow G$ is a *natural isomorphism* if there exists $\theta^{-1} \colon G \Rightarrow F$ with $\theta^{-1} \circ \theta = \mathrm{id}_F$ and $\mathrm{id}_G = \theta \circ \theta^{-1}$;
- $F \colon \mathcal{A} \to \mathcal{B}$ is an *equivalence* if there exists $F^{\bullet} \colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $\varepsilon \colon F^{\bullet} \circ F \overset{\sim}{\Rrightarrow} \mathrm{id}_{\mathcal{A}}$ and $\eta \colon \mathrm{id}_{\mathcal{B}} \overset{\sim}{\Rrightarrow} F \circ F^{\bullet}$;
- $\mathbf{F} \coloneqq (F, F^{\bullet}, \epsilon, \eta)$ is an *adjoint equivalence* if ϵ and η satisfy the zigzag/snake relations:

$$(\mathrm{id}_F \odot \varepsilon) \circ (\eta \odot \mathrm{id}_F) = \mathrm{id}_F$$
$$(\varepsilon \odot \mathrm{id}_{F^{\bullet}}) \circ (\mathrm{id}_{F^{\bullet}} \odot \eta) = \mathrm{id}_{F^{\bullet}}$$

1.2.2 Operator categories

Definition 1.2.2.1 (Operator category)**.** A C*-category is a linear category $\mathcal{A}$ equipped with:

(†) a conjugate linear contravariant involution † fixing objects, i.e. a collection of maps $\dagger \colon \overline{\mathcal{A}(A \to B)} \to \mathcal{A}(B \to A)$ satisfying $f^{\dagger\dagger} = f$ and $(g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger}$;

(C*) a sub-multiplicative norm $\|\cdot\|$ satisfying the C*-axiom, i.e. a norm on each hom-space $\mathcal{A}(A \to B)$ such that $\|g \circ f\| \leq \|g\|\|f\|$ and $\|f\| = \|f^{\dagger} \circ f\|^{1/2}$.

As a technical condition, we require algebraically positive morphisms to be spectrally positive, i.e.

(≥ 0) for $f \in \mathcal{A}(A \to B)$, there exists $g \in \mathrm{End}(A)$ such that $f^\dagger \circ f = g^\dagger \circ g$.

The previous condition holds automatically when $\mathcal{A}$ admits direct sums. We further say that a C*-category $\mathcal{A}$ is W* if:

(W*) every hom-space $\mathcal{A}(A \to B)$ admits a predual $\mathcal{A}(A \to B)_*$, i.e. a Banach space with

$$\mathcal{A}(A \to B)_*^* \cong \mathcal{A}(A \to B).$$

It automatically follows that $\circ$ is separately weak*-continuous.

Example 1.2.2.2. As described in Definition 1.1.3.5, there is a C*-category $\mathsf{Rep}(A)$ of representations of a C*-algebra A. For $A = C(T)$, we denote the C*-subcategory of finitely generated projective representations by $\mathsf{Hilb}_{\mathsf{fgp}}C(T)$.

Example 1.2.2.3. There is a C*-category $\mathsf{Hilb}(T)$ of finite-rank topological hermitian vector bundles over a compact Hausdorff space T, which we will discuss in Section §1.2.3.

Example 1.2.2.4 (Concrete operator 1-categories)**.** The prototypical example of an operator category is Hilb, the W*-category of Hilbert spaces and bounded linear transformations. Here, the norm of a morphism is given by the operator norm and the dagger of a morphism is given by its adjoint. The predual

$$\mathsf{Hilb}(H \to K)_*$$

of $\mathsf{Hilb}(H \to K) = B(H \to K)$ is given by the following Banach space of trace-class operators

$$\mathcal{L}^1(K \to H) \coloneqq \{x \colon K \to H \mid \mathrm{Tr}(|x|) \leq \infty\}$$

where $|x|$ denotes the square root $(x^\dagger x)^{1/2}$ of the element $x^\dagger \circ x$ in the C*-algebra $B(H)$.

Furthermore, every norm-closed linear subcategory of Hilb forms a C*-category, while every WOT-closed subcategory of Hilb forms a W*-category. We call such operator 1-categories *concrete*.

The previous example motivates the following notion of an isomorphism compatible with an involution.

Definition 1.2.2.5 (Unitaries)**.** We say that a morphism x in an operator category is unitary when $x^\dagger = x^{-1}$.

Fact 1.2.2.6. *Two objects $A, B \in \mathcal{A}$ in a C*-category $\mathcal{A}$ are isomorphic if and only if they are unitarily isomorphic.*

Definition 1.2.2.7 ((Normal) †-functor)**.** A †-functor $F\colon \mathcal{A} \to \mathcal{B}$ between C*-categories is a linear functor which is †-preserving, i.e.

$$F(x)^\dagger = F(x^\dagger) \qquad \text{for } x \in \mathcal{A}(A \to B).$$

Furthermore, we say that a †-functor $F\colon \mathcal{A} \to \mathcal{B}$ between W*-categories is *normal* when it is weak*-continuous on each hom-space.

Definition 1.2.2.8 (Uniformly bounded natural transformations)**.** We say that a natural transformation $\theta\colon F \Rightarrow G$ between †-functors $F, G\colon \mathcal{A} \to \mathcal{B}$ is *uniformly bounded* when

$$\|\theta\| \coloneqq \sup_{A \in \mathcal{A}} \|\theta_A\| < \infty.$$

Definition 1.2.2.9. Let $\mathbf{F} = (F, F^{\bullet}, \epsilon, \eta)$ be an adjoint equivalence where F and $F^{\bullet}$ are †-functors between C*-categories $\mathcal{A}$ and $\mathcal{B}$. We say $\mathbf{F}$ is a *unitary adjoint equivalence* whenever ϵ and η are unitaries.

Fact 1.2.2.10 (Whitehead Theorem for operator categories)**.** *A †-functor $F\colon \mathcal{A} \to \mathcal{B}$ is an equivalence between C*-categories if and only if it forms part of a unitary adjoint equivalence. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are W*, then F is automatically normal.*

1.2.3 Dualities for Abelian C*-Algebras

Here we recount the important categorical equivalences between abelian C*-algebras and topology.

Theorem 1.2.3.1 (Gelfand duality). *There is an equivalence of categories*

$$\mathsf{CHaus}^{op} \to \mathsf{AbC^*Alg_1}$$

from the opposite category of compact Hausdorff spaces and continuous maps to the category of unital, abelian C-algebras with unital $*$-homomorphisms.*

The equivalence is witnessed by sending a compact Hausdorff space T to the C*-algebra $C(T)$, and a continuous map $\Phi\colon T \to S$ is sent to the $*$-homomorphism $\Phi^*\colon C(S) \to C(T)$ given by precomposition with Φ (see Example 1.2.1.8). Throughout this work, we will use Φ^* to denote the $*$-homomorphism given by this equivalence. Conversely, given a unital $*$-homomorphism $\phi\colon C(T) \to C(S)$, we will use $\hat{\phi}\colon S \to T$ to denote its preimage under this equivalence.

The Serre–Swan Theorem relates modules over abelian C*-algebras to vector bundles over the underlying topological space. We briefly recount the definition of a complex vector bundle.

Definition 1.2.3.2. A *complex vector bundle* over a compact Hausdorff space T is a topological space E (the total space) along with a continuous surjection $p\colon E \to T$ such that the fibers $E_t := p^{-1}(\{t\})$ are vector spaces for all $t \in T$. Furthermore, the vector bundle E is asked to be locally trivial in the following sense: for each $t \in T$, there is an open set $U \subseteq T$ containing t such that $E_U := p^{-1}(U) \cong U \times \mathbb{C}^k$ (for some k). This isomorphism is required to be the identity on U (by respecting p) and is a linear isomorphism when restricted to the fibers E_t of E_U.

Because we wish to obtain Hilbert C*-modules from these bundles and not simply $C(T)$-modules, we will equip our vector bundles with Hermitian metrics.

Definition 1.2.3.3. A *Hermitian metric* on E is a continuous function $\langle\cdot|\cdot\rangle_E\colon \overline{E}\otimes E\to\mathbb{C}$ that restricts to an inner product on each fiber $E_t\times E_t$.

A module is obtained from a vector bundle by considering its continuous sections.

Definition 1.2.3.4. A (global) *continuous section* on a vector bundle E over T is a continuous function $f\colon T\to E$ such that $p\circ f=\mathrm{id}_T$. The set of all continuous sections is denoted $\Gamma(E)$. Evidently, this is a module over $C(T)$ given by pointwise multiplication (using the vector space structure on each fiber).

We now have all of the necessary background to state the Serre–Swan Theorem for abelian C*-algebras.

Theorem 1.2.3.5 ([Swa62])**.** *There is an equivalence of C*-categories*

$$\mathsf{Hilb}_{\mathsf{fd}}(T)\to\mathsf{Hilb}_{\mathsf{fgp}}C(T)$$

from the category of finite-rank topological hermitian vector bundles over T to the category of finitely generated, projective Hilbert $C(T)$-modules.

The equivalence is witnessed by sending a vector bundle E to its space of continuous global sections $\Gamma(E)$. The $C(T)$-valued inner product is given by

$$\langle f|g\rangle_{C(T)}(t)=\langle f(t)|g(t)\rangle_E$$

for the Hermitian metric $\langle\cdot|\cdot\rangle_E$. A map of vector bundles $\tau\colon E\to F$ is sent to the map $\Gamma(\tau)\colon\Gamma(E)\to\Gamma(F)$ defined by $\Gamma(\tau)(f)=\tau\circ f$ for all $f\in\Gamma(E)$.

It is known to experts that Swan's theorem is actually a monoidal equivalence of monoidal categories (with the usual tensor product of vector bundles and relative tensor product of modules). This fact will be required in subsection 4.2.2. As we have been unable to find a reference, we provide a proof in §1.3.3.

We will also be interested in the case when T has the homotopy type of a CW-complex, as isomorphism classes of line bundles are determined by a cocycle in $H^2(T;\mathbb{Z})$. We use $\mathrm{Pic}(T)$ to denote the group of isomorphism classes of (complex) line bundles over T.

Proposition 1.2.3.6 ([Hat03, Prop. 3.10])**.** *When T has the homotopy type of a CW-complex, the isomorphism class of a complex line bundle E is determined by its first Chern class $c_1(E)$ in $H^2(T;\mathbb{Z})$. The map $E \mapsto c_1(E)$ is a group isomorphism from $\mathrm{Pic}(T) \to H^2(T;\mathbb{Z})$.*

1.2.4 GNS construction for operator categories

Recall from §1.1.4 that every operator algebra can be concretely realized as an algebra of operators on a Hilbert space. The following categorification of this concreteness result was shown by [GLR85]. We also note that the GNS construction involves the Yoneda embedding for an operator category.

Theorem 1.2.4.1 (GNS)**.** *Every small C*-category $\mathcal{A}$ admits a monic*[3] *†-functor*

$$\Upsilon\colon \mathcal{A} \to \mathsf{Hilb}.$$

This construction satisfies the following property:

- *For objects $A, B \in \mathcal{A}$ and a functional $\varphi \in \mathcal{A}(A \to B)^*$, there exist $\xi \in \Upsilon(A)$ and $\eta \in \Upsilon(B)$ such that*

$$\varphi(x) = \langle \Upsilon(x)\xi, \eta\rangle, \qquad \text{for all } x \in \mathcal{A}(A \to B).$$

---

[3] By monic, we mean injective on objects and faithful

In this case we will write $\varphi = \langle \Upsilon \cdot \xi, \eta \rangle$.

Moreover, when $\mathcal{A}$ *is a small W*-category, there exists a normal monic* †*-functor* $\Upsilon^{\mathrm{W}^*} : \mathcal{A} \to \mathsf{Hilb}$.

Definition 1.2.4.2. We define the C*-category $\mathsf{GNS}(\mathcal{A})$ to be the image[4] of Υ in Hilb, for which $\Upsilon\colon \mathcal{A} \to \mathsf{GNS}(\mathcal{A})$ is an isomorphism of C*-categories. Moreover, we call operator subcategories of Hilb concrete.

Definition 1.2.4.3 (Bicommutant)**.** Given any C*-subcategory $\mathcal{A}$ of Hilb, we may take its SOT-closure in Hilb, a W*-category which we will call the bicommutant $\mathcal{A}''$ of $\mathcal{A}$.

Theorem 1.2.4.4 (Kaplansky Density Theorem)**.** *For a subset* $S \subset \mathsf{Hilb}(H \to K)$*, if* $x\colon H \to K$ *is in the SOT-closure of* S*, then there exist* $(x_\lambda) \subset S$ *with* $\|x_\lambda\| \leq \|x\|$ *such that* $x_\lambda \to x$ *SOT.*

1.2.5 Tensor products of C*-categories

Definition 1.2.5.1. For operator 1-categories $\mathcal{A}_1$ and $\mathcal{A}_2$, the algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a linear category equipped with a dagger † consisting of:

(0) objects (A_1, A_2) where $A_i \in \mathcal{A}_i$, and

(1) morphisms are given by

$$(\mathcal{A}_1 \otimes \mathcal{A}_2)((A_1, A_2) \to (B_1, B_2)) \coloneqq \mathcal{A}_1(A_1 \to B_1) \otimes \mathcal{A}_2(A_2 \to B_2).$$

Composition and † are determined on each tensorand.

---

[4] In general, the image of a functor does not form a category. It is however the case that the image under a *monic* functor forms a category, hence our insistence on this (evil) notion. The sufficiently categorified reader may notice an object "being in the image" is also *evil*, i.e. breaks the principle of equivalence [nLa25] One may instead opt to discuss the *essential image* of a functor, which does indeed always form a category. This will not however be necessary for our purposes, and so we merely refer the interested reader to [nLa20].

Definition 1.2.5.2. For small C*-categories, the *minimal tensor product* $\mathcal{A}_1 \underset{\text{min}}{\otimes} \mathcal{A}_2$ is the completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on each hom-space with respect to the spatial C*-norm

$$\|t\|_\sigma \coloneqq \|(\Upsilon_1 \otimes \Upsilon_2)(t)\|$$

where $\Upsilon_i \colon \mathcal{A}_i \to \mathsf{Hilb}$ is the universal representation of $\mathcal{A}_i$ as seen in Theorem 1.2.4.1.

Definition 1.2.5.3. The maximal tensor product $\mathcal{A}_1 \underset{\text{max}}{\otimes} \mathcal{A}_2$ is the completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on each hom-space with respect to the maximal C*-norm

$$\|t\|_\mu \coloneqq \sup\left\{\|Ft\| \mid F \colon \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathsf{Hilb} \text{ is a } \dagger\text{-functor}\right\}.$$

Universal Property 1.2.5.4. *For C*-categories $\mathcal{A}_1$ and $\mathcal{A}_2$, if $H \colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$ is a $*$-bilinear functor*[5] *into a C*-category $\mathcal{B}$, then there exists a unique $*$-functor $H \colon \mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ such that the following triangle commutes:*

$$\begin{tikzcd}
\mathcal{A}_1 \times \mathcal{A}_2 \arrow[r, "H"] \arrow[d] & \mathcal{B} \\
\mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2 \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

Theorem 1.2.5.5 (Hom-Tensor Adjunction)**.** *For C*-categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, we have that the following C*-categories are unitarily naturally equivalent:*

$$\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{max}{\otimes} \mathcal{B} \to \mathcal{C}) \cong \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})).$$

The proof of this fact is quite elementary. As we have not found a reference for this argument in the present literature, we include it here for the sake of completeness.

Proof. We define $T \colon \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\text{max}}{\otimes} \mathcal{B} \to \mathcal{C}) \to \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C}))$ as follows[6]:

---

[5] By $\mathcal{A}_1 \times \mathcal{A}_2$ we mean the cartesian product of the underlying categories, and by †-bilinear functor we mean a functor which is linear and †-preserving in each component.

[6] Helpful mnemonic: T starts with *Tensor* and is simplified into a $\widetilde{\textit{Tilde}}$.

(0) Fix a †-functor $F\colon \mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}$. We define $T(F) = \widetilde{F}\colon \mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$ to be the following †-functor:

- For an object $A \in \mathcal{A}$, we define $\widetilde{F}_A\colon \mathcal{B} \to \mathcal{C}$ by:
  - For $B \in \mathcal{B}$, we set $\widetilde{F}_A(B) \coloneqq F(A, B)$.
  - For $b \in \mathcal{B}(B \to B')$, we set $\widetilde{F}_A(b) \coloneqq F(\mathrm{id}_A \otimes b)\colon F(A, B) \to F(A, B')$.

  More succinctly, we have that $\widetilde{F}_A = F \circ (A \otimes -)$, from which it is clear that $\widetilde{F}_A$ is a †-functor since $F\colon \mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}$ and $A \otimes -\colon \mathcal{B} \to \mathcal{A} \underset{\max}{\otimes} \mathcal{B}$ are †-functors.

- For $a \in \mathcal{A}(A \to A')$, we define $\widetilde{F}_a\colon \widetilde{F}_A \Rightarrow \widetilde{F}_{A'}$ to be the uniformly bounded natural transformation with components $\big((\widetilde{F}_a)_B\big)_{B \in \mathcal{B}}$ in $\mathcal{C}$ given by

$$(\widetilde{F}_a)_B \coloneqq F(a \otimes \mathrm{id}_B)\colon \widetilde{F}_A(B) \to \widetilde{F}_{A'}(B)$$

  Indeed, $\widetilde{F}_a$ is natural since η is natural and

$$\|\widetilde{F}_a\| = \sup_{B \in \mathcal{B}} \|F(a \otimes \mathrm{id}_B)\| \leq \sup_{B \in \mathcal{B}} \|a \otimes \mathrm{id}_B\|_\sigma = \sup_{B \in \mathcal{B}} \|a\| \underbrace{\|\mathrm{id}_B\|}_{=1} = \|a\|.$$

- To see that $\widetilde{F}$ is a †-functor, note that for each $B \in \mathcal{B}$, we have that $(\widehat{F}_-)_B = F \circ (- \otimes B)$ where $F\colon \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C}$ and $- \otimes B\colon \mathcal{A} \to \mathcal{A} \underset{\max}{\otimes} \mathcal{B}$ are †-functors.

(1) Let $F, F' \in \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C})$ be †-functors and $\alpha\colon F \Rightarrow F'$ a uniformly bounded natural transformation. We define $\widetilde{\alpha}\colon \widetilde{F} \Rightarrow \widetilde{F}'$ to be the following uniformly bounded natural transformation:

- For $A \in \mathcal{A}$, we define the A^{th}-component of $\widetilde{\alpha}$ by

$$\widetilde{\alpha}_A \coloneqq \alpha_{(A,-)},$$

where $\alpha_{(A,-)}\colon \widetilde{F}_A \Rightarrow \widetilde{F}'_A$ is the uniformly bounded natural transformation whose B^{th}-component is given by $(\widetilde{\alpha}_A)_B = (\alpha_{(A,-)})_B = \alpha_{(A,B)}$ for $B \in \mathcal{B}$. To see that $\widetilde{\alpha}_A$ is natural, note that the following square commutes since α is natural.

$$\begin{tikzcd}
\widetilde{F}_A(B) \arrow[r, "\widetilde{F}_A(b)"] \arrow[d, "(\widetilde{\alpha}_A)_B"'] & \widetilde{F}_A(B') \arrow[d, "(\widetilde{\alpha}_A)_{B'}"] \\
\widetilde{F}'_A(B) \arrow[r, "\widetilde{F}'_A(b)"'] & \widetilde{F}'_A(B')
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A,B) \arrow[r, "F(\mathrm{id}_A \otimes b)"] \arrow[d, "\alpha_{(A,B)}"'] & F(A,B') \arrow[d, "\alpha_{(A,B')}"] \\
F'(A,B) \arrow[r, "F'(\mathrm{id}_A \otimes b)"'] & F'(A,B')
\end{tikzcd}$$

Moreover, $\|\widetilde{\alpha}_A\| = \sup_{B\in\mathcal{B}} \|\alpha_{(A,B)}\| \leq \|\alpha\|$ and we conclude that $\widetilde{\alpha}_A$ is a uniformly bounded natural transformation.

- We assert that $\widetilde{\alpha}$ is natural, i.e. that the following square commutes for every $a \in \mathcal{A}(A \to A')$.

$$\begin{tikzcd}
\widetilde{F}_A \arrow[r, Rightarrow, "\widetilde{F}_a"] \arrow[d, Rightarrow, "\widetilde{\alpha}_A"'] & \widetilde{F}_{A'} \arrow[d, Rightarrow, "\widetilde{\alpha}_{A'}"] \\
\widetilde{F}'_A \arrow[r, Rightarrow, "\widetilde{F}'_a"'] & \widetilde{F}'_{A'}
\end{tikzcd}$$

This is clear since, for each $A \in \mathcal{A}$, the following square commutes for any $B \in \mathcal{B}$ by the naturality of α.

$$\begin{tikzcd}
\widetilde{F}_A(B) \arrow[r, "(\widetilde{F}_a)_B"] \arrow[d, "(\widetilde{\alpha}_A)_B"'] & \widetilde{F}_{A'}(B) \arrow[d, "(\widetilde{\alpha}_{A'})_B"] \\
\widetilde{F}'_A(B) \arrow[r, "(\widetilde{F}'_a)_B"'] & \widetilde{F}'_{A'}(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A,B) \arrow[r, "F(a \otimes \mathrm{id}_B)"] \arrow[d, "\alpha_{(A,B)}"'] & F(A',B) \arrow[d, "\alpha_{(A',B)}"] \\
F'(A,B) \arrow[r, "F'(a \otimes \mathrm{id}_B)"'] & F'(A',B)
\end{tikzcd}$$

- To see that $\widetilde{\alpha}$ is uniformly bounded, observe

$$\|\widetilde{\alpha}\| = \sup_{A\in\mathcal{A}} \|\widetilde{\alpha}_A\| = \sup_{A\in\mathcal{A}} \sup_{B\in\mathcal{B}} \|\alpha_{(A,B)}\| = \|\alpha\|.$$

Therefore $\widetilde{\alpha}\colon \widetilde{F} \Rightarrow \widetilde{F}'$ is a uniformly bounded natural transformation and T is isometric.

(†) To see that T is a †-functor, consider $\lambda \in \mathbb{C}$ and parallel uniformly bounded natural transformations α, α' in $\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C})$. Notice, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have

$$\Big(\big(T(\lambda\alpha + \alpha')\big)_A\Big)_B = (\lambda\alpha + \alpha')_{(A,B)} = \lambda\alpha_{(A,B)} + \alpha'_{(A,B)} = \lambda\big((T\alpha)_A\big)_B + \big((T\alpha')_A\big)_B,$$
$$\Big(\big(T(\alpha^*)\big)_A\Big)_B = (\alpha^*)_{(A,B)} = (\alpha_{(A,B)})^* = \big((T\alpha)_A\big)^*_B.$$

Therefore $T(\lambda\alpha + \alpha') = \lambda T\alpha + T\alpha'$ and $T(\alpha^*) = (T\alpha)^*$. For composable uniformly bounded natural transformations α, α' in $\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C})$, an identical argument reveals $T(\alpha \circ \alpha') = T(\alpha) \circ T(\alpha')$. Finally, for each †-functor $F\colon \mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}$ observe

$$\big((T\mathrm{id}_F)_A\big)_B = (\mathrm{id}_F)_{(A,B)} = \mathrm{id}_{F(A,B)} = \mathrm{id}_{\widetilde{F}_A(B)}, \quad \text{for all } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Therefore $T\mathrm{id}_F = \mathrm{id}_{\widetilde{F}} = \mathrm{id}_{TF}$. We conclude that T is a †-functor.

We define $H\colon \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})) \to \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C})$ as follows[7]:

(0) Fix a †-functor $G : \mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$. We define $H(G) = \widehat{G}\colon \mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}$ to be the following †-functor:

- For an object $(A, B) \in \mathcal{A} \underset{\max}{\otimes} \mathcal{B}$, we set $\widehat{G}(A, B) \coloneqq G(A)(B)$.
- First consider morphisms of the form $\mathrm{id}_A \otimes b \in \mathcal{A} \underset{\max}{\otimes} \mathcal{B}\big((A, B) \to (A, B')\big)$. We set $\widehat{G}(\mathrm{id}_A \otimes b) \coloneqq G(A)(b)$, the image of b under the functor $G(A)$ in $\mathcal{C}$. Then consider morphisms of the form $a \otimes \mathrm{id}_B \in \mathcal{A} \underset{\max}{\otimes} \mathcal{B}\big((A, B) \to (A', B)\big)$. We set $\widehat{G}(a \otimes \mathrm{id}_B) \coloneqq G(a)_B$, the B^{th}-component of the uniformly bounded natural transformation $G(a)\colon G(A) \Rightarrow G(A')$. This determines a †-bilinear functor $\widehat{G}\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$; and by the universal property 1.2.5.4 of $\underset{\max}{\otimes}$, we obtain a unique †-functor $\widehat{G} \in \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C})$.

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[7] Helpful mnemonic: H starts with *Hom* and is simplified into a $\widehat{Hat}$.

(1) Let $G, G' \colon \mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$ be †-functors and $\beta \colon G \Rightarrow G'$ a uniformly bounded natural transformation. We define $\widehat{\beta} \colon \widehat{G} \Rightarrow \widehat{G}'$ to be the following uniformly bounded natural transformation:

- For $(A, B) \in \mathcal{A} \underset{\max}{\otimes} \mathcal{B}$, we define the $(A, B)^{th}$-component of $\widehat{\beta}$ by

  $$\widehat{\beta}_{(A,B)} \coloneqq (\beta_A)_B,$$

  where $(\beta_A)_B \colon G(A)(B) \to G'(A)(B)$ is the B^{th}-component of the uniformly bounded natural transformation $\beta_A \colon G(A) \Rightarrow G'(A)$, which is itself the A^{th}-component of the uniformly bounded natural transformation $\beta \colon G \Rightarrow G'$.

- We claim that $\widehat{\beta}$ is natural. First, for morphisms of the form $\mathrm{id}_A \otimes b$ in $\mathcal{A} \underset{\max}{\otimes} \mathcal{B}$, note that the following square commutes by the naturality of β_A.

  $$\begin{tikzcd}
  \widehat{G}(A,B) \arrow[r, "\widehat{G}(\mathrm{id}_A \otimes b)"] \arrow[d, "\widehat{\beta}_{A,B}"'] & \widehat{G}(A,B') \arrow[d, "\widehat{\beta}_{A,B'}"] \\
  \widehat{G}'(A,B) \arrow[r, "\widehat{G}'(\mathrm{id}_A \otimes b)"'] & \widehat{G}'(A,B')
  \end{tikzcd}
  \qquad
  \begin{tikzcd}
  G(A)(B) \arrow[r, "G(A)(b)"] \arrow[d, "(\beta_A)_B"'] & G(A)(B') \arrow[d, "(\beta_A)_{B'}"] \\
  G'(A)(B) \arrow[r, "G'(A)(b)"'] & G'(A)(B')
  \end{tikzcd}$$

  Similarly, for morphisms of the form $a \otimes \mathrm{id}_B$ in $\mathcal{A} \underset{\max}{\otimes} \mathcal{B}$, the following square commutes by the naturality of β.

  $$\begin{tikzcd}
  \widehat{G}(A,B) \arrow[r, "\widehat{G}(a \otimes \mathrm{id}_B)"] \arrow[d, "\widehat{\beta}_{A,B}"'] & \widehat{G}(A',B) \arrow[d, "\widehat{\beta}_{A',B}"] \\
  \widehat{G}'(A,B) \arrow[r, "\widehat{G}'(a \otimes \mathrm{id}_B)"'] & \widehat{G}'(A',B)
  \end{tikzcd}
  \qquad
  \begin{tikzcd}
  G(A)(B) \arrow[r, "G(a)(B)"] \arrow[d, "(\beta_A)_B"'] & G(A')(B) \arrow[d, "(\beta_{A'})_B"] \\
  G'(A)(B) \arrow[r, "G'(a)(B)"'] & G'(A')(B)
  \end{tikzcd}$$

  Hence, the following outer square commutes for every simple tensor $a \otimes b$ in $\mathcal{A} \otimes \mathcal{B}$.

  $$\begin{tikzcd}
  \widehat{G}(A,B) \arrow[r, "\widehat{G}(\mathrm{id}_A \otimes b)"'] \arrow[rr, bend left, "\widehat{G}(a \otimes b)"] \arrow[d, "\widehat{\beta}_{A,B}"'] & \widehat{G}(A,B') \arrow[r, "\widehat{G}(a \otimes \mathrm{id}_{B'})"'] \arrow[d, dashed, "\widehat{\beta}_{A,B'}"] & \widehat{G}(A',B') \arrow[d, "\widehat{\beta}_{A',B'}"] \\
  \widehat{G}'(A,B) \arrow[r, "\widehat{G}'(\mathrm{id}_A \otimes b)"] \arrow[rr, bend right, "\widehat{G}'(a \otimes b)"'] & \widehat{G}'(A,B') \arrow[r, "\widehat{G}'(a \otimes \mathrm{id}_{B'})"] & \widehat{G}'(A',B')
  \end{tikzcd}$$

By the linearity and continuity of composition, we conclude that $\widehat{\beta}$ is natural.

- To see that $\widehat{\beta}$ is uniformly bounded, observe

$$\|\widehat{\beta}\| = \sup_{(A,B)\in \mathcal{A}\underset{\max}{\otimes}\mathcal{B}} \|\widehat{\beta}_{(A,B)}\| = \sup_{A\in\mathcal{A}}\sup_{B\in\mathcal{B}} \|(\beta_A)_B\| = \sup_{\mathcal{A}\in A}\|\beta_A\| = \|\beta\|.$$

  This computation also shows that H is an isometry.

(†) The fact that H is a †-functor follows by formal symbolic manipulation, just like the proof that T is a †-functor.

Now that we know that H and T are †-functors between the C*-categories under consideration, we claim that they are in fact inverses.

(HT) Suppose $F\colon \mathcal{A}\underset{\max}{\otimes}\mathcal{B} \to \mathcal{C}$ is a †-functor. For an object $(A,B) \in \mathcal{A}\underset{\max}{\otimes}\mathcal{B}$, observe

$$HTF(A,B) = \widehat{TF}(A,B) = TF(A)(B) = \widetilde{F}_A(B) = F(A,B).$$

So HTF and F agree on objects. Now consider a morphism of the form $\mathrm{id}_A \otimes b$ in $\mathcal{A}\underset{\max}{\otimes}\mathcal{B}$. Observe

$$HTF(\mathrm{id}_A \otimes b) = \widehat{TF}(\mathrm{id}_A \otimes b) = TF(A)(b) = \widetilde{F}_A(b) = F(\mathrm{id}_A \otimes b).$$

Similarly, consider a morphism of the form $a \otimes \mathrm{id}_B$ in $\mathcal{A}\underset{\max}{\otimes}\mathcal{B}$ and notice

$$HTF(a \otimes \mathrm{id}_B) = \widehat{TF}(a \otimes \mathrm{id}_B) = TF(a)_B = (\widetilde{F}_a)_B = F(a \otimes \mathrm{id}_B).$$

By functoriality we obtain that HTF and F agree on all simple tensors, and by linearity and continuity we conclude that $HTF = F$.

Now suppose that $F, F'\colon \mathcal{A}\underset{\max}{\otimes}\mathcal{B} \to \mathcal{C}$ are †-functors and $\alpha\colon F \Rightarrow F'$ is a uniformly bounded natural transformation. For all objects $(A,B) \in \mathcal{A}\underset{\max}{\otimes}\mathcal{B}$, we have

$$HT\alpha_{(A,B)} = \widehat{T\alpha}_{(A,B)} = (T\alpha_A)_B = (\widetilde{\alpha}_A)_B = \alpha_{(A,B)},$$

so $HT\alpha = \alpha$. We conclude that $HT = \mathrm{id}$.

(TH) Suppose $G\colon \mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$ is a †-functor. For objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, observe

$$THG(A)(B) = \widetilde{HG}_A(B) = HG(A,B) = \widehat{G}(A,B) = G(A)(B).$$

So THG and G agree on objects. Now consider some morphism $a \in \mathcal{A}(A \to A')$. For all $B \in \mathcal{B}$, we have

$$THG(a)_B = (\widetilde{HG}_a)_B = HG(a \otimes \mathrm{id}_B) = \widehat{G}(a \otimes \mathrm{id}_B) = G(a)_B,$$

so $THG(a) = G(a)$. We conclude that $THG = G$.

Now suppose that $G, G'\colon \mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$ are †-functors and $\beta\colon G \Rightarrow G'$ is a uniformly bounded natural transformation. For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have

$$\big((TH\beta)_A\big)_B = (\widetilde{H\beta}_A)_B = H\beta_{(A,B)} = \widehat{\beta}_{(A,B)} = (\beta_A)_B,$$

so $TH\beta = \beta$. We conclude that $TH = \mathrm{id}$.

Therefore $\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}) \cong \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C}))$. We now claim that the isomorphism T is natural in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

($\mathcal{A}$) Suppose $N\colon \mathcal{A}' \to \mathcal{A}$ is a †-functor between C*-categories. We wish to show that the following square commutes.

$$\begin{tikzcd}
\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}) \arrow[r, "T"] \arrow[d, "(N \underset{\max}{\otimes} \mathcal{B})^*"'] & \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})) \arrow[d, "N^*"] \\
\mathrm{C}^*\mathsf{Cat}(\mathcal{A}' \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}) \arrow[r, "T"'] & \mathrm{C}^*\mathsf{Cat}(\mathcal{A}' \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C}))
\end{tikzcd}$$

So consider some $\alpha\colon F \Rightarrow F'$ in $\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C})$. For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, observe

$$\begin{aligned}
\big((T \circ (N \underset{\max}{\otimes} B)^*)(\alpha)_A\big)_B &= \big((N \underset{\max}{\otimes} B)^*\alpha\big)_{(A,B)} = \alpha_{(NA,B)} \\
&= (\widetilde{\alpha}_{NA})_B = \big((N^*\widetilde{\alpha})_A\big)_B = \big((N^* \circ T)(\alpha)_A\big)_B,
\end{aligned}$$

so $(T \circ (N \underset{\max}{\otimes} B)^*)(\alpha) = (N^* \circ T)(\alpha)$.

($\mathcal{B}$) Suppose $N\colon \mathcal{B}' \to \mathcal{B}$ is a †-functor between C*-categories. We wish to show that the following square commutes.

$$\begin{tikzcd}
\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}) \arrow[r, "T"] \arrow[d, "(\mathcal{A} \underset{\max}{\otimes} N)^*"'] & \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})) \arrow[d, "(N^*)_*"] \\
\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B}' \to \mathcal{C}) \arrow[r, "T"'] & \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B}' \to \mathcal{C}))
\end{tikzcd}$$

So consider some $\alpha\colon F \Rightarrow F'$ in $\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C})$. For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, observe

$$\begin{aligned}
\big((T \circ (A \underset{\max}{\otimes} N)^*)(\alpha)_A\big)_B &= \big((A \underset{\max}{\otimes} N)^*\alpha\big)_{(A,B)} = \alpha_{(A,NB)} = (\widetilde{\alpha}_A)_{NB} \\
&= \big(N^*(\widetilde{\alpha}_A)\big)_B = \big(((N^*)_*\widetilde{\alpha})_A\big)_B = \big(((N^*)_* \circ T)(\alpha)_A\big)_B,
\end{aligned}$$

so $(T \circ (A \underset{\max}{\otimes} N)^*)(\alpha) = ((N^*)_* \circ T)(\alpha)$.

($\mathcal{C}$) Suppose $N\colon \mathcal{C} \to \mathcal{C}'$ is a †-functor between C*-categories. We wish to show that the following square commutes.

$$\begin{tikzcd}
\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}) \arrow[r, "T"] \arrow[d, "N_*"'] & \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C})) \arrow[d, "N_{**}"] \\
\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C}') \arrow[r, "T"'] & \mathrm{C}^*\mathsf{Cat}(\mathcal{A} \to \mathrm{C}^*\mathsf{Cat}(\mathcal{B} \to \mathcal{C}'))
\end{tikzcd}$$

So consider some $\alpha\colon F \Rightarrow F'$ in $\mathrm{C}^*\mathsf{Cat}(\mathcal{A} \underset{\max}{\otimes} \mathcal{B} \to \mathcal{C})$. For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, observe

$$\big((TN_*)(\alpha)_A)\big)_B = (\widetilde{N_*\alpha_A})_B = (N_*\alpha)_{A,B)} = N(\alpha_{(A,B)}) = N\big((\widetilde{\alpha}_A)_B\big) = \big((N_{**}T)(\alpha)_A\big)_B,$$

so $(TN_*)(\alpha) = (N_{**}T)(\alpha)$.

With this we conclude our proof. □

1.2.6 W*-completion of a C*-category

The following section is a horizontal categorification of the results in [Are51b] and [Are51a] for C*-algebras which we discussed in §1.1.5. As we will build on these facts to construct the

W*-completion of a C*-2-category in Section §3.2.3, we will provide sketches for some of our proofs concerning these results.

For a C*-category $\mathcal{A}$, we wish to construct the enveloping W*-category $\mathrm{W}^*(\mathcal{A})$ together with a monic †-functor $\mathcal{A} \hookrightarrow \mathrm{W}^*(\mathcal{A})$. As discussed in §1.1.5, one natural candidate to consider is the double dual of $\mathcal{A}$.

Definition 1.2.6.1 (Double dual of a C*-category)**.** When $\mathcal{A}$ is a C*-category, we construct $\mathcal{A}^{**}$ as follows:

(0) objects are the same as in $\mathcal{A}$,

(1) morphisms are given by $\mathcal{A}^{**}(A \to B) := \mathcal{A}(A \to B)^{**}$;

($\circ$) we define two so-called Arens compositions on $\mathcal{A}^{**}$, which equip $\mathcal{A}^{**}$ with the structure of a (linear) category.

Remark 1.2.6.2. Notice each $\mathcal{A}^{**}(A \to B)$ admits a vector space structure and a norm, namely,

$$\|\Phi\| := \sup_{\varphi \in \mathcal{A}(A \to B)^*} \frac{|\Phi(\varphi)|}{\|\varphi\|}, \qquad \text{for } \Phi \in \mathcal{A}^{**}(A \to B).$$

Definition 1.2.6.3 (Arens compositions)**.** For $\Phi \in \mathcal{A}^{**}(A \to B)$ and $\Psi \in \mathcal{A}^{**}(B \to C)$, we define the left and right Arens compositions $\circ_\ell$ and $\circ_r$ as follows:

(ℓ) For $\varphi \in \mathcal{A}^*(A \to C)$, we set $(\Psi \circ \Phi)(\varphi) := \Psi(\Phi \triangleright \varphi)$ where $\Phi \triangleright \varphi \in \mathcal{A}(B \to C)^*$ is given by:

  ($\triangleright$) For $b \in \mathcal{A}(B \to C)$, we set $(\Phi \triangleright \varphi)(b) := \Phi(\varphi \triangleleft b)$ where $\varphi \triangleleft b \in \mathcal{A}(A \to B)^*$ is given by:

    ($\triangleleft$) For $a \in \mathcal{A}(A \to B)$, we set $(\varphi \triangleleft b)(a) := \varphi(b \circ a)$.

(r) For $\varphi \in \mathcal{A}^*(A \to C)$, we set $(\Psi \circ_r \Phi)(\varphi) \coloneqq \Phi(\varphi \triangleleft \Psi)$ where $\varphi \triangleleft \Psi \in \mathcal{A}(A \to B)^*$ is given by:

  ($\triangleleft$) For $a \in \mathcal{A}(A \to B)$, we set $(\varphi \triangleleft \Psi)(a) \coloneqq \Psi(a \triangleright \varphi)$ where $a \triangleright \varphi \in \mathcal{A}(B \to C)^*$ is given by:

    ($\triangleright$) For $b \in \mathcal{A}(B \to C)$, we set $(a \triangleright \varphi)(b) \coloneqq \varphi(b \circ a)$.

Definition 1.2.6.4. We define $\mathrm{ev} \colon \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ as follows:

(0) For an object $A \in \mathcal{A}$, $\mathrm{ev}(A) \coloneqq A$,

(1) For $x \in \mathcal{A}(A \to B)$, we define $\mathrm{ev}(x) = \mathrm{ev}_x \in \mathcal{A}^{**}(A \to B)$ by

$$\mathrm{ev}_x(\varphi) \coloneqq \varphi(x) \qquad \text{for } \varphi \in \mathcal{A}(A \to \mathcal{B}).$$

Lemma 1.2.6.5. $\mathrm{ev} \colon \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ *is* $\dagger$*-preserving when we equip* $\mathcal{A}^{**}$ *with either Arens composition.*

Definition 1.2.6.6 ($\dagger$)**.** We define a conjugate-linear contravariant map $\dagger \colon \mathcal{A}^{**}(A \to B) \to \mathcal{A}^{**}(B \to A)$ as follows:

($\dagger$) For $\Phi \in \mathcal{A}^{**}(A \to B)$, we define $\Phi^\dagger \in \mathcal{A}^{**}(B \to A)$ by

$$\Phi^\dagger(\varphi) \coloneqq \overline{\Phi(\varphi^\dagger)} \quad \text{for } \varphi \in \mathcal{A}(B \to A)^*,$$

where $\varphi^\dagger \in \mathcal{A}(A \to B)^*$ is given by $\varphi^\dagger(a) \coloneqq \overline{\varphi(a^\dagger)}$ for $a \in \mathcal{A}(A \to B)$.

We now relate the Arens compositions via the following identity, which follows from a straightforward computation.

Lemma 1.2.6.7. *For* $\Phi \in \mathcal{A}^{**}(A \to B)$ *and* $\Psi \in \mathcal{A}^{**}(B \to C)$,

$$(\Psi \circ_\ell \Phi)^\dagger = \Phi^\dagger \circ_r \Psi^\dagger.$$

From our previous result, we see that if the Arens compositions on $\mathcal{A}^{**}$ agree then:

- $(\Psi \circ \Phi)^\dagger = \Phi^\dagger \circ \Psi^\dagger$ where $\circ = \circ_\ell = \circ_r$ and
- $\mathrm{ev}\colon \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ is a †-functor by 1.2.6.5.

It turns out this is always the case for C*-categories, after which one proves that $\mathcal{A}^{**}$ satisfies the universal property required of the W*-completion of $\mathcal{A}$.

Theorem 1.2.6.8. *For a C*-category $\mathcal{A}$, the left and right Arens compositions on $\mathcal{A}^{**}$ coincide. Furthermore, these serve to equip $\mathcal{A}^{**}$ with the structure of a W*-category.*

Proof. Without loss of generality, one assumes $\mathcal{A}$ is small since compositions coincide if and only if they agree on each small subcategory. We first extend the universal representation $\Upsilon\colon \mathcal{A} \to \mathsf{Hilb}$ along ev

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Upsilon"] \arrow[d, hook, "\mathrm{ev}"'] & \mathsf{Hilb} \\ \mathcal{A}^{**} \arrow[ur, dashed, "\widetilde{\Upsilon}"'] & \end{tikzcd}$$

by declaring that for each $\Phi \in \mathcal{A}^{**}(A \to B)$, we have

$$\langle \widetilde{\Upsilon}(\Phi)\xi, \eta\rangle = \Phi(\langle \Upsilon \cdot \xi, \eta\rangle) \quad \text{for all } \xi \in \Upsilon(A) \text{ and } \eta \in \Upsilon(B).$$

In particular, this determines a bounded operator $\widetilde{\Upsilon}(\Phi)\colon \Upsilon(A) \to \Upsilon(B)$ with $\|\widetilde{\Upsilon}(\Phi)\| \leq \|\Phi\|$. Since every functional $\varphi \in \mathcal{A}(A \to B)^*$ is of the form $\varphi = \langle \Upsilon \cdot \xi, \eta\rangle$, it follows that $\Phi = 0$ if and only if $\widetilde{\Upsilon}(\Phi) = 0$. For $\Phi \in \mathcal{A}^{**}(A \to B)$ and $\Psi \in \mathcal{A}^{**}(B \to C)$, one computes

$$\langle \widetilde{\Upsilon}(\Psi \circ_\ell \Phi)\xi, \eta\rangle = \langle \widetilde{\Upsilon}(\Psi)\widetilde{\Upsilon}(\Phi)\xi, \eta\rangle = \langle \widetilde{\Upsilon}(\Psi \circ_r \Phi)\xi, \eta\rangle$$

which implies $\widetilde{\Upsilon}(\Psi \circ_\ell \Phi) = \widetilde{\Upsilon}(\Psi)\widetilde{\Upsilon}(\Phi) = \widetilde{\Upsilon}(\Psi \circ_r \Phi)$, and hence $\circ_\ell = \circ_r$.

By Lemma 1.2.6.7 we obtain that $\mathcal{A}^{**}$ is a †-category. Since Hilb is an operator category, to show $\mathcal{A}^{**}$ is a C*-category it suffices to show that $\widetilde{\Upsilon}$ is an isometric †-functor. We then compute

$$\langle \widetilde{\Upsilon}(\Phi^\dagger)\xi, \eta\rangle = \langle \widetilde{\Upsilon}(\Phi)^\dagger \xi, \eta\rangle,$$

from which we see that $\widetilde{\Upsilon}$ is †-preserving.

We now verify that $\|\widetilde{\Upsilon}(\Phi)\| = \|\Phi\|$ for $\Phi \in \mathcal{A}^{**}(A \to B)$. By construction, it is clear that $\widetilde{\Upsilon}\colon \mathcal{A}^{**} \to \mathsf{Hilb}$ is weak*-WOT continuous. Let $\varepsilon > 0$. As a property of the GNS construction for $\mathcal{A}$, we know there exist $\xi \in \Upsilon(A), \eta \in \Upsilon(B)$ with $\|\langle \Upsilon \cdot \xi, \eta\rangle\| = 1$ such that

$$|\langle \widetilde{\Upsilon}(\Phi)\xi, \eta\rangle| = |\Phi(\langle \Upsilon \cdot \xi, \eta\rangle)| \geq \|\Phi\| - \varepsilon.$$

Since $\widetilde{\Upsilon}$ is weak*-WOT continuous, $\widetilde{\Upsilon}(\Phi) \in \overline{\operatorname{Im}\Upsilon}^{\mathrm{WOT}} = \overline{\operatorname{Im}\Upsilon}^{\mathrm{SOT}} \subseteq \mathsf{Hilb}(\Upsilon A \to \Upsilon B)$. By the Kaplansky density theorem, there exist $(x_\lambda) \subset \mathcal{A}(A \to B)$ with $\|x_\lambda\| = \|\Upsilon(x_\lambda)\| \leq \|\widetilde{\Upsilon}(\Phi)\|$ such that $\Upsilon(x_\lambda) \to \widetilde{\Upsilon}(\Phi)$ SOT. Observe

$$\|\widetilde{\Upsilon}(\Phi)\| \geq \|x_\lambda\| \geq |\langle \Upsilon(x_\lambda)\xi, \eta\rangle| \to |\langle \widetilde{\Upsilon}(\Phi)\xi, \eta\rangle| \geq \|\Phi\| - \varepsilon.$$

Since $\varepsilon \geq 0$ was arbitrary, $\|\widetilde{\Upsilon}(\Phi)\| \geq \|\Phi\|$. We conclude that $\widetilde{\Upsilon}$ is isometry, and hence $\mathcal{A}^{**}$ is a W*-category. □

Universal Property 1.2.6.9. *For every †-functor $F\colon \mathcal{A} \to \mathcal{B}$ into a W*-category $\mathcal{B}$, there exists a unique normal extension $\widetilde{F}\colon \mathcal{A}^{**} \to \mathcal{B}$ making the following triangle commute.*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, hook, "\mathrm{ev}"'] & \mathcal{B} \\
\mathcal{A}^{**} \arrow[ur, dashed, "\exists!\widetilde{F}"'] &
\end{tikzcd}$$

Definition 1.2.6.10. For every †-functor $F\colon \mathcal{A} \to \mathcal{B}$ between C*-categories $\mathcal{A}$ and $\mathcal{B}$, there exists a unique normal extension $F^{**}\colon \mathcal{A}^{**} \to \mathcal{B}^{**}$ afforded by the universal property of $\mathcal{A}^{**}$, which makes the following square commute.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, hook, "\mathrm{ev}"'] & \mathcal{B} \arrow[d, hook, "\mathrm{ev}"] \\
\mathcal{A}^{**} \arrow[r, dashed, "\exists!F^{**}"'] & \mathcal{B}^{**}
\end{tikzcd}$$

Using the facts shown in the proof of Theorem 1.2.6.8, one proves the following result.

Corollary 1.2.6.11 (Sherman-Takeda for C*-categories)**.** *For a small C*-category $\mathcal{A}$, the †-functor*

$$\widetilde{\Upsilon}\colon \mathcal{A}^{**} \to \mathsf{GNS}(\mathcal{A})''$$

constructed in Theorem 1.2.6.8 is an equivalence of W-categories extending* $\Upsilon\colon \mathcal{A} \to \mathsf{GNS}(A)$ *as follows:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Upsilon"] \arrow[d, hook, "\mathrm{ev}"'] & \mathsf{GNS}(\mathcal{A}) \arrow[d, hook] \\
\mathcal{A}^{**} \arrow[r, "\widetilde{\Upsilon}"'] & \mathsf{GNS}(\mathcal{A})''
\end{tikzcd}$$

1.2.7 W*-tensor product of W*-categories

The following result is a horizontal categorification of the results in [Dau72] for W*-algebras. As we will adapt all of these results for W*-2-categories in Section 3.2.4, we will omit the details in this work. We however present the main result here for completeness.

Universal Property 1.2.7.1. *For W*-categories $\mathcal{A}_1$ and $\mathcal{A}_2$, there is a W*-category $\mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2$ equipped with separately normal †-bilinear functor $\mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2$ satisfying the following universal property:*

- *For every separately normal †-bilinear functor $H\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$ into a W*-category $\mathcal{B}$, there exists a unique normal †-functor $\overline{H}\colon \mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2 \to \mathcal{B}$ such that the following triangle commutes:*

$$\begin{tikzcd}
\mathcal{A}_1 \times \mathcal{A}_2 \arrow[r, "H"] \arrow[d] & \mathcal{B} \\
\mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2 \arrow[ru, dashed, "\overline{H}"'] &
\end{tikzcd}$$

Remark 1.2.7.2. More specifically, we construct $\mathcal{A}_1 \underset{\max}{\overline{\otimes}} \mathcal{A}_2$ as the quotient of $(\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2)^{**}$ by the polar of so-called separately normal functionals on $\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2$.

Theorem 1.2.7.3 (Hom-Tensor Adjunction)**.** *For W*-categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, we have that the following W*-categories are unitarily naturally equivalent:*

$$\mathrm{W}^*\mathsf{Cat}(\mathcal{A}\underset{max}{\overline{\otimes}}\mathcal{B}\to\mathcal{C})\cong\mathrm{W}^*\mathsf{Cat}(\mathcal{A}\to\mathrm{W}^*\mathsf{Cat}(\mathcal{B}\to\mathcal{C})).$$

1.3 Background on operator 2-categories

1.3.1 Algebraic 2-categories

Definition 1.3.1.1. A 2-category consists of the following data:

(0) A collection $\mathcal{A}$ of *objects* of $\mathcal{A}$;

(hom) For $A, B \in \mathcal{A}$, a category $\mathcal{A}(A \to B)$ where:

- The objects of $\mathcal{A}(A \to B)$ are called 1-morphisms of $\mathcal{A}$ with source A and target B,
- The arrows of $\mathcal{A}(A \to B)$ will be referred to as 2-morphisms of $\mathcal{A}$ with their same source and target. For $X, Y \in \mathcal{A}(A \to B)$, we will denote the corresponding space of 2-morphisms by $\mathcal{A}(X \Rightarrow Y)$.

($\odot$) For $A, B, C \in \mathcal{A}$, a functor $\odot\colon \mathcal{A}(A \to B) \times \mathcal{A}(B \to C) \to \mathcal{A}(A \to C)$ called *1-composition*, which we represent in Figure 1.10.

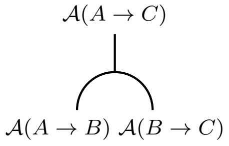

Figure 1.10: 2D string diagram for 1-composition in 2-categories

(I) For $A \in \mathcal{A}$, a functor $I_A \colon * \to \mathcal{A}(A \to A)$ which picks out an identity 1-morphism id_A, represented in Figure 1.11.

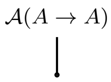

Figure 1.11: 2D string diagram for identities in 2-categories

(a) For $A, B, C, D \in \mathcal{A}$, an *"associator"* natural isomorphism α, which we represent in Figure 1.12.

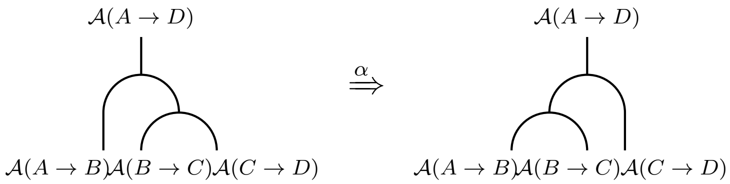

Figure 1.12: $(2+1)$D string diagram for associators in 2-categories

(u) For $A, B \in \mathcal{A}$, *left and right "unitor"* natural isomorphisms λ and ρ, which we represent in Figure 1.13.

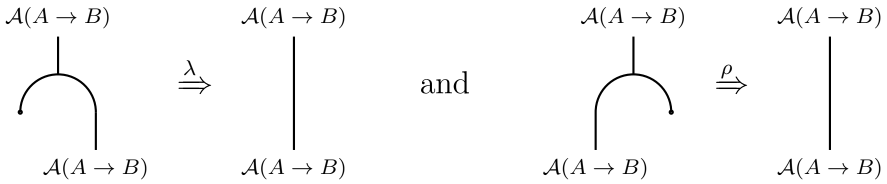

Figure 1.13: $(2+1)$D string diagram for left and right unitors in 2-categories

We say a 2-category is *strict* when 1-composition $\odot$ in $\mathcal{A}$ is associtive and unital up to equality and the constraint data α, λ, ρ are identities. We will use the terms 2-category and "bicategory" interchangeably, specifying whenever they are strict.

Furthermore, we require this data satisfies a 2-dimensional associahedron axiom for each tuple of 5 objects in $\mathcal{A}$ (non-abelian 3-cocycle condition) and an axiom for each triple of objects in $\mathcal{A}$ relating the left and right unitors with the associator (normalized cocycle condition). In particular,

(π) For $A, B, C, D, E \in \mathcal{A}$, the pentagon in Figure 1.14 commutes.

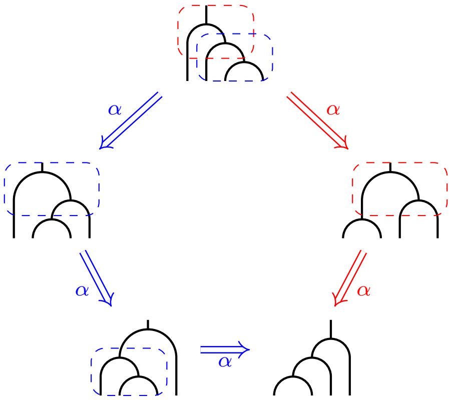

Figure 1.14: Associativity pentagon axiom for 2-categories

(coh) For $A, B, C \in \mathcal{A}$, the triangle in Figure 1.15 commutes.

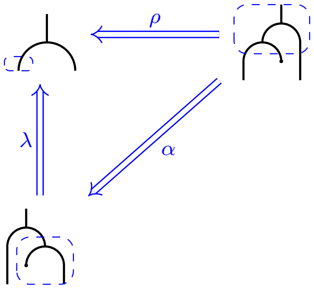

Figure 1.15: Middle unitality triangle axiom for 2-categories

Remark 1.3.1.2. At this category level, it is a coincidence that the triangles in Figure 1.16 automatically commute in any 2-category.

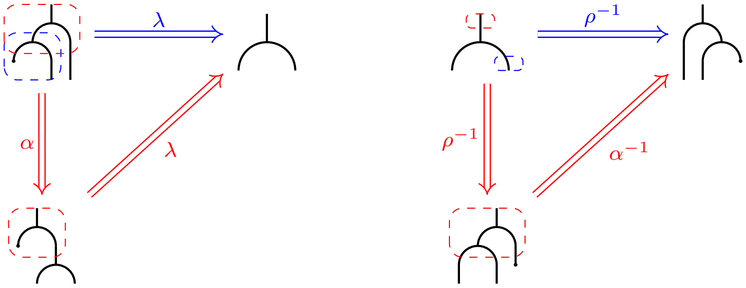

Figure 1.16: Commutative left and right unitality triangles in 2-categories

Thinking of $\alpha_{-,-,-}$ as a 3-cocycle, these commutative triangles witness the fact that α is normalized on the left, middle, and right entries respectively.

Remark 1.3.1.3. As our coherence results will heavily involve 2-categories of 2-functors between 2-categories, we will mainly use the convention that both 1-composition $\odot$ and 2-composition $\circ$ are contravariant. It is however more common in the literature to have covariant 1-composition and contravariant 2-composition–where 1-composition is denoted by $\otimes$. Since we will also build serveral tensor products of operator 2-categories (as we have done for operator categories) we have opted for this less common notation $\odot$ for the 1-composition inside a 2-category. We will however use $\otimes$ for the opposite convention of $\odot$ whenever it is convenient.

Definition 1.3.1.4. A 2-functor $F\colon \mathcal{A} \to \mathcal{A}'$ between 2-categories consists of the following data:

(0) A function from objects of $\mathcal{A}$ to objects of $\mathcal{A}'$;

(hom) For $A, B \in \mathcal{A}$, a functor $F\colon \mathcal{A}(A, B) \to \mathcal{A}'(FA, FB)$, which we represent diagrammatically in Figure 1.17.

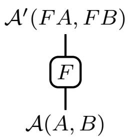

Figure 1.17: 2D string diagram for the action of 2-functors on hom-categories

($\odot$) For $A, B, C \in \mathcal{A}$, a *tensorator/compositor* natural isomorphism F^2 represented in Figure 1.18.

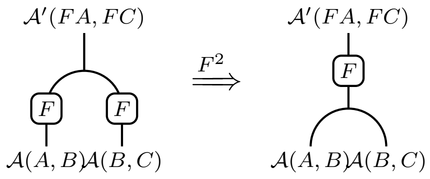

Figure 1.18: $(2 + 1)$D string diagram for tensorators/compositors of 2-functors

(I) For $A \in \mathcal{A}$, a *unitor* natural isomorphism F^0 represented in Figure 1.19.

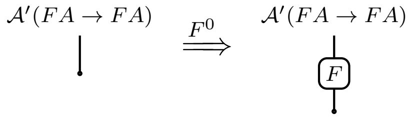

Figure 1.19: $(2 + 1)$D string diagram for unitors of 2-functors

Furthermore, we require this data satisfies an associativity axiom and two unitality axioms. In particular,

(a) For $A, B, C \in \mathcal{A}$, the hexagon in Figure 1.20 commutes.

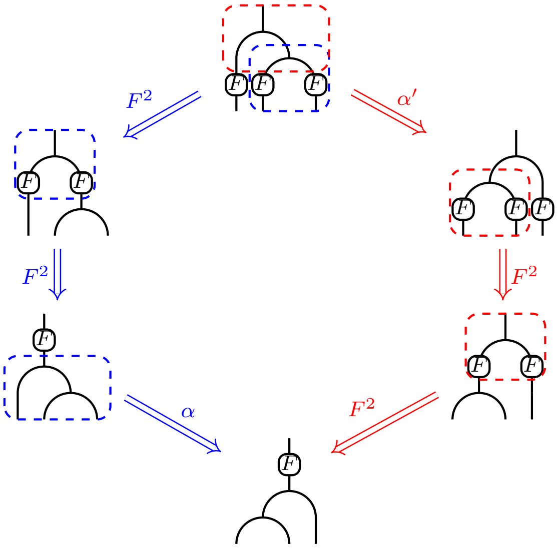

Figure 1.20: Associativity (hexagon) axiom for 2-functors

(u) For $A, B \in \mathcal{A}$, the squares in Figure 1.21 commute.

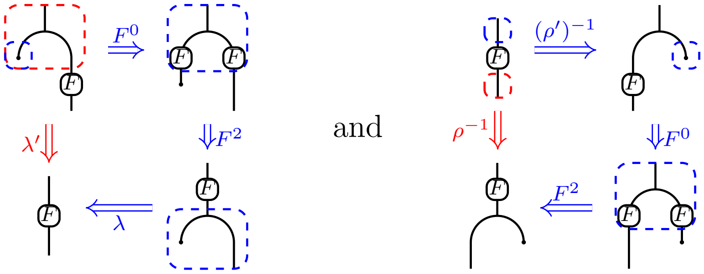

Figure 1.21: Left and right unitality axioms for 2-functors

Definition 1.3.1.5. A 2-natural transformation $\theta\colon F \Rightarrow F'$ between 2-functors $F, F'\colon \mathcal{A} \to \mathcal{A}'$ consists of the following data:

(0) A family of 1-morphisms $\theta_A \in \mathcal{A}'(FA \to F'A)$ indexed by the objects of $\mathcal{A}$, which we represent diagrammatically in Figure 1.22.

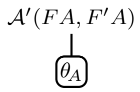

Figure 1.22: 2D string diagram for components of 2-natural transformations

(hom) For $A, B \in \mathcal{A}$, a *naturator* natural isomorphism θ represented in Figure 1.23.

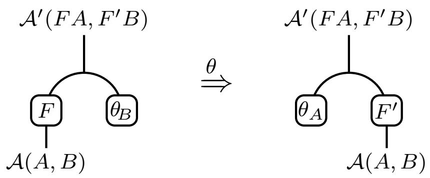

Figure 1.23: $(2+1)$D string diagram for naturators of 2-natural transformations

Furthermore, we require this data satisfy a monoidality/composability axiom and a unitality axiom. In particular,

($\odot$) For $A, B, C, D \in \mathcal{A}$, the octagon in Figure 1.24 commutes.

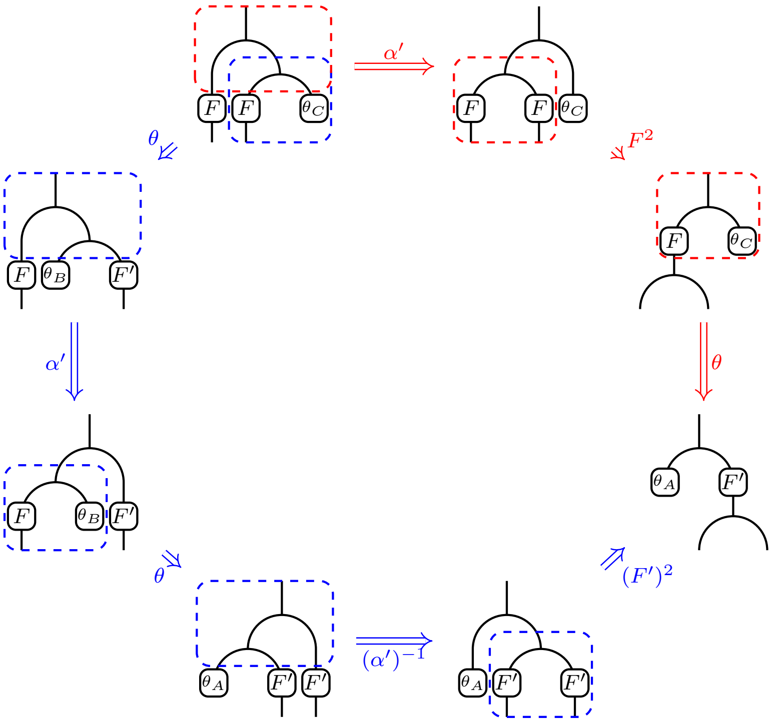

Figure 1.24: Monoidality axiom for 2-natural transformations

(I) For $A, B \in \mathcal{A}$, the pentagon in Figure 1.25 commutes.

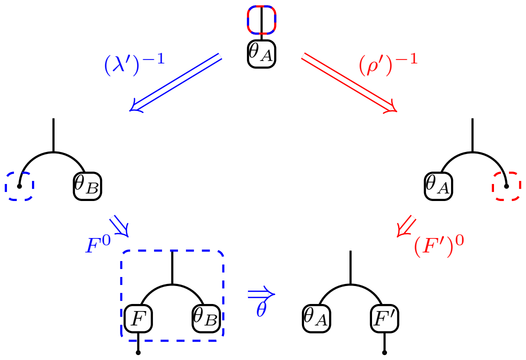

Figure 1.25: Unitality axiom for 2-natural transformations

Definition 1.3.1.6. A modification $m\colon \theta \Rrightarrow \theta'$ between 2-natural transformations $\theta, \theta'\colon F \Rightarrow F'$ consists of the following data:

(0) A family of 2-morphisms $\theta_A \in \mathcal{A}'(\theta_A \Rightarrow \theta'_A)$ indexed by the objects of $\mathcal{A}$, which we represent diagrammatically in Figure 1.26.

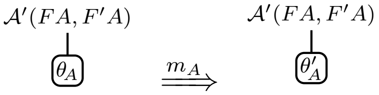

Figure 1.26: $(2+1)$D string diagram for components of modifications

Furthermore, we require that this data satisfies the obvious compatibility axiom with the higher data for θ and θ'. In particular,

(hom) For $A, B \in \mathcal{A}$, the square in Figure 1.27 commutes.

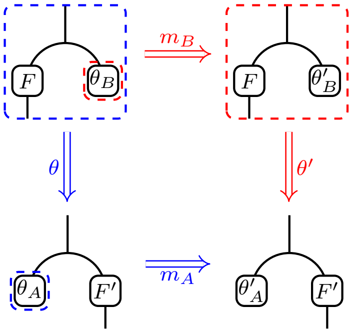

Figure 1.27: Naturality axiom for modifications

Definition 1.3.1.7. We denote the different operations for 2-natural transformations and modifications as follows:

- For 2-natural transformations $\theta\colon F \Rightarrow F'$ and $\sigma\colon G \Rightarrow G'$, we denote their 1-composition by $\sigma \circledcirc \theta\colon G \circ F \Rightarrow G' \circ F'$;
- For 2-natural transformations $\theta\colon F \Rightarrow F'$ and $\theta'\colon F' \Rightarrow F''$, we denote their 2-composition by $\theta' \odot \theta\colon F \Rightarrow F''$;
- For modifications $m\colon \theta \Rrightarrow \theta'$ and $n\colon \sigma \Rrightarrow \sigma'$ between natural transformations $\theta, \theta'\colon F \Rightarrow F'$ and $\sigma, \sigma'\colon G \Rightarrow G'$, we denote their 1-composition by $n \circledcirc m\colon \sigma \circledcirc \theta \Rightarrow \sigma' \circledcirc \theta'$;
- For modifications $m\colon \theta \Rrightarrow \theta'$ and $n\colon \sigma \Rrightarrow \sigma'$ between natural transformations $\theta, \theta'\colon F \Rightarrow F'$ and $\sigma, \sigma'\colon F' \Rightarrow F''$, we denote their 2-composition by $n \odot m\colon \sigma \odot \theta \Rightarrow \sigma' \odot \theta'$;
- For modifications $m\colon \theta \Rrightarrow \theta'$ and $n\colon \theta' \Rrightarrow \theta''$, we denote their 3-composition by $n \circ m\colon \theta \Rightarrow \theta''$. In the linear case, we may further take linear combinations of modifications.

Using these operations, we say

- $m\colon \theta \Rightarrow \theta'$ is an invertible modification when it admits an inverse with respect to $\circ$;
- $\theta\colon F \Rightarrow G$ is an equivalence if there exists $\theta^{\bullet}\colon G \Rightarrow F$ and invertible modifications $\varepsilon\colon \theta^{\bullet} \odot \theta \Rrightarrow \mathrm{id}_F$ and $\eta\colon \mathrm{id}_G \Rrightarrow \theta \odot \theta^{\bullet}$;
- $\theta := (\theta, \theta^{\bullet}, \varepsilon, \eta)$ is an adjoint equivalence if ε and η satisfy the zigzag/snake equations;
- $F\colon \mathcal{A} \to \mathcal{B}$ is an equivalence if there exists $F^{\bullet}\colon \mathcal{B} \to \mathcal{A}$ and equivalences $F^{\bullet} \circ F \Rightarrow \mathrm{id}_{\mathcal{A}}$ and $\mathrm{id}_{\mathcal{B}} \Rightarrow F \circ F^{\bullet}$. One may inductively define the notion of adjoint equivalences for 2-functors.

1.3.2 Operator 2-categories

We will follow [CHPJP22] in the following treatment of operator algebraic bicategories, except that we do not assume such 2-categories are unitarily Cauchy complete.

Definition 1.3.2.1. A C*-2-category consists of a (linear, weak) 2-category $(\mathcal{A}, \odot, 1, \alpha, \lambda, \rho)$ together with:

- ($\dagger$) a dagger structure, i.e. an involutive contravariant conjugate-linear 2-functor $\dagger : \overline{\mathcal{A}}^{\mathrm{op}} \to \mathcal{A}$, which is the identity on objects and 1-morphisms, such that the associator, left unitor, and right unitor

$$\begin{aligned} \alpha_{X,Y,Z} &: X \odot_B (Y \odot_C Z) \Rightarrow (X \odot_B Y) \odot_C Z, \\ \lambda_X &: \mathrm{id}_A \odot_A X \Rightarrow X, \\ \rho_X &: X \odot_B \mathrm{id}_B \Rightarrow X \end{aligned}$$

  are unitary ($u^\dagger = u^{-1}$) for composable 1-morphisms $A \xrightarrow{X} B \xrightarrow{Y} C \xrightarrow{Z} D$ in $\mathcal{A}$ and the composition $\odot$ of 1-morphisms in $\mathcal{A}$ is $\dagger$-preserving,

- (C*) the hom-category $\mathcal{A}(A \to B)$ is a C*-category for every pair of objects $A, B \in \mathcal{A}$.

Furthermore, we say that a C*-2-category is a W*-2-category when:

- (W*1) every hom-category $\mathcal{A}(A \to B)$ is a W*-category,
- (W*2) 1-composition $\odot$ is separately normal (weak*-continuous).

In [CHPJP22] it is shown that (W*2) is equivalent to the following condition:

- (W*2′) For every object $A \in \mathcal{A}$, the pre-composition and post-composition maps $\mathrm{id}_A \odot_A -$ and $- \odot_A \mathrm{id}_A$ are normal.

We will use the terms operator algebraic bicategory or operator 2-category to refer to C*/W*-2-categories. Moreover, we say that an operator 2-category $\mathcal{A}$ is strict if the underlying 2-category is strict, recalling that we use the terms 2-category and bicategory interchangeably, specifying whenever they are strict.

Example 1.3.2.2. Given an operator 2-category $\mathcal{A}$, we may produce related operator 2-categories $\mathcal{A}^{\odot\mathrm{p}}$ and $\mathcal{A}^{\mathrm{op}}$ given by reversing the direction of 1-morphisms and 2-morphisms respectively.

Definition 1.3.2.3. A †-2-functor $F\colon \mathcal{A} \to \mathcal{A}'$ between C*-2-categories is a (linear) 2-functor (F, F^2, F^0) such that

(†) F is dagger-preserving, i.e $F(x^\dagger) = F(x)^\dagger$ for every 2-morphism x in $\mathcal{A}$,

(coh) the compositor and unitor

$$\begin{aligned} F^2_{X,Y}\colon F(X) \odot_{F(B)} F(Y) &\Rightarrow F(X \odot_B Y), \\ F^0_A\colon \mathrm{id}_{F(A)} &\Rightarrow F(\mathrm{id}_A), \end{aligned}$$

are unitary for $A \xrightarrow{X} B \xrightarrow{Y} C$ in $\mathcal{A}$.

Furthermore, we say that a †-2-functor $F\colon \mathcal{A} \to \mathcal{A}'$ between W*-2-categories is normal when

(W*) it is normal (weak*-continuous) on hom-spaces $\mathcal{A}(X \Rightarrow Y) \to \mathcal{A}'(F(X) \Rightarrow F(Y))$.

Moreover, we say that a (normal) †-2-functor is strict if the underlying 2-functor is strict.

Definition 1.3.2.4. A †-2-natural transformation $\alpha\colon F \Rightarrow G$ between (normal) †-2-functors is a 2-natural transformation α such that (coh) The naturator component

$$\alpha_X \colon F(X) \odot_{F(B)} \alpha_B \Rightarrow \alpha_A \odot_{G(A)} G(X)$$

is unitary for every 1-morphism $A \xrightarrow{X} B$ in $\mathcal{A}$.

Definition 1.3.2.5. A uniformly bounded modification $m \colon \alpha \Rrightarrow \beta$ between †-2-natural transformations is a modification $m = (m_A \colon \alpha_A \Rightarrow \beta_A)_{A \in \mathcal{A}}$ such that

$$\|m\| \coloneqq \sup_{A \in \mathcal{A}} \|m_A\|_{\mathcal{A}'} < \infty.$$

Example 1.3.2.6. We denote the strict C*-2-category of C*-categories, †-functors, and uniformly bounded natural transformations by $\mathrm{C}^*\mathsf{Cat}$. Similarly, we denote the W*-2-category of W*-categories, normal †-functors and uniformly bounded natural transformation by $\mathrm{W}^*\mathsf{Cat}$.

Example 1.3.2.7. One may modify these definitions to obtain the notion of a C*-⊗-category $\mathcal{A}$, †-⊗-functors, and uniformly bounded ⊗-natural transformations. In particular, one may think of a C*-⊗-category as a C*-2-category with a single object $\bullet$, which we will simply omit reference to and instead denote $\odot$ by $\otimes$ and $\mathrm{id}_\bullet$ by $1_{\mathcal{A}}$. A †-⊗-functor $F \colon \mathcal{A} \to \mathcal{A}'$ is simply a †-2-functor. In contrast, a uniformly bounded ⊗-natural transformation $\theta \colon F \Rightarrow G$ consists of a uniformly bounded natural transformation θ between the maps from F and G on hom-categories, satisfying monoidality/composability and unitality axioms.

These assemble into a C*-2-category, which we will denote by $C^*\otimes\mathsf{Cat}$. There is an analogous notion of W*-⊗-categories.

Example 1.3.2.8. In Proposition 2.13 of [CP22], it is shown that for C*-2-categories $\mathcal{A}$ and $\mathcal{B}$, there is a C*-2-category $\mathrm{C}^*2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$ of †-2-functors from $\mathcal{A}$ to $\mathcal{B}$, †-2-natural transformations, and uniformly bounded modifications. Moreover, $\mathrm{C}^*2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$ is strict whenever $\mathcal{B}$ is strict. There is an analogous W*-2-category $\mathrm{W}^*2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$ of normal †-2-functors between W*-2-categories $\mathcal{A}$ and $\mathcal{B}$.

Definition 1.3.2.9. Let $\theta = (\theta, \theta^{\bullet}, \epsilon, \eta)$ be an adjoint equivalence where θ is a †-2-natural transformation in $\mathrm{C}^*2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$. We say θ is a *unitary adjoint equivalence* whenever ϵ and η are unitary modifications. One may inductively extend this notion to talk about unitary adjoint equivalences between C*-2-categories.

Fact 1.3.2.10 (Whitehead theorem for operator 2-categories)**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a †-2-functor between C*-2-categories. Then F is an equivalence of the underlying 2-categories if and only if it forms part of a unitary adjoint equivalence. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are W*, then F is automatically normal.*

1.3.3 Serre–Swan duality revisited

Swan's theorem [Swa62, p.267] states the construction Γ taking a bundle E to its space of sections $\Gamma(E)$ is an equivalence of suitable categories. Each category carries a monoidal product – the tensor product of bundles and the relative tensor product of modules, respectively – and so we will give a proof that the equivalence guaranteed by Swan's theorem is, in fact, a monoidal †-equivalence. This result is already known to experts; section 7.5 of [Con08] gives a proof of this monoidal equivalence for the smooth version of the Serre–Swan Theorem for differentiable manifolds. Our proof for Swan's theorem is essentially a modified proof of the one found in [Con08].

Returning to vector bundles, notice the rank n trivial bundle $T \times \mathbb{C}^n \to T$ yields the free n-dimensional $C(T)$-module

$$\Gamma(T \times \mathbb{C}^n) = C(T) \otimes_{\mathbb{C}} \mathbb{C}^n.$$

Of course, every free $C(T)$-module arises in this way, and each map of free modules $C(T) \otimes_{\mathbb{C}} \mathbb{C}^n \to C(T) \otimes_{\mathbb{C}} \mathbb{C}^m$ is uniquely determined by a map $\mathbb{C}^n \to \mathbb{C}^m$ which in turn induces a corresponding bundle map $T \times \mathbb{C}^n \to T \times \mathbb{C}^m$. In general, every finite rank bundle can be witnessed inside one of these trivial bundles:

Lemma 1.3.3.1. *Let E be a bundle over T. Then there is a bundle $E^\perp$ with the property that $E \oplus E^\perp$ is a trivial bundle over T.*

As a direct consequence, $\Gamma(E)$ is a finitely generated projective $C(T)$-module. The content of Swan's theorem is that every finitely generated projective module arises in this way, so that the categories of finite rank vector bundles over T and finitely generated projective modules over $C(T)$ are equivalent.

However, in this note we are interested in Hilbert C*-modules over $C(T)$. It is well-known that the data of a $C(T)$-valued inner product on the space of sections $\Gamma(E)$ is precisely the same data as a Hermitian metric on E. More specifically, $\Gamma\colon \mathsf{Hilb}_{\mathsf{fd}}(T) \to \mathsf{Hilb}_{\mathsf{fgp}}(C(T))$ is an equivalence of categories where $\mathsf{Hilb}_{\mathsf{fd}}(T)$ is the category of finite-dimensional vector bundles over T and $\mathsf{Hilb}_{\mathsf{fgp}}(C(T))$ is the category of finitely generated projective $C(T)$-modules. We remark here that Γ is a $\dagger$-functor, described precisely in the following lemma. The proof is immediate from the definition of the $C(T)$-valued inner product on $\Gamma(E)$.

Lemma 1.3.3.2. *Let E and F be vector bundles with Hermitian metrics $\langle\cdot|\cdot\rangle_E$ and $\langle\cdot|\cdot\rangle_F$. If $\sigma\colon E \to F$ is an adjointable map of bundles in the sense that there is a bundle map $\sigma^\dagger\colon F \to E$ such that $\langle\sigma(e)|f\rangle_F = \langle e|\sigma^\dagger(f)\rangle_E$, then $\Gamma(\sigma)^\dagger = \Gamma(\sigma^\dagger)$.*

However, both $\mathsf{Hilb}_{\mathsf{fd}}(T)$ and $\mathsf{Hilb}_{\mathsf{fgp}}(C(T))$ admit monoidal structures $\otimes_T$ and $\otimes_{C(T)}$ respectively. It remains to show this equivalence is monoidal; that is, we will exhibit a unitary

$$\eta\colon C(T)_{C(T)} \to \Gamma(T \times \mathbb{C})$$

and, for each pair of vector bundles E and F, a unitary

$$\mu_{E,F}\colon \Gamma(E) \otimes_{C(T)} \Gamma(F) \to \Gamma(E \otimes_T F)$$

subject to the following coherences:

- For any bundles E, F, and G, the following associativity hexagon commutes:

$$\begin{tikzcd}
(\Gamma(E) \otimes_{C(T)} \Gamma(F)) \otimes_{C(T)} \Gamma(G) \arrow[r, "\alpha"] \arrow[d, "\mu_{E,F}\otimes \mathrm{id}"'] & \Gamma(E) \otimes_{C(T)} (\Gamma(F) \otimes_{C(T)} \Gamma(G)) \arrow[d, "\mathrm{id}\otimes\mu_{F,G}"] \\
\Gamma(E \otimes_T F) \otimes_{C(T)} \Gamma(G) \arrow[d, "\mu_{E\otimes F,G}"'] & \Gamma(E) \otimes_{C(T)} \Gamma(F \otimes_T G) \arrow[d, "\mu_{E,F\otimes G}"] \\
\Gamma((E \otimes_T F) \otimes_T G) \arrow[r, "\Gamma(\alpha)"'] & \Gamma(E \otimes_T (F \otimes_T G))
\end{tikzcd} \tag{1.1}$$

- For any bundle E, the following two unitality squares commute (corresponding to left and right unitors, respectively):

$$\begin{tikzcd}
C(T) \otimes_{C(T)} \Gamma(E) \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d, "\lambda_{\Gamma(E)}"'] & \Gamma(T \times \mathbb{C}) \otimes_{C(T)} \Gamma(E) \arrow[d, "\mu_{(T\times\mathbb{C}),E}"] \\
\Gamma(E) & \Gamma((T \times \mathbb{C}) \otimes_T E) \arrow[l, "\Gamma(\lambda_E)"]
\end{tikzcd} \tag{1.2}$$

$$\begin{tikzcd}
\Gamma(E) \otimes_{C(T)} C(T) \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[d, "\rho_{\Gamma(E)}"'] & \Gamma(E) \otimes_{C(T)} \Gamma(T \times \mathbb{C}) \arrow[d, "\mu_{E,(T\times\mathbb{C})}"] \\
\Gamma(E) & \Gamma(E \otimes_T (T \times \mathbb{C})) \arrow[l, "\Gamma(\rho_E)"]
\end{tikzcd} \tag{1.3}$$

Definition 1.3.3.3. Let E and F be vector bundles over the compact Hausdorff space T. Define the linear function $\mu_{E,F} \colon \Gamma(E) \otimes_{C(T)} \Gamma(F) \to \Gamma(E \otimes_T F)$ by

$$\mu_{E,F}(f \otimes_{C(T)} g) = f \otimes_T g \qquad \text{for } f \in \Gamma(E), g \in \Gamma(F).$$

Also, define $\eta \colon C(T) \to \Gamma(T \times \mathbb{C})$ to be

$$\eta(h)(t) = (t, h(t)) \qquad \text{for } h \in C(T).$$

It is almost a tautology that η is a unitary isomorphism of $C(T)$-modules. Seeing that $\mu_{E,F}$ is unitary is more subtle. We will break this proof into multiple parts.

Proposition 1.3.3.4. *If E and F are trivial bundles equipped with Hermitian metrics $\langle\cdot|\cdot\rangle_E$ and $\langle\cdot|\cdot\rangle_F$, then $\mu_{E,F}$ is a unitary isomorphism.*

Proof. Choose (orthogonal) global sections $\{f_1, f_2, \ldots, f_n\}$ of E and $\{g_1, g_2, \ldots, g_m\}$ of F that trivialize their respective bundles. These form bases for $\Gamma(E)$ and $\Gamma(F)$, respectively. Now, note that the set

$$\{f_i \otimes_{C(T)} g_j\}_{(i,j)=(1,1)}^{(n,m)}$$

has dense $C(T)$-span in $\Gamma(E) \otimes_{C(T)} \Gamma(F)$. However, observe that

$$\{f_i \otimes_T g_j\}_{(i,j)=(1,1)}^{(n,m)}$$

is a set of (orthogonal) global sections of $E \otimes_T F$ that trivializes the bundle. Since $\mu_{E,F}$ carries the first set to the second, we see that $\mu_{E,F}$ must be surjective. It is clear that $\mu_{E,F}$ has a set-theoretic inverse. To show that $\mu_{E,F}^\dagger = \mu_{E,F}^{-1}$, we have, for all $1 \leq i, k \leq n$ and $1 \leq j, l \leq m$,

$$\begin{aligned}
\langle \mu_{E,F}(f_i \otimes_{C(T)} g_j) | f_k \otimes_T g_l \rangle(t) &= \langle (f_i \otimes_T g_j)(t) | (f_k \otimes_T g_l)(t) \rangle_{E \otimes F} \\
&= \langle f_i(t) | f_k(t) \rangle_E \langle g_j(t) | g_l(t) \rangle_F \\
&= \langle g_j(t) | \langle f_i(t) | f_k(t) \rangle_E g_l(t) \rangle_F \\
&= \langle g_j(t) | (\langle f_i | f_k \rangle \triangleright g_l)(t) \rangle_F \\
&= \langle g_j | \langle f_i | f_k \rangle \triangleright g_l \rangle(t) \\
&= \langle f_i \otimes_T g_j | \mu_{E,F}^{-1}(f_k \otimes_{C(T)} g_l) \rangle(t)
\end{aligned}$$

This shows that $\mu_{E,F}$ is adjointable and, in particular, is unitary. □

Lemma 1.3.3.5. *Let E and F be vector bundles over T. Define the maps*

$$i \colon E \to E \oplus F \qquad i(v) = (v, 0)$$

and

$$\rho\colon E \oplus F \to E \qquad \rho(v,w) = v.$$

Then $\Gamma(\rho) \circ \Gamma(i) = \mathrm{id}_{\Gamma(E)}$*. Furthermore, if* E *and* F *have Hermitian metrics* $\langle\cdot|\cdot\rangle_E$ *and* $\langle\cdot|\cdot\rangle_F$*, then* $\Gamma(\rho)$ *and* $\Gamma(i)$ *are adjointable with* $\Gamma(\rho)^\dagger = \Gamma(i)$ *when* $E \oplus F$ *is equipped with the Hermitian metric* $\langle\cdot|\cdot\rangle_{E\oplus F} = \langle\cdot|\cdot\rangle_E \oplus \langle\cdot|\cdot\rangle_F$*.*

Proof. It is clear that $\Gamma(\rho) \circ \Gamma(i) = \mathrm{id}_{\Gamma(E)}$, because $\rho \circ i = \mathrm{id}$, and Γ is functorial. It is also routine to see that $\rho^\dagger = i$, as

$$\langle \rho(f_1, g) | f_2 \rangle_E = \langle f_1 | f_2 \rangle_E = \langle (f_1, g) | (f_2, 0) \rangle_{E\oplus F} = \langle (f_1, g) | i(f_2) \rangle_{E\oplus F}.$$

Because Γ is a †-functor by Lemma 1.3.3.2, we conclude that $\Gamma(\rho)^\dagger = \Gamma(i)$. □

Lemma 1.3.3.6. *If* E *and* F *are any vector bundles equipped with Hermitian metrics, then* $\mu_{E,F}$ *is a unitary isomorphism.*

Proof. Using Lemma 1.3.3.1, it is clear that $E \otimes F$ is a direct summand of the trivial bundle $(E \oplus E^\perp) \otimes_T (F \oplus F^\perp)$. Consider the diagram

$$\begin{tikzcd}
\Gamma((E \oplus E^\perp) \otimes_T (F \oplus F^\perp)) & \Gamma(E \oplus E^\perp) \otimes_{C(T)} \Gamma(F \oplus F^\perp) \arrow[l, "\mu"'] \\
\Gamma(E \otimes_T F) \arrow[u, "\Gamma(i)"] & \Gamma(E) \otimes_{C(T)} \Gamma(F), \arrow[l, "\mu_{E,F}"] \arrow[u, "\Gamma(i)\otimes\Gamma(i)"']
\end{tikzcd} \tag{1.4}$$

where μ is the unitary isomorphism for $(E \oplus E^\perp)$ and $(F \oplus F^\perp)$ in Proposition 1.3.3.4. Note that, by Lemma 1.3.3.5, $\Gamma(i)$ is injective. Furthermore,

$$(\Gamma(\rho) \otimes_{C(T)} \Gamma(\rho)) \circ (\Gamma(i) \otimes_{C(T)} \Gamma(i)) = \Gamma(\mathrm{id}) \otimes_{C(T)} \Gamma(\mathrm{id})$$

and so $\Gamma(i) \otimes \Gamma(i)$ is injective. Thus $\mu_{E,F}$ must be injective. Similarly, we have

$$\begin{tikzcd}
\Gamma((E \oplus E^\perp) \otimes_T (F \oplus F^\perp)) \arrow[d, "\Gamma(\rho)"'] & \Gamma(E \oplus E^\perp) \otimes_{C(T)} \Gamma(F \oplus F^\perp) \arrow[l, "\mu"'] \arrow[d, "\Gamma(\rho)\otimes\Gamma(\rho)"] \\
\Gamma(E \otimes_T F) & \Gamma(E) \otimes_{C(T)} \Gamma(F), \arrow[l, "\mu_{E,F}"]
\end{tikzcd} \tag{1.5}$$

and so $\Gamma(\rho)$ and $\Gamma(\rho) \otimes \Gamma(\rho)$ are both surjective. This guarantees that $\mu_{E,F}$ is surjective as well. What remains to show is that $\mu_{E,F}$ is adjointable (and unitary), but this follows from our earlier work. Observe that Diagram (1.4) says that

$$\Gamma(i) \circ \mu_{E,F} = \mu \circ (\Gamma(i) \otimes_{C(T)} \Gamma(i)),$$

which implies

$$\mu_{E,F} = \Gamma(\rho) \circ \Gamma(i) \circ \mu_{E,F} = \Gamma(\rho) \circ \mu \circ (\Gamma(i) \otimes_{C(T)} \Gamma(i))$$

and so $\mu_{E,F}$ is a composition of adjointable maps and is therefore itself adjointable (noting that the Hermitian metrics on $E \oplus E^{\perp}$, $F \oplus F^{\perp}$, and $(E \oplus E^{\perp}) \otimes_T (F \oplus F^{\perp})$ may be chosen to be compatible with Proposition 1.3.3.4 so that μ is unitary). To verify that $\mu_{E,F}$ is unitary, we have

$$\begin{aligned} \mu_{E,F}^{\dagger} &= (\Gamma(i) \otimes_{C(T)} \Gamma(i))^{\dagger} \circ \mu^{\dagger} \circ \Gamma(\rho)^{\dagger} \\ &= (\Gamma(i)^{\dagger} \otimes_{C(T)} \Gamma(i)^{\dagger}) \circ \mu^{\dagger} \circ \Gamma(\rho)^{\dagger} \\ &= (\Gamma(\rho) \otimes_{C(T)} \Gamma(\rho)) \circ \mu^{-1} \circ \Gamma(i). \end{aligned}$$

But, Diagram (1.5) says that

$$\mu_{E,F}^{-1} \circ \Gamma(\rho) = (\Gamma(\rho) \otimes_{C(T)} \Gamma(\rho)) \circ \mu^{-1},$$

and so precomposing with $\Gamma(i)$ yields

$$\mu_{E,F}^{-1} = \mu_{E,F}^{-1} \circ \Gamma(\rho) \circ \Gamma(i) = (\Gamma(\rho) \otimes_{C(T)} \Gamma(\rho)) \circ \mu^{-1} \circ \Gamma(i) = \mu_{E,F}^{\dagger}$$

which proves that $\mu_{E,F}$ is unitary. □

Theorem 1.3.3.7. *The space of sections construction $E \mapsto \Gamma(E)$ assembles into a monoidal equivalence of C^*-$\otimes$-categories*

$$\Gamma\colon \mathsf{Hilb}_{\mathsf{fd}}(T) \to \mathsf{Hilb}_{\mathsf{fgp}}\mathsf{C(T)}.$$

Proof. We saw in Lemma 1.3.3.2 that Γ is a †-functor. We also know that η and $\mu_{E,F}$ are unitary isomorphisms from Lemma 1.3.3.6. We only need to verify the coherences for a monoidal functor. It is clear from the construction of $\mu_{E,F}$ and the fact that associators α are determined by reparenthesizing simple tensors that the associativity diagram (1.1) commutes. Indeed, for $e \in \Gamma(E)$, $f \in \Gamma(F)$, and $g \in \Gamma(G)$, observe

$$\begin{tikzcd}
(e \otimes_{C(T)} f) \otimes_{C(T)} g \arrow[r, mapsto, "\alpha"] \arrow[d, mapsto, "\mu_{E,F} \otimes \mathrm{id}"'] & e \otimes_{C(T)} (f \otimes_{C(T)} g) \arrow[d, mapsto, "\mathrm{id} \otimes \mu_{F,G}"] \\
(e \otimes_T f) \otimes_{C(T)} g \arrow[d, mapsto, "\mu_{E \otimes F, G}"'] & e \otimes_{C(T)} (f \otimes_T g) \arrow[d, mapsto, "\mu_{E, F \otimes G}"] \\
(e \otimes_T f) \otimes_T g \arrow[r, mapsto, "\Gamma(\alpha)"'] & e \otimes_T (f \otimes_T g)
\end{tikzcd}$$

We next need to check Diagrams (1.2) and (1.3) for unitality. For $h \in C(T)$ and $e \in \Gamma(E)$, observe

$$\begin{tikzcd}
h(t) \otimes_{C(T)} e(t) \arrow[r, mapsto, "\eta \otimes \mathrm{id}"] \arrow[d, mapsto, "\lambda_{\Gamma(E)}"'] & (t, h(t)) \otimes_{C(T)} e(t) \arrow[d, mapsto, "\mu_{(T \times \mathbb{C}), E}"] \\
h(t)e(t) & (t, h(t)) \otimes_T e(t) \arrow[l, mapsto, "\Gamma(\lambda_E)"]
\end{tikzcd}$$

$$\begin{tikzcd}
e(t) \otimes_{C(T)} h(t) \arrow[r, mapsto, "\mathrm{id} \otimes \eta"] \arrow[d, mapsto, "\rho_{\Gamma(E)}"'] & e(t) \otimes_{C(T)} (t, h(t)) \arrow[d, mapsto, "\mu_{E, (T \times \mathbb{C})}"] \\
e(t)h(t) & e(t) \otimes_T (t, h(t)) \arrow[l, mapsto, "\Gamma(\rho_E)"]
\end{tikzcd}$$

Thus, Γ is a monoidal †-equivalence $\mathsf{Hilb}_{\mathsf{fd}}(T) \to \mathsf{Hilb}_{\mathsf{fgp}}(C(T))$ □

1.3.4 Coherence and concreteness for operator 2-categories

Definition 1.3.4.1. For a C*-2-category $\mathcal{A}$, the Yoneda embedding

$$\text{よ} : \mathcal{A} \to \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}^{\odot\mathrm{p}} \to \mathrm{C}^*\mathsf{Cat})$$

is the †-2-functor given by:

(0) For an object $A \in \mathcal{A}$, we define the †-2-functor $\text{よ}_A : \mathcal{A} \to \mathrm{C}^*\mathsf{Cat}$ by:

- For an object $B \in \mathcal{A}$, we set the C*-category $\text{よ}_A(B) \coloneqq \mathcal{A}(B \to A)$,
- For a morphism ${}_BX_C$ in $\mathcal{A}$, we set the †-functor $\text{よ}_A({}_BX_C) \coloneqq - \odot_B X_C$,
- For a 2-morphism $x\colon {}_BX_C \to {}_BX'_C$, we set the uniformly bounded natural transformation $\text{よ}_A(x) \coloneqq - \odot_B x$. Notice $\|\text{よ}^A(x)\| \leq \|x\|$.
- We define the tensorator component

  $$(\text{よ}_A)^2_{X,X'} : \text{よ}_A(X') \odot_{\text{よ}_A(C)} \text{よ}_A(X) \Rightarrow \text{よ}_A(X' \odot_C X)$$

  for two composable morphisms ${}_BX_C$ and ${}_CX'_D$ in $\mathcal{A}$ to be the natural unitary $(\text{よ}_A)^2_{X,X'} \coloneqq \alpha_{-,X',X}$.
- We define the unitor component $(\text{よ}_A)^1 : \mathrm{id}_{\text{よ}_A(B)} \Rightarrow \text{よ}_A(\mathrm{id}_B)$ for an object $B \in \mathcal{A}$ to be the natural unitary $(\text{よ}_A)^1_B \coloneqq \rho^\dagger$.

(1) For a morphism ${}_{A'}Y_A$ in $\mathcal{A}$, we define the †-2-natural transformation $\text{よ}_Y\colon \text{よ}_A \Rightarrow \text{よ}_{A'}$ by:

- For an object $B \in \mathcal{A}$, we set the †-functor $(\text{よ}_Y)_B \coloneqq {}_{A'}Y \odot_A -$.
- For a morphism ${}_BX_C$ in $\mathcal{A}$, we set the uniformly bounded natural transformation $(\text{よ}_Y)_X \coloneqq \alpha_{Y,-,X}$.

(2) For a 2-morphism $y\colon {}_{A'}Y_A \Rightarrow {}_{A'}Y'_A$, we define the uniformly bounded modification $\text{よ}_y\colon \text{よ}_Y \Rrightarrow \text{よ}_{Y'}$ by:

- For an object $B \in \mathcal{A}$, we set the uniformly bounded natural transformation $(\text{よ}_y)_B \coloneqq y \odot_A -$.

($\odot$) We define the compositor component $\text{よ}^2_{Y',Y} : \text{よ}_{Y'} \odot \text{よ}_Y \Rightarrow \text{よ}_{Y' \odot Y}$ for two composable morphisms ${}_{A''}Y'_{A'}$ and ${}_{A'}Y_A$ in $\mathcal{A}$ to be $\text{よ}^2_{Y',Y} \coloneqq \alpha_{Y',Y,-}$,

(I) We define the unitor component $\text{よ}^1_A : \mathrm{id}_{\text{よ}_A} \Rightarrow \text{よ}^{\mathrm{id}_A}$ for an object $A \in \mathcal{A}$ to be $\text{よ}^1_A := \lambda^\dagger$.

Remark 1.3.4.2. When $\mathcal{A}$ is a strict C*-2-category, it is clear from construction that the Yoneda embedding for $\mathcal{A}$ is a strict †-2-functor which lands in $\mathrm{C}^*\mathsf{2Cat}_{\mathsf{st}}(\mathcal{A}^{\odot\mathrm{p}} \to \mathrm{C}^*\mathsf{Cat})$.

Theorem 1.3.4.3. *For a C*-2-category $\mathcal{A}$, the Yoneda embedding*

$$\text{よ} : \mathcal{A} \to \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}^{\odot\mathrm{p}} \to \mathrm{C}^*\mathsf{Cat})$$

is fully faithful. Hence, every C-2-category is equivalent to a strict one.*

Proof. This follows by the Yoneda embedding theorem for ordinary 2-categories. □

Theorem 1.3.4.4. *For a C*-2-category $\mathcal{A}$, the universal representation*

$$\text{よ}_{\amalg} \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}^{\odot\mathrm{p}} \to \mathrm{C}^*\mathsf{Cat}) \quad \textit{given by} \quad \text{よ}_{\amalg} := \coprod_{A \in \mathcal{A}} \text{よ}_A$$

is monic. Thus, every C-2-category can be realized as a norm closed †-2-subcategory of* $\mathrm{C}^*\mathsf{Cat}$. *Moreover*

- *when $\mathcal{A}$ is a strict, $\text{よ}_{\amalg} \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}^{\odot\mathrm{p}} \to \mathrm{C}^*\mathsf{Cat})$ is a strict †-2-functor; and*
- *when $\mathcal{A}$ is small, we have that $\text{よ}_{\amalg} \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}^{\odot\mathrm{p}} \to \mathrm{C}^*\mathsf{Cat}_{\mathsf{small}})$.*

Proof. To see that $\text{よ}_{\amalg}$ is injective, suppose $\text{よ}_{\amalg}(A) = \text{よ}_{\amalg}(B)$ for $A, B \in \mathcal{A}$. Then $\mathrm{id}_A \in \text{よ}_{\amalg}(B)$, which occurs only when $A = B$. To see F is injective on 1-morphisms, suppose $\text{よ}_{\amalg}({}_AX_B) = \text{よ}_{\amalg}({}_AX'_B)$ and observe

$${}_AX_B = \text{よ}_{\amalg}({}_AX_B)(\mathrm{id}_A) = \text{よ}_{\amalg}({}_AX'_B)(\mathrm{id}_A) = {}_AY_B.$$

An identical argument yields that $\text{よ}_{\amalg}$ is injective at level of 2-morphisms. Furthermore, when $\mathcal{A}$ is strict, よ is a strict †-2-functor, hence $\text{よ}_{\amalg}$ is strict as well. Finally, when $\mathcal{A}$ is small, the objects $B \in \mathcal{A}$ form a set and each hom C*-category $\mathcal{A}(A \to B)$ is small, so

$$\text{よ}_{\amalg}(B) = \coprod_{A \in \mathcal{A}} \mathcal{A}(B \to A)$$

is a small C*-category for every $A \in \mathcal{A}$. $\square$

Definition 1.3.4.5. Recall that, for a small C*-category $\mathcal{B}$, there exists a universal representation

$$\mathcal{B} \xrightarrow[\sim]{\Upsilon} \mathsf{GNS}(\mathcal{B}) \hookrightarrow \mathsf{GNS}(\mathcal{B})''$$

where $\widetilde{\Upsilon} : \mathcal{B}^{**} \xrightarrow{\sim} \mathsf{GNS}(\mathcal{B})''$ is an isomorphism by Corollary 1.2.6.11. We may upgrade this construction to a †-2-functor

$$\mathsf{GNS}''\colon \mathrm{C}^*\mathsf{Cat}_{\mathsf{small}} \to \mathsf{Hilb}^{\mathsf{W}^*}$$

as follows:

(0) On a small C*-category $\mathcal{B}$, $\mathsf{GNS}''(\mathcal{B}) = \mathsf{GNS}(\mathcal{B})''$.

(1) For a †-functor $F\colon \mathcal{B}_1 \to \mathcal{B}_2$ between small C*-categories, we define the normal †-functor

$$\mathsf{GNS}''(F)\colon\ \mathsf{GNS}(\mathcal{B}_1)'' \to \mathsf{GNS}(\mathcal{B}_2)''$$

as follows:

(F0) On a Hilbert space $\Upsilon B \in \mathsf{GNS}(\mathcal{B}_1)''$ where $B \in \mathcal{B}_1$, we set

$$\mathsf{GNS}''(F)(\Upsilon B) \coloneqq \Upsilon(FB) \in \mathsf{GNS}(\mathcal{B}_2)''.$$

(F1) On an operator $S \in \mathsf{GNS}(\mathcal{B}_1)''(\Upsilon B \to \Upsilon B')$, consider $\widetilde{\Upsilon}^{-1}S \in \mathcal{B}_1^{**}(B \to B')$. As seen in Definition 1.2.6.10, consider the morphism $F^{**}\widetilde{\Upsilon}^{-1}S \in \mathcal{B}_2^{**}(FB \to FB')$. We then define the morphism $\mathsf{GNS}''(F)(S) \in \mathsf{GNS}(\mathcal{B}_2)''(\Upsilon FB \to \Upsilon FB')$ by

$$\mathsf{GNS}''(F)(S) \coloneqq \widetilde{\Upsilon} F^{**} \widetilde{\Upsilon}^{-1} S.$$

Notice $\mathsf{GNS}''(F)$ is a normal †-functor since $\widetilde{\Upsilon}$ and F^{**} are normal †-functors. Moreover, GNS'' is strictly 1-composition-preserving since $(-)^{**}$ is strict.

(2) For a uniformly bounded natural transformation $\alpha \in \mathrm{C^*}\mathsf{Cat}_{\mathsf{small}}({}_{\mathcal{B}_1}F_{\mathcal{B}_2} \Rightarrow {}_{\mathcal{B}_1}G_{\mathcal{B}_2})$, we define the uniformly bounded natural transformation

$$\mathsf{GNS}''(\alpha)\colon\ \mathsf{GNS}''(F) \Rightarrow \mathsf{GNS}''(G)$$

as follows:

(α0) On a Hilbert space $\Upsilon B \in \mathsf{GNS}(\mathcal{B}_1)''$ where $B \in \mathcal{B}_1$, we define the component

$$\mathsf{GNS}''(\alpha)_{\Upsilon B}\colon \Upsilon(FB) \to \Upsilon(GB)$$

in $\mathsf{GNS}(\mathcal{B}_2)''$ by

$$\mathsf{GNS}''(\alpha)_{\Upsilon B} \coloneqq \Upsilon(\alpha_B) = \widetilde{\Upsilon}(\mathrm{ev}_{\alpha_B}).$$

Lemma 1.3.4.6. $\mathsf{GNS}''\colon \mathrm{C^*}\mathsf{Cat}_{\mathsf{small}} \to \mathsf{Hilb}^{\mathsf{W^*}}$ *is monic.*

Proof. Quite pedantically, it is clear that GNS'' is injective on objects. Now consider †-functors $F, G\colon \mathcal{B}_1 \to \mathcal{B}_2$ between small C*-categories such that $\mathsf{GNS}''(F) = \mathsf{GNS}''(G)$. Then, for every $B \in \mathcal{B}_1$, we have $\Upsilon(FB) = \Upsilon(GB)$, which implies $FB = GB$ since $\Upsilon\colon \mathcal{B}_2 \to \mathsf{Hilb}$ is a monic †-functor. For $b \in \mathcal{B}_1(B \to B')$, consider $\widetilde{\Upsilon}(\mathrm{ev}_b) \in \mathsf{GNS}(\mathcal{B}_1)''(\Upsilon B \to \Upsilon B')$ and observe

$$\widetilde{\Upsilon}F(b) = \widetilde{\Upsilon}F^{**}\,\mathrm{ev}_b = \mathsf{GNS}''(F)(\widetilde{\Upsilon}(\mathrm{ev}_b)) = \mathsf{GNS}''(G)(\widetilde{\Upsilon}(\mathrm{ev}_b)) = \widetilde{\Upsilon}G(b).$$

Thus $F = G$ and we conclude GNS'' is injective on 1-morphisms. Finally, consider $\alpha \in \mathrm{C^*}\mathsf{Cat}_{\mathsf{small}}(F \Rightarrow G)$ between arbitrary †-functors F, G such that $\mathsf{GNS}''(\alpha) = 0$. For $B \in \mathcal{B}_1$, observe

$$\Upsilon(\alpha_B) = \mathsf{GNS}''(\alpha)_{\Upsilon B} = 0,$$

hence $\alpha_B = 0$ since $\Upsilon : \mathcal{B}_2 \to \mathsf{Hilb}$ is a monic †-functor. Therefore $\alpha = 0$ and we conclude that GNS'' is injective on 2-morphisms. $\square$

Theorem 1.3.4.7 (Gelfand-Naimark for C*-2-categories)**.** *For a small C*-2-category $\mathcal{A}$, the universal representation*

$$\Upsilon_2 \colon \mathcal{A} \to \mathsf{Hilb}^{\mathsf{W}^*}$$

given by $\Upsilon_2 := \mathsf{GNS}'' \circ \text{よ}^{\mathrm{II}}$ is monic. Thus, every C-2-category $\mathcal{A}$ can be realized as a norm-closed †-2-category $\mathsf{GNS}_2(\mathcal{A}) := \operatorname{Im}\Upsilon_2$ of weakly closed †-categories of Hilbert spaces and operators. Moreover, if $\mathcal{A}$ is strict, then Υ_2 is a strict †-2-functor.*

Proof. Both †-functors $\text{よ}^{\mathrm{II}} \colon \mathcal{A} \to \mathrm{C}^*\mathsf{Cat}_{\mathsf{small}}$ and $\mathsf{GNS}'' \colon \mathrm{C}^*\mathsf{Cat}_{\mathsf{small}} \to \mathsf{Hilb}^{\mathsf{W}^*}$ are monic, so Υ_2 is monic. Recall that GNS'' is always strict, whereas よ^{II} is strict whenever $\mathcal{A}$ is. Therefore Υ_2 is a strict †-2-functor if $\mathcal{A}$ is a strict C*-2-category. $\square$

1.3.5 Cofibrant replacement for operator 2-categories

Proposition 1.3.5.1. *For every C*-2-category $\mathcal{A}$, there exists a strict C*-2-category $\widehat{\mathcal{A}}$ together with an epic 2-equivalence $\mathrm{ev}_{\mathcal{A}} \colon \widehat{\mathcal{A}} \to \mathcal{A}$. Moreover, when $\mathcal{A}$ is a W*-2-category we have that $\widehat{\mathcal{A}}$ is a W*-2-category and hence $\mathrm{ev}_{\mathcal{A}}$ is automatically normal.*

Proof. We provide the construction in [Gur13, §2.2.3], noting that all the relevant data is compatible with dagger structures.

Let $\mathcal{A}$ be a C*-2-category. We define the C*-2-category $\widehat{\mathcal{A}}$ to have the same objects as $\mathcal{A}$, and free paths of 1-morphisms in $\mathcal{A}$ as 1-morphisms. More specifically, $X \in \widehat{\mathcal{A}}(A \to A')$ is some finite tuple

$$(A' \xrightarrow{X_n} A_n, \ldots, A_2 \xrightarrow{X_1} A_1, A_1 \xrightarrow{X_0} A)$$

of composable 1-morphisms in $\mathcal{A}$ starting on A and ending on A'. We note that for every object $A \in \mathcal{A}$ there is an empty string $\varnothing_A \in \widehat{\mathcal{A}}(A \to A)$. Composition $\widehat{\odot}$ of 1-morphisms in

$\widehat{\mathcal{A}}$ is given by concatenation, which is strictly associative and unital with $\varnothing_A$ acting as the identity on $\mathcal{A}$.

Before we define what 2-morphisms are in $\widehat{\mathcal{A}}$, we define $\mathrm{ev}_{\mathcal{A}}$ on objects to act as the identity. For $X \in \widetilde{\mathcal{A}}(A \to A')$ we define

$$\mathrm{ev}_{\mathcal{A}}(X) = (\cdots((X_0 \odot X_1) \odot X_2)\cdots) \odot X_n,$$

that is, $\mathrm{ev}_{\mathcal{A}}$ acts on a free path by evaluating its left-most parenthesization in $\mathcal{A}$. In particular, when $X = \varnothing_A$, we have that $\mathrm{ev}_{\mathcal{A}}(\varnothing_A) = \mathrm{id}_A$. We may thus choose the unitor $\mathrm{ev}^1_A : \mathrm{id}_{\mathrm{ev}_{\mathcal{A}}(A)} \Rightarrow \mathrm{ev}_{\mathcal{A}}(\varnothing_A)$ to be $\mathrm{id}_{\mathrm{id}_{\mathcal{A}}}$. Note that $\mathrm{ev}_{\mathcal{A}}$ is surjective on 1-morphisms since every 1-morphism $X_0 \in \mathcal{A}(A \to A')$ forms a path (X_0) of length 1.

For $X, Y \in \widehat{\mathcal{A}}(A \to A')$, we now define the space of 2-morphisms from X to Y to be

$$\widehat{\mathcal{A}}(X \Rightarrow Y) := \mathcal{A}(\mathrm{ev}_{\mathcal{A}}(X) \Rightarrow \mathrm{ev}_{\mathcal{A}}(Y)).$$

Notice that $\widehat{\mathcal{A}}$ inherits compositions, linear and dagger structures, and norms from $\mathcal{A}$, equipping each $\mathcal{A}(A \to A')$ with the structure of a C*-category. When $\mathcal{A}$ is a W*-2-category, we further have that each $\mathcal{A}(A \to A')$ is a W*-category.

By setting $\mathrm{ev}_{\mathcal{A}}$ to act as the identity on 2-morphisms, we immediately have that $\mathrm{ev}_{\mathcal{A}}$ is locally a fully-faithful †-functor.

For composable 1-morphisms $X = (X_n, \ldots, X_0)$ and $Y = (Y_m, \ldots, Y_0)$ in $\widehat{\mathcal{A}}$, we set the tensorator $\mathrm{ev}^2_{X,Y} : \mathrm{ev}_{\mathcal{A}}(X) \odot \mathrm{ev}_{\mathcal{A}}(Y) \Rightarrow \mathrm{ev}_{\mathcal{A}}(X \odot Y)$ to be the unique coherence unitary

$$\big((\cdots(X_0 \odot X_1)\cdots) \odot X_n\big) \odot \big((\cdots(Y_0 \odot Y_1)\cdots) \odot Y_m\big) \Rightarrow (\cdots((((\cdots(X_0 \odot X_1)\cdots) \odot X_n) \odot Y_0) \odot Y_1)\cdots) \odot Y_m,$$

given by coherence for 2-categories. The 1-composition $\widehat{\odot}$ of composable 2-morphisms $a \in \widehat{\mathcal{A}}(X \Rightarrow Y)$ and $a' \in \widehat{\mathcal{A}}(X' \Rightarrow Y')$ is given by the unique 2-morphisms in $a \,\widehat{\odot}\, a' \in$

$\widehat{\mathcal{A}}((X \odot X') \Rightarrow (Y \circ Y'))$ such that the following square commutes

$$\begin{tikzcd}
\mathrm{ev}_{\mathcal{A}}(X) \odot \mathrm{ev}_{\mathcal{A}}(Y) \arrow[r, "\mathrm{ev}^2_{X,Y}"] \arrow[d, "a \odot a'"'] & \mathrm{ev}_{\mathcal{A}}(X \odot Y) \arrow[d, dashed, "a \mathbin{\widehat{\odot}} a'"] \\
\mathrm{ev}_{\mathcal{A}}(X') \odot \mathrm{ev}_{\mathcal{A}}(Y') \arrow[r, "\mathrm{ev}^2_{X',Y'}"'] & \mathrm{ev}_{\mathcal{A}}(X' \odot Y')
\end{tikzcd}$$

This automatically implies that the tensorator $\mathrm{ev}^2_{X,Y}$ is natural in X and Y. By coherence for 2-categories it follows that $\mathrm{ev}_{\mathcal{A}}$ satisfies all coherence axioms for †-2-functors and $\widehat{\mathcal{A}}$ satisfies all coherence axioms for C*-2-categories. Finally, when $\mathcal{A}$ is a W*-2-category, we see that $\widehat{\odot}$ is separately normal since both $\circ$ and $\odot$ are separately normal in $\mathcal{A}$. In this case we conclude that $\widehat{\mathcal{A}}$ is also a W*-2-category and that $\mathrm{ev}_{\mathcal{A}} : \widehat{\mathcal{A}} \to \mathcal{A}$ is automatically normal as an equivalence between W*-2-categories. □

Proposition 1.3.5.2. *For each †-2-functor $F\colon \mathcal{A} \to \mathcal{B}$ between C*-2-categories, there exists a strict †-2-functor $\widehat{F}\colon \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$ and a unitary icon u^F as follows:*

$$\begin{tikzcd}
\widehat{\mathcal{A}} \arrow[r, "\widehat{F}"] \arrow[d, "\mathrm{ev}_{\mathcal{A}}"'] & \widehat{\mathcal{B}} \arrow[d, "\mathrm{ev}_{\mathcal{B}}"] \arrow[dl, Rightarrow, shorten=4mm, "u^F"'] \\
\mathcal{A}_1 \arrow[r, "F"'] & \mathcal{A}_2
\end{tikzcd}$$

Moreover, when F is a normal †-2-functor between W-2-categories, then $\widehat{F}$ is also normal.*

Proof. We provide the construction in [Gur13, §2.3.3], noting that all the relevant data is compatible with dagger structures. Let $F : \mathcal{A} \to \mathcal{B}$ be a †-2-functor between C*-2-categories. We define $\widehat{F}$ to act as F on objects, and on 1-morphisms by

$$\widehat{F}(A' \xrightarrow{X_n} A_n, \ldots, A_2 \xrightarrow{X_1} A_1, A_1 \xrightarrow{X_0} A) = (FA' \xrightarrow{FX_n} FA_n, \ldots, FA_2 \xrightarrow{FX_1} FA_1, FA_1 \xrightarrow{FX_0} FA)$$

In particular $\widehat{F}(\varnothing_A) = \varnothing_{FA}$, so F is strictly unital. From construction it is clear that $\widehat{F}$ is also strictly $\odot$-preserving. For a 2-morphism $a \in \widehat{A}(X \Rightarrow Y)$ we define $\widehat{F}(a) \in \widehat{\mathcal{B}}(\widehat{F}X \Rightarrow \widehat{F}Y)$ to be the unique 2-morphism in $\mathcal{A}$ such that the following square commutes.

$$\begin{tikzcd}
F((\cdots(X_0 \odot X_1)\cdots) \odot X_n) \arrow[r, "\sim"] \arrow[d, "Fa"'] & (\cdots(FX_0 \odot FX_1)\cdots) \odot FX_n \arrow[d, dashed, "\widehat{F}a"] \\
F((\cdots(Y_0 \odot Y_1)\cdots) \odot Y_m) \arrow[r, "\sim"'] & (\cdots(FY_0 \odot FY_1)\cdots) \odot FY_m
\end{tikzcd}$$

Here the horizontal morphisms are the unique constraint unitaries provided by the coherence theorem for 2-functors [Gur13, §2.3]. By this uniqueness it follows that $\widehat{F}$ satisfies the coherence axioms for a strict 2-functor. Moreover, $\widehat{F}$ is †-preserving since F is †-preserving and the constraints for F are unitary. From our definition of $\widehat{F}$ on 2-morphisms, it is also clear that $\widehat{F}$ is linear. When F is a normal †-2-functor between W*-2-categories, it follows that $\widehat{F}$ is normal since F is and 2-composition $\circ$ is separately normal.

Notice that $\mathrm{ev}_{\mathcal{B}} \circ \widehat{F}$ and $\mathrm{ev}_{\mathcal{A}} \circ F$ both act like F on objects, so they agree at the level of objects. For $X = (X_n, \dots, X_0) \in \widehat{\mathcal{A}}(A \to A')$, we define the unitary

$$u_X^F \in \mathcal{B}\big(\,\mathrm{ev}(FX_n, \dots, FX_0) \Rightarrow F(\mathrm{ev}(X_n, \dots, X_0))\big)$$

to be the unique constraint unitary provided by the coherence theorem for 2-functors. Note that u_X^F is natural in X since constraints are natural, and u^F satisfies the axioms of a unitary icon due to uniqueness of constraints. □

Proposition 1.3.5.3. *For C*-2-categories $\mathcal{A}_1$ and $\mathcal{A}_2$, there exists a cubical †-2-functor*

$$C \colon \widehat{\mathcal{A}}_1 \underset{max}{\otimes} \widehat{\mathcal{A}}_2 \to \widehat{\mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2}$$

which is the identity on objects, and a unitary icon u as follows:

$$\begin{tikzcd}
 & \widehat{\mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2} \arrow[dr, "\mathrm{ev}"] \arrow[d, phantom, "\overset{u^{\mathrm{ev}}}{\Downarrow}"] & \\
\widehat{\mathcal{A}}_1 \underset{max}{\otimes} \widehat{\mathcal{A}}_2 \arrow[ur, "C"] \arrow[rr, "\mathrm{ev} \underset{max}{\otimes} \mathrm{ev}"'] & & \mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2
\end{tikzcd}$$

Moreover, when $\mathcal{A}_1$ and $\mathcal{A}_2$ are W-2-categories, we may upgrade C to a separately normal cubical †-2-functor*

$$\overline{C} \colon \widehat{\mathcal{A}}_1 \underset{max}{\overline{\otimes}} \widehat{\mathcal{A}}_2 \to \widehat{\mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2}$$

and u to a unitary icon $\overline{u}$ as follows:

$$\begin{tikzcd}
 & \widehat{\mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2} \arrow[dr, "\mathrm{ev}"] \arrow[d, phantom, "\overline{u}^{\mathrm{ev}} \Downarrow"] & \\
\widehat{\mathcal{A}}_1 \underset{max}{\overline{\otimes}} \widehat{\mathcal{A}}_2 \arrow[ur, "\overline{C}"] \arrow[rr, "\mathrm{ev} \underset{max}{\overline{\otimes}} \mathrm{ev}"'] & & \mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2
\end{tikzcd}$$

Proof. We adapt [Gur13, §8.1, Prop. 8.5]. To define C, we will provide the equivalent data seen in Proposition 3.2.2.2. For objects $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ we define

$$C_{A_1}(A_2) = (A_1, A_2) = C_{A_2}(A_1).$$

Recall that we denote the identity of A in $\mathcal{A}_1$ by id_A, and its identity in $\widehat{\mathcal{A}_1}$ by $\varnothing_{A_1}$. For $X = (X_n, \dots, X_0) \in \widehat{\mathcal{A}}_2(A_2 \to A_2')$, we define

$$C_{A_1}(X) \coloneqq \big((\mathrm{id}_{A_1}, X_n), \dots, (\mathrm{id}_{A_1}, X_0)\big).$$

In particular $C_{A_1}(\varnothing_{A_2}) = \varnothing_{(A_1, A_2)}$, so C_{A_1} is strictly unital. It is clear from definition that C_{A_1} is also strictly $\odot$-preserving. For $k \geq 0$, let E_1^k be the 1-morphism in $\mathcal{A}_1$ given by

$$E_1^k \coloneqq \mathrm{ev}_{\mathcal{A}_1}(\underbrace{\mathrm{id}_{A_1}, \dots, \mathrm{id}_{\mathcal{A}_1}}_{k \text{ times}}).$$

For $a \in \widehat{\mathcal{A}}_2(X \Rightarrow Y)$ where $X = (X_n, \dots, X_0)$ and $Y = (Y_m, \dots, Y_0)$, we define the 2-morphism

$$C_{A_1}(a) \in \widehat{\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2}\big(C_{A_1}(X) \Rightarrow C_{A_1}(Y)\big)$$

as follows. First note that $C_{A_1}(a)$ must be a 2-morphism in $\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2$ with source $(E_1^{n+1}, \mathrm{ev}_{\mathcal{A}_1}(X))$ and target $(E_1^{m+1}, \mathrm{ev}_{\mathcal{A}_1}(Y))$. We may then define

$$C_{A_1}(a) \coloneqq \gamma_{n+1,m+1} \otimes a$$

where $\gamma_{n+1,m+1} \in \mathcal{A}_1(E_1^{n+1} \Rightarrow E_1^{m+1})$ is the unique constraint unitary given by coherence for 2-categories. From this definition, it is immediate that C_{A_1} is linear. Since $\upsilon^{\dagger}_{n+1,m+1} = \upsilon^{-1}_{n+1,m+1} = \upsilon_{m+1,n+1}$ by uniqueness of constraints, we conclude that C_{A_1} is a strict †-2-functor. When $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ are W*-2-categories and C is normal, we may identically define $\overline{C}_{A_1} \colon \mathcal{A}_2 \to \widehat{\mathcal{A}_1 \underset{\max}{\overline{\otimes}} \mathcal{A}_2}$, which is normal since $\otimes$ is separately normal in $\mathcal{A}_1 \underset{\max}{\overline{\otimes}} \mathcal{A}_2$.

One defines the (normal) strict †-2-functor F_{A_2} in a similar fashion. We now define the unitary $\Sigma_{X,Y}$ for 1-morphisms $X = (X_n, \dots, X_0) \in \widehat{\mathcal{A}}_1(A_1 \to A_1')$ and $Y = (Y_m, \dots, Y_0) \in \widehat{\mathcal{A}}_2(A_2 \to A_2')$ as follows.

$$\begin{tikzcd}
(A_1, A_2) \arrow[r, "C_{A_1}(Y)"] \arrow[d, "C_{A_2}(X)"'] & (A_1, A_2') \arrow[d, "C_{A_2'}(X)"] \arrow[dl, Rightarrow, "\Sigma_{X,Y}"'] \\
(A_1', A_2) \arrow[r, "C_{A_1'}(Y)"'] & (A_1', A_2')
\end{tikzcd}$$

First note that $\Sigma_{X,Y}$ must be a 2-morphism in $\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2$ with source

$$\mathrm{ev}_{\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2}\big((X_n, \mathrm{id}_{A_2'}), \dots, (X_0, \mathrm{id}_{A_2'}), (\mathrm{id}_{A_1}, Y_m), \dots, (\mathrm{id}_{A_1}, Y_0)\big)$$

and target

$$\mathrm{ev}_{\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2}\big((\mathrm{id}_{A_1'}, Y_m), \dots, (\mathrm{id}_{A_1'}, Y_0), (X_n, \mathrm{id}_{A_2}), \dots, (X_0, \mathrm{id}_{A_2})\big).$$

We may then define $\Sigma_{X,Y}$ to be the unique constraint unitary in $\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2$ given by coherence for 2-categories. Note that (Σ_1) follows by naturality of constraints, while (Σ_2) and (Σ_3) follows from their uniqueness. Hence, this data serves to determine a cubical †-2-functor

$$C \colon \widehat{\mathcal{A}}_1 \underset{\max}{\otimes} \widehat{\mathcal{A}}_2 \to \widehat{\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2},$$

and in the W* case, a separately normal cubical †-2-functor

$$\overline{C} \colon \widehat{\mathcal{A}}_1 \underset{\max}{\overline{\otimes}} \widehat{\mathcal{A}}_2 \to \widehat{\mathcal{A}_1 \underset{\max}{\overline{\otimes}} \mathcal{A}_2}.$$

Note that $\mathrm{ev} \circ C$ and $\mathrm{ev} \underset{\max}{\otimes} \mathrm{ev}$ act as the identity on objects, so they agree at the level of objects. For a 1-morphism (X,Y) in $\widehat{A}_1 \underset{\max}{\otimes} \widehat{A}_2$ where $X = (X_n, \dots, X_0)$ and $Y = (Y_m, \dots, Y_0)$ we define the unitary

$$u^{\mathrm{ev}}_{X,Y} \in (\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2)\Big(\mathrm{ev}\left(F_{A_1}(Y) \odot F_{A_2}(X)\right) \Rightarrow \left(\mathrm{ev}(X), \mathrm{ev}(Y)\right)\Big)$$

as follows. Notice $\mathrm{ev}\left(F_{A_1}(Y) \odot F_{A_2}(X)\right) = (E_1^{n+1} \odot \mathrm{ev}(X), \mathrm{ev}(Y) \odot E_2^{m+1})$ We may then define $u^{\mathrm{ev}}_{X,Y} := v_1 \otimes v_2$ for $v_1 \in \mathcal{A}_1(E_1^{n+1} \odot \mathrm{ev}(X) \Rightarrow \mathrm{ev}(X))$, $v_2 \in \mathcal{A}_2(\mathrm{ev}(Y) \odot E_2^{m+1} \Rightarrow \mathrm{ev}(X))$ to be the unique constraint unitaries given by coherence for 2-categories. We see that $u^{\mathrm{ev}}_{X,Y}$ is natural in X and Y since constraints are natural, and u satisfies the axioms for unitary icons by uniqueness of constraints. In the W* case, we define $\overline{u}^{\mathrm{ev}}$ identically to u^{ev}. □

1.4 Background on algebraic 3-categories

1.4.1 Algebraic 3-categories

Definition 1.4.1.1. A 3-category consists of the following data:

(0) A collection $\mathcal{A}$ of *objects* of $\mathcal{A}$;

(hom) For $A, B \in \mathcal{A}$, a 2-category $\mathcal{A}(A \to B)$ where:

- The objects of $\mathcal{A}(A \to B)$ are called *1-morphisms* of $\mathcal{A}$ with source A and target B,
- The arrows of $\mathcal{A}(A \to B)$ will be referred to as *2-morphisms* of $\mathcal{A}$ with their same source and target, and
- The 2-morphisms of $\mathcal{A}(A \to B)$ are called *3-morphisms* of $\mathcal{A}$, also with their same source and target.

($\circledcirc$) For $A, B, C \in \mathcal{A}$, a 2-functor $\circledcirc\colon \mathcal{A}(A \to B) \times \mathcal{A}(B \to C) \to \mathcal{A}(A \to C)$ called *1-composition*, which we represent in Figure 1.28.

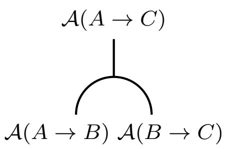

Figure 1.28: 2D string diagram for 1-composition in 3-categories

(I) For $A \in \mathcal{A}$, a 2-functor $I_A\colon * \to \mathcal{A}(A \to A)$ which we represent in Figure 1.29.

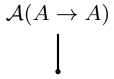

Figure 1.29: 2D string diagram for identities in 3-categories

(a) For $A, B, C, D \in \mathcal{A}$, an *"associator"* adjoint equivalence $\mathbf{a} = (a, a^{\bullet}, \epsilon^a, \eta^a)$ where a is a 2-natural transformation represented in Figure 1.30.

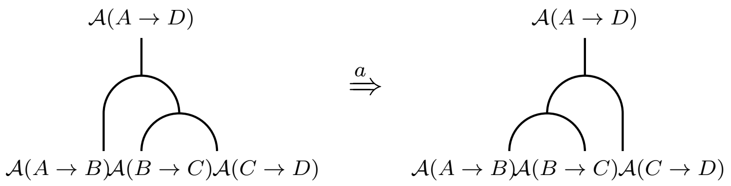

Figure 1.30: (2+1)D string diagram for associators in 3-categories

(u) For $A, B \in \mathcal{A}$, *left and right "unitor"* adjoint equivalences $\mathbf{l}$ and $\mathbf{r}$ where l and r are 2-natural transformations as in Figure 1.31.

Figure 1.31: $(2+1)$D string diagrams for left and right unitors in 3-categories

(π) For $A, B, C, D, E \in \mathcal{A}$, a *"pentagonator"* modification as in Figure 1.32.

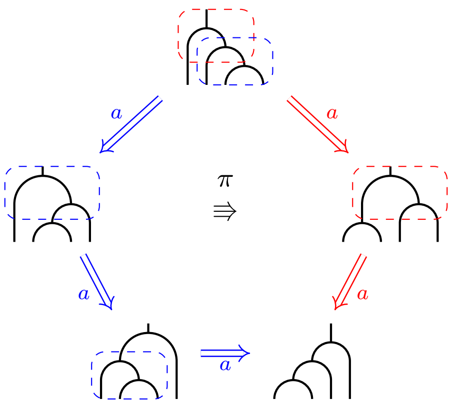

Figure 1.32: (2+2)D string diagram for pentagonators in 3-categories

(coh) For $A, B, C \in \mathcal{A}$, *middle, left, and right unity coheretor* modifications as in Figure 1.33.

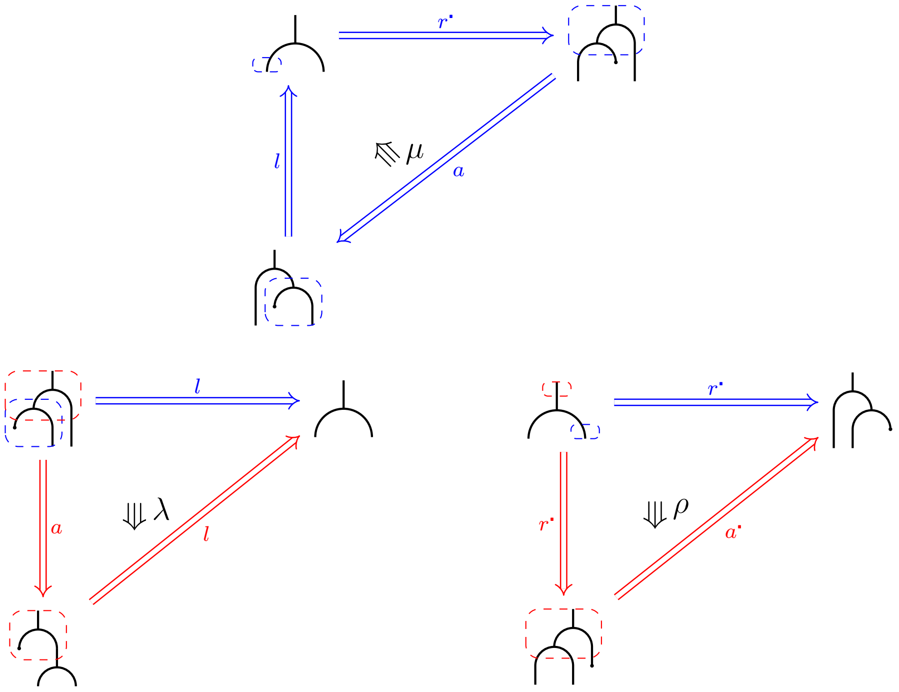

Figure 1.33: (2+2)D string diagrams for middle, left, and right unity coheretors in 3-categories

Furthermore, we require this data satisfies a 3-dimensional associahedron axiom for each tuple of 6 objects in $\mathcal{A}$ (non-abelian 4-cocycle condition) and two axioms for each quadruple of objects in $\mathcal{A}$ relating the unity coheretors with the associator and pentagonators (normalized cocycle conditions). We refer the interested reader to [Gur13, Definition 4.1].

Remark 1.4.1.2. As our coherence results in Chapter 3 will involve 3-categories of 3-functors between 3-categories, we will mainly use the convention that 1-composition $\circledcirc$, 2-composition $\odot$, and 3-composition $\circ$ are contravariant. It is however more common in the literature to have covariant 1-composition and 2-composition. Due to this, we will also use the notation $\boxtimes$ for the opposite convention of $\circledcirc$, and $\otimes$ for $\odot$ respectively whenever it is convenient.

Example 1.4.1.3. Given a topological space X, there is an associated algebraic 3-category $\Pi_3(X)$ of points, paths, homotopies, and homotopies between homotopies in X. In this 3-categories, all 3-morphisms are invertible, 2-morphisms are 2-equivalences, and morphisms are equivalences; i.e., $\Pi_3(X)$ is a 3-groupoid. The homotopy hypothesis states that the data of any such algebraic 3-groupoid is equivalent to that of a homotopy 3-type.

The previous example motivates defining homotopy groups in an arbitrary algebraic 3-category.

Definition 1.4.1.4. Let $\mathcal{A}$ be a 3-category and $A \in \mathcal{A}$.

1. $\pi_0(\mathcal{A})$ is defined to be the set of objects in $\mathcal{A}$ up up to equivalence.

2. $\pi_1(\mathcal{A}, A)$ is the group of invertible 1-morphisms $A \to A$ up 2-equivalence.

3. $\pi_2(\mathcal{A}, A)$ is the group of invertible 2-morphisms $\mathrm{id}_A \Rightarrow \mathrm{id}_A$ up to 3-isomorphism.

4. $\pi_3(\mathcal{A}, A)$ is the group of invertible 3-morphisms $\mathrm{id}_{\mathrm{id}_A} \Rrightarrow \mathrm{id}_{\mathrm{id}_A}$.

Definition 1.4.1.5. A 3-functor $F\colon \mathcal{A} \to \mathcal{A}'$ between 3-categories consists of the following data:

(0) A function from objects of $\mathcal{A}$ to objects of $\mathcal{A}'$;

(hom) For $A, B \in \mathcal{A}$, a 2-functor $F\colon \mathcal{A}(A,B) \to \mathcal{A}'(FA, FB)$, which we represent diagrammatically in Figure 1.34.

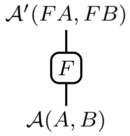

Figure 1.34: 2D string diagram for the action of 3-functors on hom-2-categories

(⊚) A "tensorator" adjoint equivalence $\mathbf{F^2}$ where F^2 is a 2-natural transformation as in Figure 1.35.

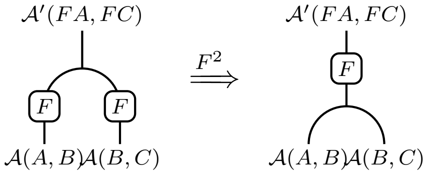

Figure 1.35: $(2+1)$D string diagram for tensorators of 3-functors

(I) A "unitor" adjoint equivalence $\mathbf{F^0}$ where F^0 is a 2-natural transformation as in Figure 1.36

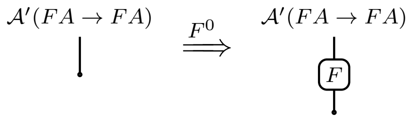

Figure 1.36: $(2+1)$D string diagram for unitors of 3-functors

(a) An invertible “associator” modification F^a as in Figure 1.37.

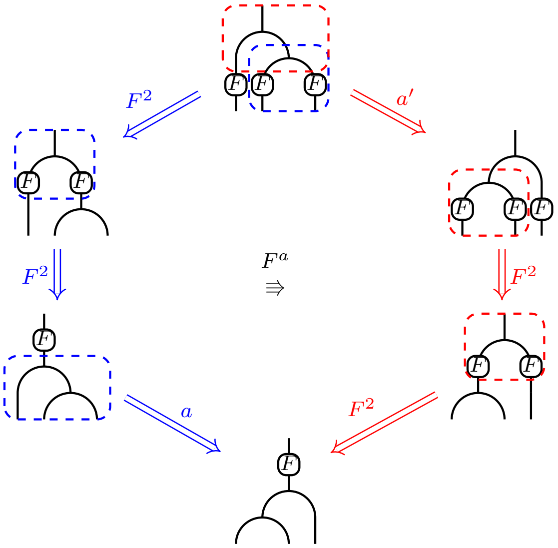

Figure 1.37: $(2+2)$D string diagram for associators (hexagonators) of 3-functors

(u) Invertible “left unitor” and “right unitor” modifications F^l and F^r as in Figure 1.38.

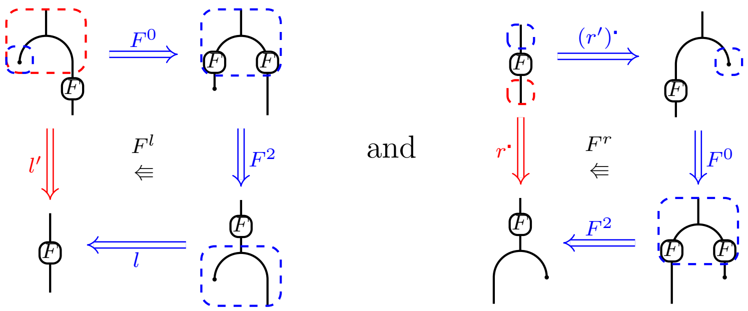

Figure 1.38: $(2+2)$D string diagrams for left and right unitors of 3-functors

Furthermore, we require this data satisfies a higher associativity (or "pentagonity") axiom relating the pentagonators in $\mathcal{A}$ and $\mathcal{A}'$, and a unitality axiom relating the middle unity coheretors in $\mathcal{A}$ and $\mathcal{A}'$. We refer the interested reader to [Gur13, Definition 4.10].

Definition 1.4.1.6. A 3-natural transformation $\theta\colon F \Rightarrow F'$ between 3-functors $F, F'\colon \mathcal{A} \to \mathcal{A}'$ consists of the following data:

(0) A family of 1-morphisms $\theta_A \in \mathcal{A}'(FA \to F'A)$ indexed by the objects of $\mathcal{A}$, which we represent diagrammatically in Diagram 1.39.

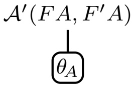

Figure 1.39: 2D string diagram for components of 3-natural transformations

(hom) For $A, B \in \mathcal{A}$, an adjoint equivalence $\boldsymbol{\theta}$ where θ is a 2-natural transformation as in Figure 1.40.

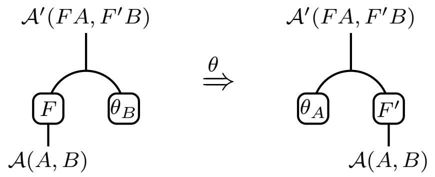

Figure 1.40: $(2+1)$D string diagram for naturators of 3-natural transformations

(◎) An invertible "tensorator" modification θ^2 as in Figure 1.41

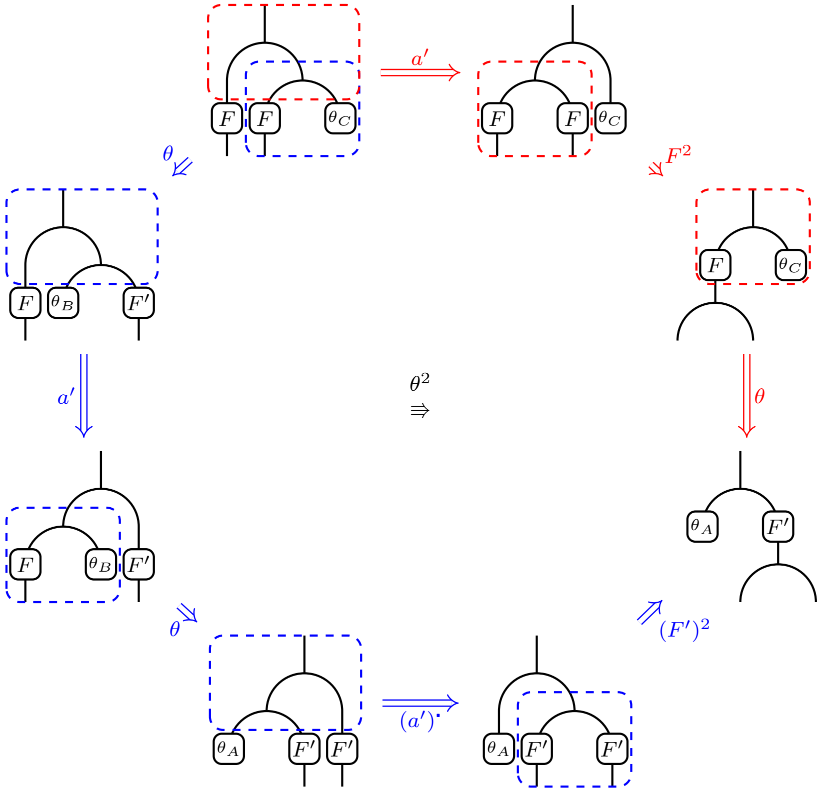

Figure 1.41: $(2+2)$D string diagram for tensorators of 3-natural transformations

(I) An invertible "unitor" modification θ^0 as in Figure 1.42.

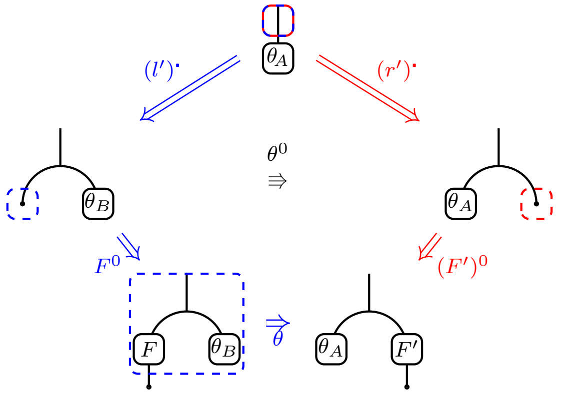

Figure 1.42: $(2+2)$D string diagram for unitors of 3-natural transformations

Furthermore, we require this data satisfies an associativity axiom relating F^a and $(F')^a$, and two unitality axiom relating F^l, $(F')^l$ and F^r, $(F')^r$ respectively. We refer the interested reader to [Gur13, Definition 4.16].

Definition 1.4.1.7. A 3-modification $m\colon \theta \Rrightarrow \theta'$ between 3-natural transformations $\theta, \theta'\colon F \Rightarrow F'$ consists of the following data:

(0) A family of 2-morphisms $m_A \in \mathcal{A}'(\theta_A \Rightarrow \theta'_A)$ indexed by the objects of $\mathcal{A}$, which we represent diagrammatically as in Figure 1.43.

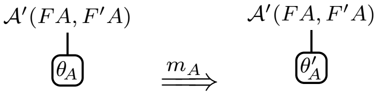

Figure 1.43: 2D string diagram for components of 3-modifications

(hom) For $A, B \in \mathcal{A}$, an invertible "naturator" modification m as in Figure 1.44.

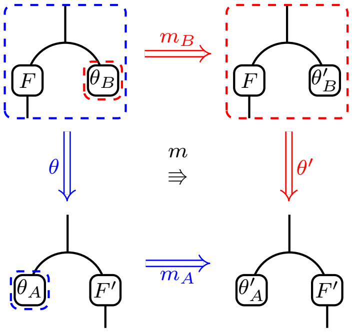

Figure 1.44: $(2 + 2)$D string diagram for naturators of 3-modifications

Furthermore, we require this data satisfies a monoidality/composability axiom and a unitality axiom. We refer the interested reader to [Gur13, Definition 4.18].

Definition 1.4.1.8. A perturbation $\sigma\colon m \Rrightarrow m'$ between 3-modifications consists of a family of 3-morphisms $\sigma_A \in \mathcal{A}'(m_a \Rrightarrow n_a)$ indexed by objects in $\mathcal{A}$, satisfying the obvious compatibility axiom with the higher data for m and n. We refer the interested reader to [Gur13, Definition 4.21].

We may summarize this subsection through the following tables.

3-category $\mathcal{A}$	3-functor F	3-nat. trans. θ	3-modification m	perturbation σ
hom 2-category $\mathcal{A}(A \to B)$	map of homs F	naturator θ	naturator θ	naturality axiom
1-composition $\circledcirc$	tensorator F^2	compositor θ^2	composability axiom	coherence theorem
associator a	associator F^a	associativity axiom	coherence theorem	coherence theorem
pentagonator π	pentagonity axiom	coherence theorem	coherence theorem	coherence theorem
3D associahedron axiom	coherence theorem	coherence theorem	coherence theorem	coherence theorem

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F</th><th colspan="12">3-nat. trans.
θ</th><th colspan="12">3-modification
m</th><th colspan="12">perturbation
σ</th></tr>
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F^0</td><td colspan="12">unitor
θ^0</td><td colspan="12">unitality
axiom</td><td colspan="12">coherence
theorem</td></tr>
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l</td><td colspan="6">right unitor
r</td><td colspan="6">left unitor
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F^r</td><td colspan="6">left unit.
axiom</td><td colspan="6">right unit.
axiom</td><td colspan="6">coh.
thm.</td><td colspan="6">coh.
thm.</td><td colspan="6"></td><td colspan="6"></td></tr>
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μ</td><td colspan="4">right unitor
ρ</td><td colspan="4">left unital
(c. thm.)</td><td colspan="4">mid. unital
axiom</td><td colspan="4">right unital
(c. thm.)</td><td colspan="4">coh.
thm.</td><td colspan="4">coh.
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thm.</td><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td></tr>
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axiom</td><td colspan="3">mid right
axiom</td><td colspan="3">far right
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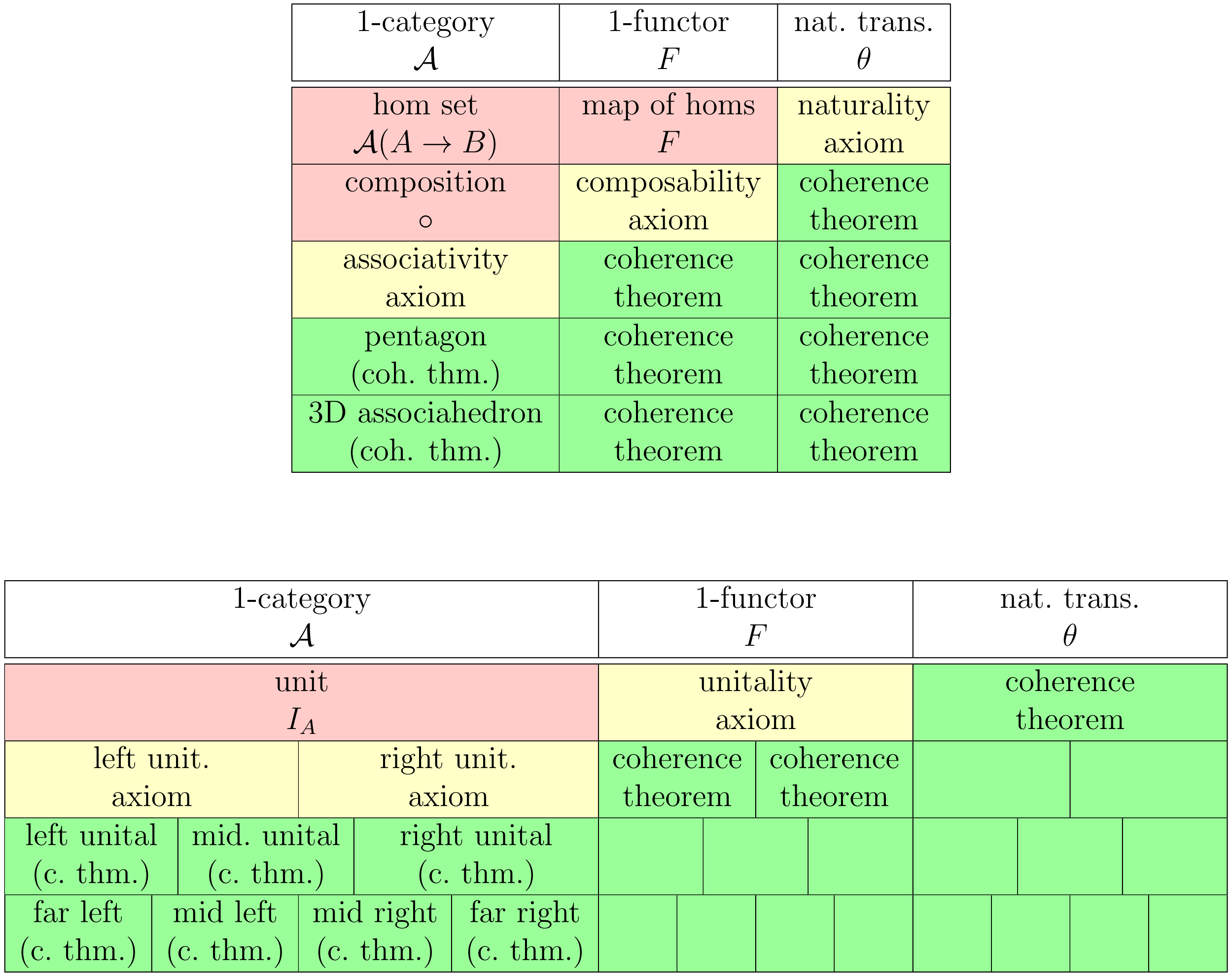
</table>

Table 1.1: Data and axioms of three-dimensional category theory

2-category $\mathcal{A}$	2-functor F	2-nat. trans. θ	modification m
hom category $\mathcal{A}(A \to B)$	map of homs F	naturator θ	naturality axiom
1-composition $\odot$	compositor F^2	composability axiom	coherence theorem
associator α	associativity axiom	coherence theorem	coherence theorem
pentagon axiom	coherence theorem	coherence theorem	coherence theorem
3D associahedron (coh. thm.)	coherence theorem	coherence theorem	coherence theorem

2-category $\mathcal{A}$	2-functor F	2-nat. trans. θ	modification m
unit I_A	unitor F^0	unitality axiom	coherence theorem
left unitor l right unitor r	left unit. axiom right unit. axiom	coh. thm. coh. thm.	🟩 🟩
left unital (c. thm.) mid. unital axiom right unital (c. thm.)	coh. thm. coh. thm. coh. thm.	🟩 🟩 🟩	🟩 🟩 🟩
far left (c. thm.) mid left (c. thm.) mid right (c. thm.) far right (c. thm.)	🟩 🟩 🟩 🟩	🟩 🟩 🟩 🟩	🟩 🟩 🟩 🟩

Table 1.2: Data and axioms of two-dimensional category theory



1-category $\mathcal{A}$	1-functor F	nat. trans. θ
hom set $\mathcal{A}(A \to B)$	map of homs F	naturality axiom
composition $\circ$	composability axiom	coherence theorem
associativity axiom	coherence theorem	coherence theorem
pentagon (coh. thm.)	coherence theorem	coherence theorem
3D associahedron (coh. thm.)	coherence theorem	coherence theorem

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$\mathcal{A}$</th><th colspan="12">1-functor
F</th><th colspan="12">nat. trans.
θ</th></tr>
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I_A</td><td colspan="12">unitality
axiom</td><td colspan="12">coherence
theorem</td></tr>
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axiom</td><td colspan="6">right unit.
axiom</td><td colspan="6">coherence
theorem</td><td colspan="6">coherence
theorem</td><td colspan="6"></td><td colspan="6"></td></tr>
<tr><td colspan="4">left unital
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(c. thm.)</td><td colspan="4">right unital
(c. thm.)</td><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td></tr>
<tr><td colspan="3">far left
(c. thm.)</td><td colspan="3">mid left
(c. thm.)</td><td colspan="3">mid right
(c. thm.)</td><td colspan="3">far right
(c. thm.)</td><td colspan="3"></td><td colspan="3"></td><td colspan="3"></td><td colspan="3"></td><td colspan="3"></td><td colspan="3"></td><td colspan="3"></td><td colspan="3"></td></tr>
</table>

Table 1.3: Data and axioms of one-dimensional category theory

1.4.2 Gray-categories

In this section we discuss the strictest version of algebraic 3-categories which still capture all examples (up to equivalence). Indeed, there are four levels of strictness for 3-categories:

- In a generic algebraic 3-category, each hom 2-category $\mathcal{A}(A \to B)$ need not be a strict 2-category. Moreover, the composition and unit 2-functors need not be strict 2-functors, and the associators, unitors, pentagonators, and unity coheretors need not be trivial.
- We say an algebraic 3-category is *cubical* if the hom 2-categories are strict, the unit 2-functors I_A are strict and 1-composition $\circledcirc$ is cubical, i.e. each $A \circledcirc -$, $- \circledcirc A$ is a strict 2-functor and the components of the tensorator for $\circledcirc$ are completely determined by 3-isomorphisms of the form:

  $$\Sigma_{X,Y} \colon (Y \circledcirc \mathrm{id}) \odot (\mathrm{id} \circledcirc X) \Rrightarrow (\mathrm{id} \circledcirc X) \odot (Y \circledcirc \mathrm{id}).$$

  These so-called interchangers $\Sigma_{X,Y}$ are stricly unital and monoidal in X and Y. The associators, unitors, pentagonators, and unity coheretors need not be trivial in a cubical category.
- Gray-categories, as we will define below, have strict hom 2-categories, strict units I_A, cubical 1-composition $\circledcirc$, and furthermore the associators, unitors, pentagonators, and unity coheretors are trivial. It is a result from Gordon, Powell, and Street that every algebraic 3-category is equivalent to a Gray-category [GPS95].
- An algebraic 3-category is called *strict* if each hom 2-category $\mathcal{A}(A \to B)$ is strict 2-category, the unit functors are strict, 1-composition $\circledcirc$ is cubical, and the associators, unitors, pentagonators, and unity coheretors are trivial. Not every algebraic 3-category

is equivalent to a strict one. For example, any braided category which is *not* symmetric gives rise to an algebraic 3-category that cannot be strictified.

We summarize this discussion in the following table.

	algebraic 3-cats	cubical 3-cats	Gray-cats	strict 3-cats
hom 2-cats	weak	strict	strict	strict
units	weak	strict	strict	strict
composition	weak	cubical	cubical	strict
constraints	nontrivial	nontrivial	trivial	trivial

Table 1.4: Degrees of strictness for 3-categories

Let 2Cat be the category of strict 2-categories and strict 2-functors. The symmetric monoidal category Gray is the category 2Cat equipped with the Gray monoidal structure [Gur13]. A Gray-category is then a category enriched in Gray in the sense of [Kel05]. We denote the 1-category of Gray-categories and Gray-functors by GrayCat.

We will concretely present the data and axioms of a Gray-category in order to fix notation for §1.4.3 and §1.4.4, using the covariant conventions $\boxtimes$ and $\otimes$ for 1-composition and 2-composition respectively.

Notation 1.4.2.1. A Gray-category $\mathcal{C}$ consists of the following data:

(D0) a collection of objects $\mathcal{C}_0$, denoted by lower case letters a, b, c

(D1) for $a, b \in \mathcal{C}_0$, a strict 2-category $\mathcal{C}(a, b)$ where we write $f\colon a \to b$ whenever $f \in \mathcal{C}(a, b)$, composition of 1-morphisms (called 2-morphisms in $\mathcal{C}$) is denoted by $\otimes$, and composition of 2-morphisms (called 3-morphisms in $\mathcal{C}$) is denoted by $\circ$;

(D2) for each $a \in \mathcal{C}_0$, an *identity* $\mathrm{id}_a\colon a \to a$;

(D3) for objects $a, b, c, d \in \mathcal{C}_0$ and 1-morphisms $g : c \to d$, $f \colon a \to b$, strict covariant and contravariant *hom-2-functors* $g_* = g \boxtimes -$ and $f^* = - \boxtimes f$:

$$g_* \colon \mathcal{C}(b, c) \to \mathcal{C}(b, d) \quad \text{and} \quad f^* \colon \mathcal{C}(b, c) \to \mathcal{C}(a, c),$$

(D4) an *interchanger* 3-isomorphism $\Sigma_{\gamma,\xi}$ for each pair of "horizontally composable" 2-morphisms $\xi \colon g \Rightarrow g'$ with $g, g' \colon b \to c$ and $\gamma : f \Rightarrow f'$ with $f, f' : a \to b$:

$$\Sigma_{\xi,\gamma} \colon\ (\xi \boxtimes \mathrm{id}_{f'}) \otimes (\mathrm{id}_g \boxtimes \gamma) \Rrightarrow (\mathrm{id}_{g'} \boxtimes \gamma) \otimes (\xi \boxtimes \mathrm{id}_f)$$

subject to the following conditions:

(C1) for composable 1-morphisms $g : b \to c$ and $f \colon a \to b$,

$$g_* f = f^* g = g \boxtimes f;$$

(C2) $\boxtimes$ is strictly unital and associative, i.e., the following hold whenever they make sense:

$$\begin{aligned} (\mathrm{id}_b)_* = \mathrm{id}_{\mathcal{C}(a,b)} &= (\mathrm{id}_a)^* \\ g_* f_* &= (g \boxtimes f)_* \\ f^* g^* &= (g \boxtimes f)^* \\ g_* f^* &= f^* g_*; \end{aligned}$$

(C3) the interchanger Σ respects identities, i.e., for a 1-morphism $f \colon b \to c$ and 2-morphisms ξ, γ, the following hold whenever they make sense:

$$\Sigma_{\xi,\mathrm{id}_f} = \mathrm{id}_{\xi \boxtimes f} \qquad \text{and} \qquad \Sigma_{\mathrm{id}_f,\gamma} = \mathrm{id}_{f \boxtimes \gamma}$$

(C4) the interchanger Σ respects $\otimes$, i.e., for $g \overset{\xi}{\Rightarrow} g' \overset{\xi'}{\Longrightarrow} g''$ and $f \overset{\gamma}{\Rightarrow} f' \overset{\gamma'}{\Longrightarrow} f''$, the following hold whenever they make sense:

$$\Sigma_{\xi'\otimes\xi,\gamma} = (\Sigma_{\xi',\gamma} \otimes (\xi \boxtimes f)) \circ ((\xi' \boxtimes f') \otimes \Sigma_{\xi,\gamma})$$

$$\Sigma_{\xi,\gamma'\otimes\gamma} = ((g' \boxtimes \gamma') \otimes \Sigma_{\xi,\gamma}) \circ (\Sigma_{\xi,\gamma'} \otimes (g \boxtimes \gamma))$$

(C5) the interchanger Σ is natural, i.e., for $g, g'\colon b \to c$, $\xi, \xi'\colon g \Rightarrow g'$ and $\Xi\colon \xi \Rrightarrow \xi'$; and $f, f'\colon a \to b$, $\gamma, \gamma'\colon f \Rightarrow f'$ and $\Gamma : \gamma \Rrightarrow \gamma'$:

$$\Sigma_{\xi',\gamma} \circ ((\Xi \boxtimes f') \otimes \mathrm{id}_{g\boxtimes\gamma}) = (\mathrm{id}_{g'\boxtimes\gamma} \otimes (\Xi \boxtimes f)) \circ \Sigma_{\xi,\gamma}$$

$$\Sigma_{\xi,\gamma'} \circ (\mathrm{id}_{\xi\boxtimes f'} \otimes (g \boxtimes \Gamma)) = ((g' \boxtimes \Gamma) \otimes \mathrm{id}_{\xi\boxtimes f}) \circ \Sigma_{\xi,\gamma}$$

(C6) the interchanger Σ respects $\boxtimes$, i.e., for 1-morphisms f, g, h and 2-morphisms σ, ξ, γ, the following hold whenever they make sense:

$$\Sigma_{h\boxtimes\xi,\gamma} = h \boxtimes \Sigma_{\xi,\gamma} \qquad \Sigma_{\sigma\boxtimes g,\gamma} = \Sigma_{\sigma,g\boxtimes\gamma} \qquad \Sigma_{\sigma,\xi\boxtimes f} = \Sigma_{\sigma,\xi} \boxtimes f$$

1.4.3 Gurski's Gr construction

We recall [Gur13, Def. 10.7] specialized to the case of a Gray-category.

Construction 1. *The* Gray*-category* $\mathsf{Gr}\,\mathcal{A}$ *of a* Gray*-category* $\mathcal{A}$ *is constructed as follows.*

(Gr*0)* $\mathsf{Gr}\,\mathcal{A}$ *has the same objects as* $\mathcal{A}$*, i.e.* $(\mathsf{Gr}\,\mathcal{A})_0 = \mathcal{A}_0$.

(Gr*1) For* $a, b \in \mathcal{A}_0$*, the objects in the 2-category* $(\mathsf{Gr}\,\mathcal{A})(a, b)$ *are (finite) strings* $\{f_i\}$ *of composable 1-cells of* $\mathcal{A}$ *(starting at* a *and ending at* b*). In particular, such a string* $\{f_i\}_{i=1}^{n}$ *takes order and multiplicity into account, thereby representing a path in* $\mathcal{A}$*:*

$$a \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} x_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} b.$$

For composable lists $\{g_j\}_{j=1}^{n_2}\colon b \to c$ *and* $\{f_i\}_{i=1}^{n_1}\colon a \to b$*, we define* $\{g_j\} \boxtimes \{f_i\}$ *to be their concatenation* $\{f_i, g_j\}$*, which represents the following path:*

$$a \xrightarrow{f_1} \cdots \xrightarrow{f_{n_1}} b \xrightarrow{g_1} \cdots \xrightarrow{g_{n_2}} c.$$

Notice the identity for an object $a \in \mathsf{Gr}\,\mathcal{A}$ *will be the empty string* $\varnothing_a$ *starting and ending in* a*.*

(Gr*2)* *A morphism* $\overline{\alpha}$ *in* $(\mathsf{Gr}\,\mathcal{A})(a,b)$ *consists of a composable string* $(\overline{\alpha}_n, \dots, \overline{\alpha}_1)$ *of generator morphisms* $\overline{\alpha_k}\colon \{f_i\}_{i=1}^{n_1} \to \{g_j\}_{j=1}^{n_2}$ *which themselves consist of:*

(a) Three numbers k, ℓ_1, ℓ_2 *with* $k \leq \ell_i$ *such that*

- *If* $m < k$*, then* $f_m = g_m$ *and*
- *If* $m > 0$*, then* $f_{\ell_1+m} = g_{\ell_2+m}$ *if either side exists. (so* $n_1 - \ell_1 = n_2 - \ell_2$*)*

(b) A pair (σ, τ) *where* $\sigma = (\sigma, D)$ *and* $\tau = (\tau, E)$ *are so-called associations for* $\{f_i\}_{i=k}^{\ell_1}$ *and* $\{g_j\}_{j=k}^{\ell_2}$ *respectively. Since* $\mathcal{A} \in \mathsf{GrayCat}$*, the composition* $\boxtimes$ *of 1-morphisms in* $\mathcal{A}$ *is associative, so this data is superfluous and we will not present more detail.*

(c) A 2-morphism between the "evaluated associations" for $\{f_i\}_{i=k}^{\ell_1}$ *and* $\{g_j\}_{j=k}^{\ell_2}$*. In our case, this amounts to a 2-morphisms* $\alpha\colon f_{\ell_1} \boxtimes \cdots \boxtimes f_k \Rightarrow g_{\ell_2} \boxtimes \cdots \boxtimes g_k$ *in* $\mathcal{A}$*.*

The composition $\otimes$ *of* 1*-morphisms in* $(\mathsf{Gr}\,T)(a,b)$ *is given by concatenation, so the empty 1-morphism* $\varnothing_{\{f_i\}}$ *is the identity on* $\{f_i\}$*.*

For a basic 2-morphism $\overline{\alpha}$ *and a 1-morphism* $\{h_k\}_{k=1}^{n_3}$ *in* $\mathsf{Gr}\,\mathcal{A}$ *we define their composites* $\overline{\alpha} \boxtimes \{h_k\}\colon \{h_k, f_i\} \Rightarrow \{h_k, g_j\}$ *and* $\{h_k\} \boxtimes \overline{\alpha}\colon \{f_i, h_k\} \Rightarrow \{g_j, h_k\}$ *(whenever they make sense) by*

$$\overline{\alpha} \boxtimes \{h_k\} \coloneqq (k + n_3, \ell_1 + n_3, \ell_2 + n_3, \sigma, \tau, \alpha).$$

$$\{h_k\} \boxtimes \overline{\alpha} \coloneqq (k, \ell_1, \ell_2, \sigma, \tau, \alpha).$$

We extend $\boxtimes$ *for arbitrary 2-morphisms* $\overline{\alpha} = (\overline{\alpha}_n, \dots, \overline{\alpha}_1)$ *by*

$$\overline{\alpha} \boxtimes \{h_k\} \coloneqq (\overline{\alpha}_n \boxtimes \{h_k\}, \dots, \overline{\alpha}_1 \boxtimes \{h_k\}),$$

$$\{h_k\} \boxtimes \overline{\alpha} \coloneqq (\{h_k\} \boxtimes \overline{\alpha}_n, \dots, \{h_k\} \boxtimes \overline{\alpha}_1).$$

With the data presented thus far, $\mathsf{Gr}\,\mathcal{A}$ *forms a sesquicategory which is free on a computad. Once we finish constructing the* Gray*-category* $\mathsf{Gr}\,\mathcal{A}$*, [Lac11, Corollary 9.4] will yield that* $\mathsf{Gr}\,\mathcal{A}$ *is cofibrant.*

(Gr*3) For generator 1-morphisms* $\overline{\alpha}, \overline{\beta}$ *in* $(\mathsf{Gr}\,\mathcal{A})(a,b)$*, a 2-morphism* $\Gamma\colon \overline{\alpha} \Rightarrow \overline{\beta}$ *in* $(\mathsf{Gr}\,\mathcal{A})(a,b)$ *is simply a 3-morphism* $\Gamma\colon [\alpha] \Rrightarrow [\beta]$ *in* $\mathcal{A}$ *where* $[\alpha], [\beta]\colon f_{n_1} \boxtimes \dots \boxtimes f_1 \Rightarrow g_{n_2} \boxtimes \dots \boxtimes g_1$ *are given by*

$$[\alpha] \coloneqq f_{n_1} \boxtimes \dots \boxtimes f_{\ell_1+1} \boxtimes \alpha \boxtimes f_{k-1} \boxtimes \dots \boxtimes f_1,$$

$$[\beta] \coloneqq g_{n_2} \boxtimes \dots \boxtimes g_{\ell_2+1} \boxtimes \beta \boxtimes g_{k-1} \boxtimes \dots \boxtimes g_1.$$

For general 1-morphisms $\overline{\alpha} = \overline{\alpha}_n \otimes \dots \otimes \overline{\alpha}_1$ *and* $\overline{\beta} = \overline{\beta}_{n'} \otimes \dots \otimes \overline{\beta}_1$ *in* $\mathsf{Gr}\,\mathcal{A}(a,b)_1$*,* $\Gamma\colon \overline{\alpha} \Rightarrow \overline{\beta}$ *is simply a 3-morphism* $\Gamma\colon [\alpha_n] \otimes \dots \otimes [\alpha_1] \Rrightarrow [\beta_{n'}] \otimes \dots \otimes [\beta_1]$ *in* $\mathcal{A}$*.*

The vertical composition $\circ$ *of 2-morphisms in* $(\mathsf{Gr}\,\mathcal{A})(a,b)$ *is inherited from* $\mathcal{A}$ *and is thus strictly associative and unital. The horizontal composition* $\otimes$ *of 2-morphisms in* $(\mathsf{Gr}\,\mathcal{A})(a,b)$ *is also inherited from* T*, so this composition satisfies the interchange law and is strictly associative. Thus* $\mathsf{Gr}\,T(a,b)$ *is actually a strict 2-category.*

For a 1-morphism $\{h_k\}$ *in* $\mathsf{Gr}\,\mathcal{A}$ *we define* $\{h_k\} \boxtimes \Gamma\colon \{h_k\} \boxtimes \overline{\alpha} \Rrightarrow \{h_k\} \boxtimes \overline{\beta}$ *(whenever it makes sense) by*

$$(\boxtimes_k h_k) \boxtimes ([\alpha_n] \otimes \dots \otimes [\alpha_1]) \xRightarrow{(\boxtimes_k h_k) \boxtimes \Gamma} (\boxtimes_k h_k) \boxtimes ([\beta_{n'}] \otimes \dots \otimes [\beta_1]).$$

We similarly define $\Gamma \boxtimes \{h_k\}$*.*

(GrΣ) *For basic 2-morphisms $\overline{\alpha}$ and $\overline{\beta}$ in $\mathsf{Gr}\,\mathcal{A}$ we may define the interchanger $\Sigma_{\overline{\alpha},\overline{\beta}}$ (whenever it makes sense) to be the interchanger $\Sigma_{[\alpha],[\beta]}$ in $\mathcal{A}$. One then extends Σ to general 2-morphisms in $\mathsf{Gr}\,\mathcal{A}$ by the same formula in (C4).*

By [Gur13, Thm. 10.8], this data serves to equip $\mathsf{Gr}\,\mathcal{A}$ with the structure of a Gray-category. We continue recalling [Gur13, Def. 10.7] and [Gur13, Thm. 10.9]:

Construction 2. *We define the Gray-functor $\mathrm{ev}_{\mathcal{A}}\colon \mathsf{Gr}\,\mathcal{A} \to \mathcal{A}$ as*

(ev0) On objects, $\mathrm{ev}_{\mathcal{A}}$ is the identity.

(ev1) On 1-morphisms,

$$\mathrm{ev}(\{f_i\}) = [f_i] \coloneqq f_n \boxtimes \cdots \boxtimes f_1$$

where $[\quad] = I_a$ for the empty 1-cell $\varnothing_a\colon a \to a$. We will also use the notation $[b_j, a_i] \coloneqq [b_j] \boxtimes [a_i]$.

(ev2) For a basic 2-morphism $\overline{\alpha}\colon \{f_i\} \Rightarrow \{g_j\}$, we define $\mathrm{ev}_{\mathcal{A}}(\overline{\alpha}) = [\alpha]$ where we are using the same notation as in Construction 1. We then extend $\mathrm{ev}_{\mathcal{A}}$ to general 2-morphisms in $\mathcal{A}$ by

$$\mathrm{ev}(\overline{\alpha}_n \otimes \cdots \otimes \overline{\alpha}_1) \coloneqq \mathrm{ev}(\overline{\alpha}_n) \otimes \cdots \otimes \mathrm{ev}(\overline{\alpha}_1).$$

and set $\mathrm{ev}(\varnothing_{\{f_i\}}) \coloneqq \mathrm{id}_{[f_i]}$.

(ev3) On 3-morphisms, $\mathrm{ev}_{\mathcal{A}}$ is the identity.

Remark 1.4.3.1. By construction $\mathrm{ev}_{\mathcal{A}}$ is surjective at all levels and (fully) faithful at the top level, i.e., $\mathrm{ev}_{\mathcal{A}}$ is a trivial fibration in Lack's model structure for $\mathsf{GrayCat}$. So $\mathsf{Gr}\,\mathcal{A} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A}$ is indeed a cofibrant replacement for $\mathcal{A}$ in $\mathsf{GrayCat}$.

We now recall [Gur13, §10.6].

Construction 3. *For a weak 3-functor $F\colon \mathcal{A} \to \mathcal{B}$ between* Gray*-categories, we define the* Gray*-functor* $\mathsf{Gr}\, F\colon \mathsf{Gr}\,\mathcal{A} \to \mathsf{Gr}\,\mathcal{B}$ *together with an equivalence pseudo-icon* (φ, M, Π) *where*

$$\begin{tikzcd}
\mathsf{Gr}\,\mathcal{A} \arrow[r, "\mathsf{Gr}\,F"] \arrow[d, "\mathrm{ev}_{\mathcal{A}}"'] \arrow[dr, phantom, "\varphi \Downarrow"] & \mathsf{Gr}\,\mathcal{B} \arrow[d, "\mathrm{ev}_{\mathcal{B}}"] \\
\mathcal{A} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}$$

($\mathsf{Gr}\,F0$) $\mathsf{Gr}\,F(s) = F(s)$ *for every object* $s \in S$.

($\mathsf{Gr}\,F1$) $\mathsf{Gr}\,F(\{f_i\}) = \{Ff_i\}$ *for a string* $\{f_i\} \in \mathsf{Gr}\,S$.

($\varphi 1$) *Let* $\{f_i\}_{i=1}^n$ *be a 1-morphism in* $\mathsf{Gr}\,\mathcal{A}$. *If* $n = 0$, *so that* $\{f_i\} = \varnothing_a$, *we define the equivalence* $\varphi_{\{f_i\}}\colon I_{F(a)} \Rightarrow F(\varnothing_a)$ *to be the unitor* $\varphi_{\{f_i\}} \coloneqq F_a^0$ *of* F. *When* $n = 1$, *we define* $\varphi_{\{f_i\}}\colon F(f_1) \Rightarrow F(f_1)$ *to be the identity* $\varphi_{\{f_i\}} \coloneqq \mathrm{id}_{F(f_1)}$. *When* $n > 2$, *we define* $\varphi_{\{f_i\}}\colon Ff_n \boxtimes \cdots \boxtimes Ff_1 \Rightarrow F\big(f_n \boxtimes \cdots \boxtimes f_1\big)$ *to be the leftmost composition of tensorators* F^2 *of* F *tensored with identities*

$$F^2_{f_n \boxtimes \cdots \boxtimes f_2, f_1} \otimes \cdots \otimes (F^2_{f_n \boxtimes f_{n-1}, f_{n-2}} \boxtimes Ff_{n-3} \boxtimes \cdots \boxtimes Ff_1) \otimes (F^2_{f_n, f_{n-1}} \boxtimes Ff_{n-2} \boxtimes \cdots \boxtimes Ff_1).$$

We also choose the obvious adjoint $\varphi^{\bullet}$, *unit, and counit. For simplicity, we will continue to denote this n-fold tensor by* $[\,\cdot\,]$, *so that*

$$\varphi_{\{f_i\}}\colon [Ff_i] \Rightarrow F[f_i].$$

($\mathsf{Gr}\,F2$) *For a basic* $\overline{\alpha} = (k, \ell_1, \ell_2, (\sigma, D), (\tau, E), \alpha)$, *we define*

$$\mathsf{Gr}\,F(\overline{\alpha}) = (k, \ell_1, \ell_2, (\sigma, FD), (\tau, FE), \mathsf{Gr}\,F\alpha).$$

where we recall the data of (σ, FD) *and* (τ, FE) *is superfluous and define* $\mathsf{Gr}\,F\alpha$ *so that the following diagram commutes.*

$$\begin{tikzcd}
{[Ff_i]} \arrow[r, Rightarrow, "\mathsf{Gr}\, F\alpha"] \arrow[d, Rightarrow, "{[\varphi_{\{f_{\ell_1 \geq i \geq k}\}}]}"'] & {[Fg_j]} \\
{[Ff_{i>\ell_1}, F[f_{\ell_1 \geq i \geq k}], Ff_{k>i}]} \arrow[r, Rightarrow, "{[F\alpha]}"'] & {[Fg_{j>\ell_2}, F[g_{\ell_2 \geq j \geq k}], Fg_{k>j}]} \arrow[u, Rightarrow, "{[\varphi^{\bullet}_{\{g_{\ell_2 \geq j \geq k}\}}]}"']
\end{tikzcd}$$

We extend this for general 2-morphisms by $\mathsf{Gr}\, F(\overline{\alpha}_n \otimes \ldots \otimes \overline{\alpha}_1) \coloneqq \mathsf{Gr}\, F(\overline{\alpha}_n) \otimes \ldots \otimes \mathsf{Gr}\, F(\overline{\alpha}_1)$.

($\varphi 2$) *For a basic 2-morphism* $\overline{\alpha}$ *in* $\mathsf{Gr}\,\mathcal{A}$, *we again denote* $f_{n_1} \boxtimes \cdots \boxtimes f_{\ell_1} \boxtimes \alpha \boxtimes f_{k-1} \boxtimes \cdots \boxtimes f_1$ *by* $[\alpha]$. *We then define the naturality isomorphism* $\varphi_{\overline{\alpha}}$ *to be the following composite of adjoint equivalence data from* F

$$\begin{tikzcd}
{[Ff_i]} \arrow[rr, Rightarrow, "{[\varphi_{\{f_{\ell_1 \geq i \geq k}\}}]}"] \arrow[d, Rightarrow, "\varphi_{\{f_i\}}"'] \arrow[dr, phantom, "\cong", near start] & & {[Ff_{i>\ell_1}, F[f_{\ell_1 \geq i \geq k}], Ff_{k>i}]} \arrow[d, Rightarrow, "{[F\alpha]}"] \arrow[dll, Rightarrow, "{\varphi_{\{f_{i>\ell_1}, [f_{\ell_1 \geq i \geq k}], f_{k>i}\}}}" description] \\
{F[f_i]} \arrow[d, Rightarrow, "{F[\alpha]}"'] & \cong & {[Fg_{j>\ell_2}, F[g_{\ell_2 \geq j \geq k}], Fg_{k>j}]} \arrow[d, Rightarrow, "{[\varphi^{\bullet}_{\{g_{\ell_2 \geq j \geq k}\}}]}"] \arrow[dll, Rightarrow, "{\varphi_{\{g_{j>\ell_2}, [g_{\ell_2 \geq j \geq k}], g_{k>j}\}}}" description] \arrow[d, phantom, "\cong"', shift right=12] \\
{F[g_j]} & & {[Fg_j]} \arrow[ll, Rightarrow, "\varphi_{\{g_j\}}"]
\end{tikzcd}$$

We then extend this to general 2-morphisms in $\mathsf{Gr}\,\mathcal{A}$ *as usual.*

(M) *For every object* $a \in \mathsf{Gr}\,\mathcal{A}_0 = \mathcal{A}_0$, *we define the invertible 3-cell* M_a

$$\begin{tikzcd}
\mathrm{id}_{Fa} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "M_a \Downarrow"] & \mathrm{id}_{Fa} \arrow[d, Rightarrow, "\varphi_{\varnothing_a}"] \\
\mathrm{id}_{Fa} \arrow[r, Rightarrow, "F^0_a"'] & F(\varnothing_a)
\end{tikzcd}$$

to be the identity $\mathrm{id}_{F^0_a}$.

(Π) *We define the modification* Π *to be the unique coherence isomorphism given by* F. *In particular,* Π *has component invertible 3-cells* $\Pi_{\{f_i\},\{g_j\}}$ *for* $a \xrightarrow{\{f_i\}} b \xrightarrow{\{g_j\}} c$ *in* $\mathsf{Gr}\,\mathcal{A}$

$$\begin{tikzcd}
{[Fg_j] \boxtimes [Ff_i]} \arrow[rr, equal] \arrow[d, Rightarrow, "\varphi_{\{g_j\}} \boxtimes \varphi_{\{f_i\}}"'] & \Pi_{f,g} \Downarrow & {[Ff_i, Fg_j]} \arrow[d, Rightarrow, "\varphi_{\{g_j\} \boxtimes \{f_i\}}"] \\
{F[g_j] \boxtimes F[f_i]} \arrow[rr, Rightarrow, "F^2_{[f_i],[g_j]}"'] & & {F[f_i, g_j]}
\end{tikzcd}$$

($\mathsf{Gr}\,F3$) *For a 3-morphism* $\Gamma\colon \overline{\alpha} \Rrightarrow \overline{\beta}$ *in* $\mathsf{Gr}\,\mathcal{A}$ *we define* $\mathsf{Gr}\,F(\Gamma)\colon \mathsf{Gr}\,F(\overline{\alpha}) \Rrightarrow \mathsf{Gr}\,F(\overline{\beta})$ *so that the following diagram commutes*

$$\begin{tikzcd}
{[\varphi^{\bullet}_{\{f_{\ell_2 \geq j \geq k}\}}] \otimes [F\alpha] \otimes [\varphi_{\{f_{\ell_1 \geq i \geq k}\}}]} \arrow[r, "\mathsf{Gr}\,F(\Gamma)"] \arrow[d, "\phi_{\overline{\alpha}}"'] & {[\varphi^{\bullet}_{\{f_{\ell_2 \geq j \geq k}\}}] \otimes [F\beta] \otimes [\varphi_{\{f_{\ell_1 \geq i \geq k}\}}]} \\
\varphi^{\bullet}_{\{g_j\}} \otimes F[\alpha] \otimes \varphi_{\{f_i\}} \arrow[r, "\mathrm{id} \otimes F\Gamma \otimes \mathrm{id}"'] & \varphi^{\bullet}_{\{g_j\}} \otimes F[\beta] \otimes \varphi_{\{f_i\}} \arrow[u, "\phi^{\bullet}_{\overline{\beta}}"']
\end{tikzcd}$$

where the 3-isomorphisms $\phi_{\overline{\alpha}}$ *and* $\phi^{\bullet}_{\overline{\beta}}$ *are similar to* $\varphi_{\overline{\alpha}}$ *and* $\varphi^{\bullet}_{\overline{\beta}}$ *in* ($\varphi 2$).

Properties 1. *We now review the properties outlined in Proposition 2.2.0.1.*

$\widehat{1}$ *The fact that* (φ, M, Π) *forms a pseudo-icon follows from the coherence theorem for weak 3-functors. We refer the interested reader to [Gur13, Thm. 10.13] and its corollary [Gur13, Cor. 10.15].*

$\widehat{2}$ *When* F *is* Gray, F *is strict so for every 1-morphism* $\{f_i\}$ *in* $\mathsf{Gr}\,A$, $\varphi_{\{f_i\}} = \mathrm{id}$ *by construction. Now consider a generator 2-morphism* $\overline{\alpha} = (k, \ell_1, \ell_2, (\sigma, D), (\tau, E), \alpha)$ *from* $\{f_i\}$ *to* $\{g_j\}$,

$$\mathsf{Gr}\,F(\overline{\alpha}) = (k, \ell_1, \ell_2, (\sigma, FD), (\tau, FE), F\alpha).$$

Therefore, the following diagram strictly commutes

$$\begin{tikzcd}
\mathrm{ev}_{\mathcal{B}} \circ \mathsf{Gr}\,F\{f_i\} \arrow[r, Rightarrow, "\mathrm{ev}_{\mathcal{B}} \circ \mathsf{Gr}\,F(\overline{\alpha})"] \arrow[d, Rightarrow, "\varphi_{\{f_i\}}"'] & \mathrm{ev}_{\mathcal{B}} \circ \mathsf{Gr}\,F\{g_j\} \arrow[d, Rightarrow, "\varphi_{\{g_j\}}"] \\
F \circ \mathrm{ev}_{\mathcal{A}}\{f_i\} \arrow[r, Rightarrow, "F \circ \mathrm{ev}_{\mathcal{A}}(\overline{\alpha})"'] & F \circ \mathrm{ev}_{\mathcal{A}}\{g_j\}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[Ff_i]} \arrow[r, Rightarrow, "{[F\alpha]}"] \arrow[d, equal] & {[Fg_j]} \arrow[d, equal] \\
{F[f_i]} \arrow[r, Rightarrow, "{F[\alpha]}"'] & {F[g_i]}
\end{tikzcd}$$

In other words, $\mathrm{ev}_{\mathcal{B}} \circ \mathsf{Gr}\, F(\overline{\alpha}) = F \circ \mathrm{ev}_{\mathcal{A}}(\overline{\alpha})$ *and since both functors are strict, this equality holds for general 2-morphisms. Finally, since* $\mathsf{Gr}\, F$ *and* $\mathrm{ev}_{\mathcal{A}}$ *are uniquely determined on 3-morphisms by coherence, it is immediate that* $\mathrm{ev}_{\mathcal{B}} \circ \mathsf{Gr}\, F = F \circ \mathrm{ev}_{\mathcal{A}}$ *at the level of 3-morphisms.*

$(\widehat{3})$ *Since* $\mathrm{id}_{\mathcal{A}}$ *is* Gray, *the result is immediate from our discussion in* $(\widehat{2})$.

$(\widehat{4})$ *From our discussion in* $(\widehat{1})$ *we see that*

$$\begin{tikzcd}
 & \mathsf{Gr}\,\mathcal{A} \arrow[r, "\mathsf{Gr}\,F"] \arrow[d, "\mathrm{ev}_{\mathcal{A}}"'] \arrow[dr, phantom, "\varphi^F \Downarrow"] & \mathsf{Gr}\,\mathcal{B} \arrow[d, "\mathrm{ev}_{\mathcal{B}}"] & \\
 & \mathcal{A} \arrow[r, "F"'] \arrow[dr, "GF"] & \mathcal{B} \arrow[d, "G"'] \arrow[dr, phantom, "\Leftarrow \varphi^G"] & \mathsf{Gr}\,\mathcal{B} \arrow[l, "\mathrm{ev}_{\mathcal{B}}"'] \arrow[d, "\mathsf{Gr}\,G"] \\
\mathsf{Gr}\,\mathcal{A} \arrow[ur, "\mathrm{ev}_{\mathcal{A}}"] \arrow[dr, "\mathsf{Gr}(GF)"'] \arrow[rr, phantom, "\varphi^{GF} \Nearrow"] & & \mathcal{C} & \mathsf{Gr}\,\mathcal{C} \arrow[l, "\mathrm{ev}_{\mathcal{C}}"] \\
 & \mathsf{Gr}\,\mathcal{C} \arrow[ur, "\mathrm{ev}_{\mathcal{C}}"'] & &
\end{tikzcd}$$

Therefore,

$$\begin{aligned}
&\mathsf{Gr}\,\mathcal{A} \xrightarrow{\mathsf{Gr}(GF)} \mathsf{Gr}\,\mathcal{C} \\
\cong\ &\mathsf{Gr}\,\mathcal{A} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \xrightarrow{\mathrm{ev}^{\bullet}_{\mathcal{C}}} \mathsf{Gr}\,\mathcal{C} \\
\cong\ &\mathsf{Gr}\,\mathcal{A} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A} \xrightarrow{\mathrm{ev}^{\bullet}_{\mathcal{A}}} \mathsf{Gr}\,\mathcal{A} \xrightarrow{\mathsf{Gr}\,F} \mathsf{Gr}\,B \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \xrightarrow{\mathrm{ev}^{\bullet}_{\mathcal{B}}} \mathsf{Gr}\,\mathcal{B} \xrightarrow{\mathsf{Gr}\,G} \mathsf{Gr}\,\mathcal{C} \xrightarrow{\mathrm{ev}_{\mathcal{C}}} \mathcal{C} \xrightarrow{\mathrm{ev}^{\bullet}_{\mathcal{C}}} \mathsf{Gr}\,\mathcal{C} \\
\cong\ &\mathsf{Gr}\,\mathcal{A} \xrightarrow{\mathsf{Gr}\,F} \mathsf{Gr}\,\mathcal{B} \xrightarrow{\mathsf{Gr}\,G} \mathsf{Gr}\,\mathcal{C},
\end{aligned}$$

where $\mathrm{ev}^{\bullet}_{\mathcal{A}}, \mathrm{ev}^{\bullet}_{\mathcal{B}}, \mathrm{ev}^{\bullet}_{\mathcal{C}}$ *are pseudo-inverse to* $\mathrm{ev}_{\mathcal{A}}, \mathrm{ev}_{\mathcal{B}}, \mathrm{ev}_{\mathcal{C}}$ *respectively.*

$(\widehat{5})$ *Notice that* $\mathsf{Gr}(G \circ F)$ *and* $\mathsf{Gr}\, G \circ \mathsf{Gr}\, F$ *always agree on objects and 1-morphisms in* $\mathsf{Gr}\,\mathcal{A}$*. We now suppose* F *is a* Gray*-functor. Consider a generator 2-morphism* $\overline{\alpha} = (k, \ell_1, \ell_2, (\sigma, F), (\tau, E), \alpha)$ *in* $\mathsf{Gr}\,\mathcal{A}$*. We must show* $\mathsf{Gr}(G)\,\mathsf{Gr}(F)\overline{\alpha} = \mathsf{Gr}(G \circ F)\overline{\alpha}$*. As we saw in* $(\widehat{2})$*,* $\mathsf{Gr}(F)\overline{\alpha} = (k, \ell_1, \ell_2, (\sigma, FD), (\tau, FE), F\alpha)$ *and*

$$\mathsf{Gr}(G)\,\mathsf{Gr}(F)\alpha = (\varphi^G)^{\cdot}_{\{Fg_{\ell_2} \geq j \geq k\}} \otimes GF\alpha \otimes \varphi^G_{\{Ff_{\ell_1} \geq i \geq k\}}.$$

This agrees with

$$\mathsf{Gr}(G \circ F)\alpha = (\varphi^{GF})^{\cdot}_{\{g_{\ell_2} \geq j \geq k\}} \otimes GF\alpha \otimes \varphi^{GF}_{\{f_{\ell_1} \geq i \geq k\}}.$$

because in this case the constraint data for GF *is simply that of* G *restricted to the image of* F*. Since* $\mathsf{Gr}(G) \circ \mathsf{Gr}(F)$ *and* $\mathsf{Gr}(G \circ F)$ *are strict, we conclude that these functors agree on general 2-morphisms in* $\mathsf{Gr}\,\mathcal{A}$*. We conclude by a similar argument that these functors agree on 3-morphisms as well. Thus* $\mathsf{Gr}(G \circ F) = \mathsf{Gr}(G) \circ \mathsf{Gr}(F)$*.*

Now suppose G *is* Gray*. We must again show* $\mathsf{Gr}(F)\,\mathsf{Gr}(G)\overline{\alpha} = \mathsf{Gr}(G \circ F)\overline{\alpha}$ *for a generator 2-morphism* $\overline{\alpha}$ *in* $\mathsf{Gr}\,\mathcal{A}$*. Recall that*

$$\mathsf{Gr}(F)\alpha = (\varphi^F)^{\cdot}_{\{g_{\ell_2} \geq j \geq k\}} \otimes F\alpha \otimes \varphi^F_{\{f_{\ell_1} \geq i \geq k\}}.$$

Thus

$$\mathsf{Gr}(G)\,\mathsf{Gr}(F)\alpha = G(\varphi^F)^{\cdot}_{\{g_{\ell_2} \geq j \geq k\}} \otimes GF\alpha \otimes G\varphi^F_{\{f_{\ell_1} \geq i \geq k\}}.$$

This agrees with

$$\mathsf{Gr}(G \circ F)\alpha = (\varphi^{GF})^{\cdot}_{\{g_{\ell_2} \geq j \geq k\}} \otimes GF\alpha \otimes \varphi^{GF}_{\{f_{\ell_1} \geq i \geq k\}}.$$

because in this case, the constraint data for GF *is that of* F *pushed through the functor* G*. We then conclude that* $\mathsf{Gr}(G \circ F)$ *and* $\mathsf{Gr}(G) \circ \mathsf{Gr}(F)$ *agree on arbitrary 2-morphisms in* $\mathsf{Gr}\,\mathcal{A}$ *as usual. A similar argument reveals that these functors also agree on 3-morphisms, and we again conclude* $\mathsf{Gr}(G \circ F) = \mathsf{Gr}(G) \circ \mathsf{Gr}(F)$*.*

1.4.4 Lack's path object construction

To show (P1), we recall [Lac11, Prop. 4.1] and include details which were originally left to the reader.

Construction 4. *The path* Gray*-category* $\mathbb{P}\mathcal{B}$ *of a* Gray*-category* $\mathcal{B}$ *is constructed as follows.*

($\mathbb{P}0$) *An object* $a \in \mathbb{P}\mathcal{B}$ *is a biequivalence* $\vec{a}\colon Sa \to Ta$ *in* $\mathcal{B}$.

($\mathbb{P}1$) *A 1-morphism* $f\colon a \to b$ *in* $\mathbb{P}\mathcal{B}$ *consists a tuple* $f = (Sf, Tf, \vec{f}\,)$ *where* $Sf\colon Sa \to Sb$ *and* $Tf\colon Ta \to Tb$ *are 1-morphisms in* $\mathcal{B}$, *and* $\vec{f}$ *is an equivalence in* $\mathcal{B}$ *with:*

$$\begin{tikzcd}
Sa \arrow[r, "Sf"] \arrow[d, "\vec{a}"'] \arrow[dr, phantom, "\Downarrow \vec{f}"] & Sb \arrow[d, "\vec{b}"] \\
Ta \arrow[r, "Tf"'] & Tb
\end{tikzcd}$$

($\mathbb{P}2$) *A 2-morphism* $\theta\colon f \Rightarrow g$ *in* $\mathbb{P}\mathcal{B}$ *consists of a tuple* $(S\theta, T\theta, \vec{\theta}\,)$ *where* $S\theta\colon Sf \Rightarrow Sg$ *and* $T\theta\colon Tf \Rightarrow Tg$ *are 2-morphisms in* $\mathcal{B}$, *and* $\vec{\theta}$ *is an invertible 3-morphisms in* $\mathcal{B}$ *with:*

$$\begin{tikzcd}
Sa \arrow[r, bend left=50, "Sf"] \arrow[r, phantom, bend left=25, "\Downarrow S\theta"] \arrow[r, "Sg"'] \arrow[d, "\vec{a}"'] \arrow[dr, phantom, "\Downarrow \vec{g}"] & Sb \arrow[d, "\vec{b}"] \arrow[drr, phantom, "\overset{\vec{\theta}}{\Rrightarrow}"] & & Sa \arrow[r, "Sf"] \arrow[d, "\vec{a}"'] \arrow[dr, phantom, "\Downarrow \vec{f}"] & Sb \arrow[d, "b_\alpha"] \\
Ta \arrow[r, "Tg"'] & Tb & & Ta \arrow[r, "Tf"] \arrow[r, bend right=50, "Tg"'] \arrow[r, phantom, bend right=25, "\Downarrow T\theta"] & Tb
\end{tikzcd}$$

($\mathbb{P}3$) *A 3-morphism* $\Gamma\colon \theta \Rrightarrow \sigma$ *in* $\mathbb{P}\mathcal{B}$ *consists of a tuple* $(S\Gamma, T\Gamma)$ *where* $S\Gamma\colon S\theta \Rrightarrow S\sigma$ *and* $T\Gamma\colon T\theta \Rrightarrow T\sigma$ *are 3-morphisms in* $\mathcal{B}$ *that commute with* $\vec{\theta}$ *and* $\vec{\sigma}$ *in the obvious way:*

$$\begin{tikzcd}
\vec{g} \otimes (\vec{b} \boxtimes S\theta) \arrow[r, "\vec{\theta}"] \arrow[d, "\vec{g}\otimes(\vec{b}\boxtimes S\Gamma)"'] & (T\theta \boxtimes \vec{a}) \otimes \vec{f} \arrow[d, "(T\Gamma\boxtimes\vec{a})\otimes\vec{f}"] \\
\vec{g} \otimes (\vec{b} \boxtimes S\sigma) \arrow[r, "\vec{\sigma}"'] & (T\sigma \boxtimes \vec{a}) \otimes \vec{f}
\end{tikzcd}$$

There is an organic way of equipping $\mathbb{P}\mathcal{B}$ with the structure of a Gray*-category by inheriting composites and interchangers from the* Gray*-category $\mathcal{B}$. We now define the "source" and "target"* Gray*-functors $S, T \colon \mathbb{P}\mathcal{B} \to \mathcal{B}$.*

$(ST0)$ *For an object $a \in \mathbb{P}\mathcal{B}_0$, we set $S(a) \coloneqq Sa$ and $T(a) \coloneqq Ta$,*

$(ST1)$ *For a 1-morphism $f \in \mathbb{P}\mathcal{B}_1$, we set $S(f) \coloneqq Sf$ and $T(f) \coloneqq Tf$,*

$(ST2)$ *For a 2-morphism $\theta \in \mathbb{P}\mathcal{B}_2$, we set $S(\theta) \coloneqq S\theta$ and $T(\theta) \coloneqq T\theta$,*

$(ST3)$ *For a 3-morphism $\Gamma \in \mathbb{P}\mathcal{B}_3$, we set $S(\Gamma) \coloneqq S\Gamma$ and $T(\Gamma) \coloneqq T\Gamma$*

The fact that S and T are Gray*-functors is due to the fact that composites in $\mathbb{P}\mathcal{B}$ are inherited from those in $\mathcal{B}$. We now define the "constant"* Gray*-functor $C \colon \mathcal{B} \to \mathbb{P}\mathcal{B}$.*

(C0) For an object $b \in \mathcal{B}_0$, we set $C(b)$ to be the trivial biequivalence of b with itself, i.e. $C(b) \coloneqq \mathrm{id}_b$,

(C1) For a 1-morphism $f \in \mathcal{B}_1$, we set $C(f) \coloneqq (f, f, \mathrm{id}_f)$,

(C2) For a 2-morphism $\theta \in \mathcal{B}_2$, we set $C(\theta) \coloneqq (\theta, \theta, \mathrm{id}_\theta)$,

(C3) For a 3-morphism $\Gamma \in \mathcal{B}_3$, we set $C(\Gamma) \coloneqq (\Gamma, \Gamma)$.

The fact that $C \colon \mathcal{B} \to \mathbb{P}\mathcal{B}$ is a Gray*-functor is also due to the fact that composites in $\mathbb{P}\mathcal{B}$ are inherited from those in $\mathcal{B}$. Furthermore, we have $S \circ C = T \circ C = \mathrm{id}_{\mathcal{B}}$ by construction.*

From this we also see that S and T are surjective on objects and full at all levels. To show that S and T are trivial fibrations, it thus suffices to show these are faithful on 3-morphisms. Consider when $S\Gamma = S\Gamma'$ or $T\Gamma = T\Gamma'$ for 3-morphisms $\Gamma, \Gamma' : \theta \Rrightarrow \sigma : f \Rightarrow g : a \to b$ in $\mathbb{P}\mathcal{B}$. Using the fact that $\vec{\theta}, \vec{\sigma}$ are invertible, $\vec{f}, \vec{g}$ are equivalences, and $\vec{a}$, $\vec{b}$ are bi-equivalences in $\mathcal{B}$, *the axiom which 3-morphisms in $\mathbb{P}\mathcal{B}$ must satisfy reveals that $\Gamma = \Gamma'$. Hence S and T are trivial fibrations and, by the 2-out-of-3 property, it follows that C is a weak equivalence.*

The fact that $\binom{S}{T}\colon \mathcal{B}^I \to \mathcal{B} \times \mathcal{B}$ is a fibration follows from a simple characterization of isomorphisms, equivalences, and bi-equivalences in $\mathcal{B}^I$. As we have not provided much treatment for fibrations in Lack's model structure, we refer the interested reader to [Lac11] for more details.

Remark 1.4.4.1. It is easy to show (P2) from our previous construction. Indeed, for a Gray-functor $F\colon \mathcal{B}_1 \to \mathcal{B}_2$, notice there is an organic Gray-functor $\mathbb{P}F\colon \mathbb{P}\mathcal{B}_1 \to \mathbb{P}\mathcal{B}_2$ which is obtained by passing the data of a k-morphism $(0 \le k \le 3)$ in $\mathbb{P}\mathcal{B}_1$ through F.

Remark 1.4.4.2. When $F, G\colon \mathcal{A} \to \mathcal{B}$ are pseudo-naturally equivalent Gray-functors, there exists a tritransformation $\alpha\colon F \to G$ consisting of:

(α0) for each object $a \in \mathcal{A}$, a biequivalence $\alpha_a\colon Fa \to Ga$ and a 3-isomorphism

$$M_a\colon \alpha_{\mathrm{id}_a} \Rrightarrow \mathrm{id}_{\alpha_a}$$

in $\mathcal{B}$, where

(α1) for each 1-morphism $f \in \mathcal{A}$ these is an equivalence $\alpha_f\colon \alpha_b \boxtimes Ff \Rightarrow Gf \boxtimes \alpha_a$ in $\mathcal{B}$ and for each composable pair f, g of 1-morphisms, a 3-isomorphism

$$\Pi_{gf}\colon (Gg \boxtimes \alpha_f)(\alpha_g \boxtimes Ff) \Rrightarrow \alpha_{g \boxtimes f}$$

in $\mathcal{B}$, and

(α2) for each 2-morphism $\theta\colon f \Rightarrow g$ with $f, g\colon a \to b$, a 3-isomorphism

$$\alpha_\theta\colon \alpha_g \otimes (\alpha_b \boxtimes F\theta) \Rrightarrow (G\theta \boxtimes \alpha_a) \otimes \alpha_f$$

in $\mathcal{B}$.

This data is subject to various axioms, including naturality conditions for α, M, and Π; an associativity condition Π; and left and right unitality conditions relating M and Π. We refer the interested reader to [Gur13, Def. 4.16] for more details. We now prove (P3).

Construction 5. *Suppose $F, G\colon \mathcal{A} \to \mathcal{B}$ are pseudo-naturally equivalent* Gray*-functors, so that we have data as in Remark 1.4.4.2. We define the weak 3-functor $\langle F, G\rangle\colon \mathcal{A} \to \mathbb{P}\mathcal{B}$ as follows.*

- *(0) For objects $a \in \mathcal{A}$, we set $\langle F, G\rangle(a) \coloneqq (Fa \xrightarrow{\alpha_a} Ga)$,*
- *(1) For 1-morphisms $f \in \mathcal{A}$, we set $\langle F, G\rangle(f) \coloneqq (Ff, Fg, \alpha_f)$,*
- *(2) For 2-morphisms $\theta \in \mathcal{A}$, we set $\langle F, G\rangle(\theta) \coloneqq (F\theta, G\theta, \alpha_\theta)$,*
- *(3) For 3-morphism $\Gamma \in \mathcal{A}$, we set $\langle F, G\rangle(\Gamma) \coloneqq (F\Gamma, G\Gamma)$.*

For a 3-morphism $\Gamma : \theta \Rrightarrow \sigma : f \Rightarrow g : a \to b$ in $\mathcal{B}$, the fact that $\langle F, G\rangle(\Gamma)$ is a 3-morphism in $\mathbb{P}\mathcal{B}$ follows by the naturality axiom α satisfies, i.e. the following diagram commutes:

$$\begin{tikzcd}
\alpha_g \otimes (\alpha_b \boxtimes F\theta) \arrow[r, "\alpha_\theta"] \arrow[d, "\alpha_g\otimes(\alpha_b\boxtimes F\Gamma)"'] & (G\theta \boxtimes \alpha_a) \otimes \alpha_f \arrow[d, "(G\Gamma\boxtimes\alpha_a)\otimes\alpha_f"] \\
\alpha_g \otimes (\alpha_b \boxtimes F\sigma) \arrow[r, "\alpha_\sigma"'] & (G\sigma \boxtimes \alpha_a) \otimes \alpha_f
\end{tikzcd}$$

We now equip this map with the structure of a weak 3-functor.

- *(χ) For $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{A}$, we define an adjoint equivalence $(\chi_{gf}, \chi^{\bullet}_{gf}, \epsilon_{\chi_{gf}}, \eta_{\chi_{gf}})$ where*

$$\chi_{gf}\colon \langle F, G\rangle(g) \boxtimes \langle F, G\rangle(f) \Rrightarrow \langle F, G\rangle(g \boxtimes f),$$

by $\chi_{gf} \coloneqq (\mathrm{id}, \mathrm{id}, \Pi_{gf})$, $\chi^{\bullet}_{gf} \coloneqq (\mathrm{id}, \mathrm{id}, \Pi^{-1}_{gf})$, and $\epsilon_{\chi_{gf}} = \eta_{\chi_{gf}} \coloneqq (\mathrm{id}, \mathrm{id})$. Notice $\epsilon_{\chi_{gf}}$ and $\eta_{\chi_{gf}}$ immediately satisfy the axioms for 3-morphisms in $\mathbb{P}\mathcal{B}$.

For $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\theta"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\sigma"] & c \end{tikzcd}$ *in $\mathcal{A}$, we define the invertible 3-morphisms*

$$\begin{tikzcd}
\langle F,G\rangle(g)\boxtimes\langle F,G\rangle(f) \arrow[r, Rightarrow, "\chi_{gf}"] \arrow[d, Rightarrow, "\langle F,G\rangle(\sigma)\boxtimes\langle F,G\rangle(\theta)"'] \arrow[dr, phantom, "\Uparrow\chi_{\sigma\theta}"] & \langle F,G\rangle(g\boxtimes f) \arrow[d, Rightarrow, "\langle F,G\rangle(\sigma\boxtimes\theta)"] \\
\langle F,G\rangle(g')\boxtimes\langle F,G\rangle(f') \arrow[r, Rightarrow, "\chi_{g'f'}"'] & \langle F,G\rangle(g'\boxtimes f')
\end{tikzcd}$$

$$\begin{tikzcd}
\langle F,G\rangle(g\boxtimes f) \arrow[r, Rightarrow, "\chi^{\bullet}_{gf}"] \arrow[d, Rightarrow, "\langle F,G\rangle(\sigma\boxtimes\theta)"'] \arrow[dr, phantom, "\Uparrow\chi^{\bullet}_{\sigma\theta}"] & \langle F,G\rangle(g)\boxtimes\langle F,G\rangle(f) \arrow[d, Rightarrow, "\langle F,G\rangle(\sigma)\boxtimes\langle F,G\rangle(\theta)"] \\
\langle F,G\rangle(g'\boxtimes f') \arrow[r, Rightarrow, "\chi^{\bullet}_{g'f'}"'] & \langle F,G\rangle(g')\boxtimes\langle F,G\rangle(f')
\end{tikzcd}$$

simply by $\chi_{\sigma\theta} = \chi^{\bullet}_{\sigma\theta} \coloneqq (\mathrm{id},\mathrm{id})$. *The fact that* $\chi_{\sigma\theta}$ *and* $\chi^{\bullet}_{\sigma\theta}$ *satisfy the axioms for 3-morphisms in* $\mathbb{P}\mathcal{B}$ *follows from the naturality of* Π.

(ι) *For each object* $a \in \mathcal{A}$, *we define an adjoint equivalence* $(\iota_a, \iota^{\bullet}_a, \epsilon_{\iota_a}, \eta_{\iota_a})$ *where*

$$\iota_a \colon \mathrm{id}_{\langle F,G\rangle(a)} \Rightarrow \langle F,G\rangle(\mathrm{id}_a),$$

by $\iota_a \coloneqq (\mathrm{id},\mathrm{id},M_a^{-1})$, $\iota^{\bullet}_a \coloneqq (\mathrm{id},\mathrm{id},M_a)$, *and* $\epsilon_{\iota_a} = \eta_{\iota_a} \coloneqq (\mathrm{id},\mathrm{id})$. *Notice* ϵ_{ι_a} *and* η_{ι_a} *immediately satisfy the axioms for 3-morphisms in* $\mathbb{P}\mathcal{B}$. *We must also choose invertible 3-morphisms*

$$\iota_{\mathrm{id}_a} \colon \iota_a \otimes \mathrm{id}_{\mathrm{id}_{\langle F,G\rangle(a)}} \overset{\sim}{\Rrightarrow} \langle F,G\rangle(\mathrm{id}_{\mathrm{id}_a}) \otimes \iota_a$$

$$\iota^{\bullet}_{\mathrm{id}_a} \colon \iota^{\bullet}_a \otimes \langle F,G\rangle(\mathrm{id}_{\mathrm{id}_a}) \overset{\sim}{\Rrightarrow} \mathrm{id}_{\mathrm{id}_{\langle F,G\rangle a}} \otimes \iota^{\bullet}_a$$

which we will simply take to be $\iota_{\mathrm{id}_a} \coloneqq (\mathrm{id},\mathrm{id})$ *and* $\iota^{\bullet}_{\mathrm{id}_a} \coloneqq (\mathrm{id},\mathrm{id})$. *The fact that* ι_{id_a} *and* $\iota^{\bullet}_{\mathrm{id}_a}$ *are 3-isomorphisms in* $\mathbb{P}\mathcal{B}$ *follows from the naturality condition* M *satisfies.*

(ω) *For* $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ *in* $\mathcal{A}$, *we define the invertible 3-morphism*

$$\begin{tikzcd}
\langle F,G\rangle(h)\boxtimes\langle F,G\rangle(g)\boxtimes\langle F,G\rangle(f) \arrow[r, Rightarrow, "\chi_{hg}\boxtimes\langle F,G\rangle(f)"] \arrow[d, Rightarrow, "\langle F,G\rangle(h)\boxtimes\chi_{gf}"'] \arrow[dr, phantom, "\Downarrow\omega_{hgf}"] & \langle F,G\rangle(h\boxtimes g)\boxtimes\langle F,G\rangle(f) \arrow[d, Rightarrow, "\chi_{hg,f}"] \\
\langle F,G\rangle(h)\boxtimes\langle F,G\rangle(g\boxtimes f) \arrow[r, Rightarrow, "\chi_{h,gf}"'] & \langle F,G\rangle(h\boxtimes g\boxtimes f)
\end{tikzcd}$$

simply by $\omega_{hgf} = (\mathrm{id}, \mathrm{id})$*. In this case, the axiom for 3-morphisms in* $\mathbb{P}\mathcal{B}$ *translates to the associativity of* Π*.*

(γ) For $a \xrightarrow{f} b$ *in* $\mathcal{A}$*, we define the invertible 3-morphism*

$$\begin{tikzcd}
\mathrm{id}_{\langle F,G\rangle(b)} \boxtimes \langle F,G\rangle(f) \arrow[rr, Rightarrow, "\iota_b \boxtimes \langle F,G\rangle(f)"] \arrow[dr, equal] & \Rrightarrow \gamma_f & \langle F,G\rangle(\mathrm{id}_b) \boxtimes \langle F,G\rangle(f) \arrow[dl, Rightarrow, "\chi_{\mathrm{id}_b,f}"] \\
& \langle F,G\rangle(f) &
\end{tikzcd}$$

simply by $\gamma_f = (\mathrm{id}, \mathrm{id})$*. In this case, the axiom for 3-morphisms in* $\mathbb{P}\mathcal{B}$ *translates to the left unitality axiom relating* Π *and* M*.*

(δ) For $a \xrightarrow{f} b$ *in* $\mathcal{A}$*, we define the invertible 3-morphism* $\delta_f \colon \mathrm{id}_{\langle F,G\rangle(f)} \overset{\sim}{\Rrightarrow} \chi_{f,\mathrm{id}_a}$

$$\begin{tikzcd}
\langle F,G\rangle(f) \boxtimes \mathrm{id}_{\langle F,G\rangle(a)} \arrow[rr, Rightarrow, "\langle F,G\rangle(f) \boxtimes \iota_a"] \arrow[dr, equal] & \Rrightarrow \delta_f & \langle F,G\rangle(f) \boxtimes \langle F,G\rangle(\mathrm{id}_a) \arrow[dl, Rightarrow, "\chi_{f,\mathrm{id}_a}"] \\
& \langle F,G\rangle(f) &
\end{tikzcd}$$

by $\delta_f = (\mathrm{id}, \mathrm{id})$*. In this case, the axiom for 3-morphisms in* $\mathbb{P}\mathcal{B}$ *translates to the right unitality axiom relating* Π *and* M*.*

These collections of morphisms in $\mathbb{P}\mathcal{B}$ *assemble themselves into pseudonatural transformations or modifications due to the naturality conditions* Π *and* M *satisfy. Furthermore, these pseudonatural transformations and modifications trivially satisfy the two axioms of weak 3-functors. Thus* $\langle F,G\rangle$ *is a weak 3-functor and it is clear that* $S \circ \langle F,G\rangle = F$ *and* $T \circ \langle F,G\rangle = G$ *by construction.*

Chapter 2: Gray-categories model algebraic tricategories

The content of this chapter is based on [Fer22]. The author would like to thank David Penneys for his guidance in this project and for many helpful conversations. The author would also like to thank Nick Gurski, Niles Johnson, and David Reutter for useful suggestions.

Lack described a Quillen model structure on the category $\mathsf{GrayCat}$ of Gray-categories and Gray-functors, for which the weak equivalences are the weak 3-equivalences. Restricted to Gray-groupoids, the resulting homotopy category is equivalent to the homotopy category of 3-types. In this chapter, we adapt the technique of Gurski, Johnson, and Osorno to show the localization of $\mathsf{GrayCat}$ at the weak equivalences is equivalent to the category of algebraic tricategories and pseudo-natural equivalence classes of weak 3-functors. This finishes establishing the homotopy hypothesis for algebraic trigroupoids.

2.1 Introduction

We first discuss the model category structure on $\mathsf{GrayCat}$ from [Lac11].

Definition 2.1.0.1. A Gray-functor $F\colon \mathcal{A} \to \mathcal{B}$ between Gray-categories is called:

- a *weak equivalence* if it is a weak 3-equivalence of $\mathcal{A}$ and $\mathcal{B}$ as weak 3-categories,
- a *fibration* if it is a fibration on each hom-2-category and satisfies a so-called biadjoint-biequivalence-lifting property (from which one sees all Gray-categories are fibrant),

- in particular, a *trivial fibration* if it is surjective on objects, full at all levels, and faithful at the top level,

- a *cofibration* if it has the left lifting property against trivial fibrations. More specifically, $F\colon \mathcal{A} \to \mathcal{B}$ is a cofibration if and only if for every trivial fibration $F'\colon \mathcal{A}' \to \mathcal{B}'$ and Gray-functors $a\colon \mathcal{A} \to \mathcal{A}'$ and $b\colon \mathcal{B} \to \mathcal{B}'$ such that the following square commutes,

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "a"] \arrow[d, "F"'] & \mathcal{A}' \arrow[d, "F'"] \\
\mathcal{B} \arrow[r, "b"'] \arrow[ur, dashed, "\ell"] & \mathcal{B}'
\end{tikzcd}$$

  there exists a Gray-functor lift $\ell : \mathcal{B} \to \mathcal{A}'$ such that the two triangles commute.

Definition 2.1.0.2. We denote by $\mathcal{W}$ the collection of weak equivalences in $\mathsf{GrayCat}$. The category $\mathsf{GrayCat}[\mathcal{W}^{-1}]$ is the localization of $\mathsf{GrayCat}$ at the weak equivalences, which is determined (up to unique isomorphism) by the following universal property [GZ67]:

- For any functor $F : \mathsf{GrayCat} \to \mathcal{D}$ of categories which maps weak equivalences to isomorphisms, F uniquely factors through $\mathsf{GrayCat}[\mathcal{W}^{-1}]$, i.e., there exists a unique $\Phi\colon \mathsf{GrayCat}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that the following diagram commutes on the nose:

$$\begin{tikzcd}
\mathsf{GrayCat} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathcal{D} \\
\mathsf{GrayCat}[\mathcal{W}^{-1}] \arrow[ur, dashed, "\exists!\,\Phi"']
\end{tikzcd}$$

Remark 2.1.0.3. In [GZ67], an explicit construction is provided in which an object of $\mathsf{GrayCat}[\mathcal{W}^{-1}]$ is simply a Gray-category and a morphism from $\mathcal{A} \to \mathcal{B}$ in $\mathsf{GrayCat}[\mathcal{W}^{-1}]$ is a "zigzag", i.e. a finite chain of Gray-categories and Gray-functors

$$\mathcal{A} \xleftarrow{w_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{w_2} X_3 \xrightarrow{f_2} \cdots \xleftarrow{w_n} X_{2n-1} \xrightarrow{f_n} \mathcal{B}$$

where the ← and → alternate and each morphism w_i "in the wrong direction" is a weak equivalence. These zigzags are considered equal up to a certain equivalence relation. We refer the reader to the previous citation for more details.

Definition 2.1.0.4. We denote by $\mathsf{hoTriCat}$ the 1-category whose objects are Gray-categories and morphisms are pseudo-natural equivalence classes of weak 3-functors. More specifically, we say two weak 3-functors $F, G \colon \mathcal{A} \to \mathcal{B}$ are pseudo-naturally equivalent and write $F \cong G$ if they are (internally) biequivalent in the weak 3-category $\mathsf{TriCat}(\mathcal{A}, \mathcal{B})$ of weak 3-functors. Since each weak 3-category (algebraic tricategory) is triequivalent to a Gray-category [Gur13, §10.4], $\mathsf{hoTriCat}$ is equivalent to the 1-category of weak 3-categories and pseudo-natural equivalence classes of weak 3-functors.

The main result of this note shows that Lack's model structure models algebraic tricategories in the following sense.

Theorem 2.1.0.5 (Theorem 2.3.0.3)**.** *The categories* $\mathsf{GrayCat}[\mathcal{W}^{-1}]$ *and* $\mathsf{hoTriCat}$ *are isomorphic.*

The proof is a straightforward adaptation of [GO19, Prop. 3.31] using cofibrant replacement and path objects. When restricted to Gray-groupoids, we obtain the following consequence of [Lac11, Thm. 5.4].

Corollary 2.1.0.6 (Homotopy hypothesis for algebraic trigroupoids)**.** *The 1-category of algebraic trigroupoids and natural equivalence classes of weak 3-functors is equivalent to the category of homotopy 3-types and homotopy classes of continuous maps.*

2.2 Constructions for Gray-categories

In this section we provide the two necessary constructions needed to adapt the technique of [GO19, Prop. 3.31] to show Theorem 2.3.0.3.

Proposition 2.2.0.1 (Cofibrant replacement in $\mathsf{GrayCat}$)**.** *For every $\mathcal{D} \in \mathsf{GrayCat}$, there is a cofibrant $\widehat{\mathcal{D}} \in \mathsf{GrayCat}$ together with an "evaluation" Gray-functor $\mathrm{ev}_{\mathcal{D}} \colon \widehat{\mathcal{D}} \to \mathcal{D}$ which is a trivial fibration. For every $\mathcal{A}, \mathcal{B} \in \mathsf{GrayCat}$ and weak 3-functor $F \colon \mathcal{A} \to \mathcal{B}$, there is a Gray-functor $\widehat{F} \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$ which satisfies the following properties:*

$(\widehat{1})$ *For every weak 3-functor $F \colon \mathcal{A} \to \mathcal{B}$, the diagram* $\begin{tikzcd} \widehat{\mathcal{A}} \arrow[r, "\widehat{F}"] \arrow[d, "\mathrm{ev}_{\mathcal{A}}"'] & \widehat{\mathcal{B}} \arrow[d, "\mathrm{ev}_{\mathcal{B}}"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{B} \end{tikzcd}$ *weakly commutes.*

$(\widehat{2})$ *When $F \colon \mathcal{A} \to \mathcal{B}$ is a Gray-functor, the diagram in $(\widehat{1})$ strictly commutes.*

$(\widehat{3})$ $\widehat{\mathrm{id}_{\mathcal{A}}} = \mathrm{id}_{\widehat{\mathcal{A}}}$.

$(\widehat{4})$ *If $F \colon \mathcal{A} \to \mathcal{B}$ and $G \colon \mathcal{B} \to \mathcal{D}$ are weak 3-functors, then $\widehat{G \circ F} \cong \widehat{G} \circ \widehat{F}$.*

$(\widehat{5})$ *In $(\widehat{4})$, if either F or G is a Gray-functor, then $\widehat{G \circ F} = \widehat{G} \circ \widehat{F}$.*

Proof. Gurski's Gr construction in [Gur13, §10.4] satisfies the properties required of $\widehat{\cdot}$. We recall this construction and go over its desired properties in §1.4.3. □

Proposition 2.2.0.2 (Path objects exist in $\mathsf{GrayCat}$)**.** *For every $\mathcal{B} \in \mathsf{GrayCat}$, there exists a path object $\mathcal{B}^I \in \mathsf{GrayCat}$ in the sense of [Qui67, Ch. 1, Def. 4] such that:*

(P1) *There exist Gray-functors* $\begin{tikzcd} \mathcal{B} \arrow[r, "C"] & \mathcal{B}^I \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] & \mathcal{B} \end{tikzcd}$ *where C is a weak equivalence and $\binom{S}{T} \colon \mathcal{B}^I \to \mathcal{B} \times \mathcal{B}$ is a fibration such that $S \circ C = T \circ C = \mathrm{id}_{\mathcal{B}}$.*

(P2) *If $F \colon \mathcal{B}_1 \to \mathcal{B}_2$ is a Gray-functor, then there exists a Gray-functor $F^I \colon \mathcal{B}_1^I \to \mathcal{B}_2^I$ which makes the corresponding C, S, and T squares commute:*

$$\begin{tikzcd}
\mathcal{B}_1 \arrow[r, "C"] \arrow[d, "F"'] & \mathcal{B}_1^I \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[d, "F^I"'] & \mathcal{B}_1 \arrow[d, "F"] \\
\mathcal{B}_2 \arrow[r, "C"] & \mathcal{B}_2^I \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] & \mathcal{B}_2
\end{tikzcd}$$

(P3) *Moreover, $\mathcal{B}^I$ satisfies the property that for every two pseudo-naturally equivalent* Gray*-functors $F, G\colon \mathcal{A} \to \mathcal{B}$, there is a weak 3-functor $\langle F, G\rangle\colon \mathcal{A} \to \mathcal{B}^I$ such that $S \circ \langle F, G\rangle = F$ and $T \circ \langle F, G\rangle = G$.*

Proof. Lack's path space construction $\mathbb{P}\mathcal{B}$ for $\mathcal{B} \in \mathsf{GrayCat}$ in [Lac11, Prop. 4.1] satisfies the desired properties. As [GO19] note in their Remark 3.33, there have been some mistakes in the literature regarding path objects and transferred model structures. We provide a careful treatment of this by recalling Lack's construction and showing it satisfies the properties of $\mathcal{B}^I$ in §1.4.4. □

Remark 2.2.0.3. We note that the existence of such path objects in $\mathsf{GrayCat}$ satisfying (P3) is crucial in proving our main result, as it relates (right) homotopies in the model theoretic sense with pseudo-natural equivalences.

2.3 The main theorem

In this section we implement the technique of [GO19]. There is an obvious functor $Q\colon \mathsf{GrayCat} \to \mathsf{hoTriCat}$ which maps each Gray-category to itself and maps $F\colon \mathcal{A} \to \mathcal{B}$ to its equivalence class $[F]$ in $\mathsf{hoTriCat}$. Clearly this functor maps equivalences to isomorphisms, and thus uniquely factors through $\mathsf{GrayCat}[\mathcal{W}^{-1}]$. We will denote this factorization by $\Phi\colon \mathsf{GrayCat}[\mathcal{W}^{-1}] \to \mathsf{hoTriCat}$.

Definition 2.3.0.1. Define $\Psi\colon \mathsf{hoTriCat} \to \mathsf{GrayCat}[\mathcal{W}^{-1}]$ to be the identity on objects and, for a weak 3-functor $F\colon \mathcal{A} \to \mathcal{B}$, define

$$\Psi([F]) \coloneqq \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{F}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B}.$$

Lemma 2.3.0.2. *Ψ is a well-defined functor.*

Proof. Suppose $F, G\colon \mathcal{A} \to \mathcal{B}$ are pseudo-naturally equivalent. Then

$$\begin{align*}
\Psi([F]) &= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{F}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{S\circ\langle F,G\rangle}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \tag{P3} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{\langle F,G\rangle}} \widehat{\mathcal{B}^I} \xrightarrow{\widehat{S}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \tag{$\widehat{5}$} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{\langle F,G\rangle}} \widehat{\mathcal{B}^I} \xrightarrow{\mathrm{ev}_{\mathcal{B}^I}} \mathcal{B}^I \xrightarrow{S} \mathcal{B} \tag{$\widehat{2}$} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{\langle F,G\rangle}} \widehat{\mathcal{B}^I} \xrightarrow{\mathrm{ev}_{\mathcal{B}^I}} \mathcal{B}^I \xleftarrow{C} \mathcal{B} \xrightarrow{C} \mathcal{B}^I \xrightarrow{S} \mathcal{B} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{\langle F,G\rangle}} \widehat{\mathcal{B}^I} \xrightarrow{\mathrm{ev}_{\mathcal{B}^I}} \mathcal{B}^I \xleftarrow{C} \mathcal{B} \xrightarrow{C} \mathcal{B}^I \xrightarrow{T} \mathcal{B} \tag{P1} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{\langle F,G\rangle}} \widehat{\mathcal{B}^I} \xrightarrow{\mathrm{ev}_{\mathcal{B}^I}} \mathcal{B}^I \xrightarrow{T} \mathcal{B} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{\langle F,G\rangle}} \widehat{\mathcal{B}^I} \xrightarrow{\widehat{T}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \tag{$\widehat{2}$} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{T\circ\langle F,G\rangle}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \tag{$\widehat{5}$} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{G}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} = \Psi([G]). \tag{P3}
\end{align*}$$

Hence Ψ is well-defined. Observe

$$\begin{align*}
\Psi([\mathrm{id}_{\mathcal{A}}]) &= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{\mathrm{id}_{\mathcal{A}}}} \widehat{\mathcal{A}} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\mathrm{id}_{\widehat{\mathcal{A}}}} \widehat{\mathcal{A}} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A} \tag{$\widehat{3}$} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A} \\
&= \mathcal{A} \xrightarrow{\mathrm{id}_{\mathcal{A}}} \mathcal{A} = \mathrm{id}_{\Psi(\mathcal{A})}.
\end{align*}$$

Since Ψ is well-defined and for composable weak 3-functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{D}$ we have $[\widehat{G} \circ \widehat{F}] = [\widehat{G \circ F}]$, it follows

$$\begin{aligned}
\widehat{\mathcal{A}} \xrightarrow{\widehat{F}} \widehat{\mathcal{B}} \xrightarrow{\widehat{G}} \widehat{\mathcal{D}} \underset{\widehat{(2)}}{=} \widehat{\mathcal{A}} \xleftarrow{\mathrm{ev}_{\widehat{\mathcal{A}}}} \widehat{\widehat{\mathcal{A}}} \xrightarrow{\widehat{\widehat{G}\circ\widehat{F}}} \widehat{\widehat{\mathcal{D}}} \xrightarrow{\mathrm{ev}_{\widehat{\mathcal{D}}}} \widehat{\mathcal{D}} &= \Psi([\widehat{G} \circ \widehat{F}]) && \widehat{(4)} \\
&= \Psi([\widehat{G \circ F}]) = \widehat{\mathcal{A}} \xleftarrow{\mathrm{ev}_{\widehat{\mathcal{A}}}} \widehat{\widehat{\mathcal{A}}} \xrightarrow{\widehat{\widehat{G\circ F}}} \widehat{\widehat{\mathcal{D}}} \xrightarrow{\mathrm{ev}_{\widehat{\mathcal{D}}}} \widehat{\mathcal{D}} && \widehat{(2)} \\
&= \widehat{\mathcal{A}} \xrightarrow{\widehat{G\circ F}} \widehat{\mathcal{D}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Psi([G]) \circ \Psi([F]) &= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{F}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \xleftarrow{\mathrm{ev}_{\mathcal{B}}} \widehat{\mathcal{B}} \xrightarrow{\widehat{G}} \widehat{\mathcal{D}} \xrightarrow{\mathrm{ev}_{\mathcal{D}}} \mathcal{D} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{F}} \widehat{\mathcal{B}} \xrightarrow{\widehat{G}} \widehat{\mathcal{D}} \xrightarrow{\mathrm{ev}_{\mathcal{D}}} \mathcal{D} \\
&= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{G\circ F}} \widehat{\mathcal{D}} \xrightarrow{\mathrm{ev}_{\mathcal{D}}} \mathcal{D} = \Psi([G \circ F]),
\end{aligned}$$

and we conclude Ψ is a functor. □

We are now ready to prove the main result of this note.

Theorem 2.3.0.3. *The functors Φ and Ψ exhibit an isomorphism of categories.*

Proof. Since both functors are the identity on objects, it suffices to show that Φ and Ψ are mutual inverses on hom sets. The fact that Φ is surjective on hom sets follows from $\widehat{(1)}$. We now show $(\Psi \circ \Phi)([F]) = [F]$ and conclude Φ is an isomorphism with inverse Ψ. Indeed, consider the following diagram:

$$\begin{tikzcd}
\mathsf{GrayCat} \arrow[r, "Q"] \arrow[d, "\pi"'] & \mathsf{hoTriCat} \arrow[d, "\Psi"] \\
\mathsf{GrayCat}[\mathcal{W}^{-1}] \arrow[ur, "\Phi"] \arrow[r, "\mathrm{id}"'] & \mathsf{GrayCat}[\mathcal{W}^{-1}]
\end{tikzcd}$$

whose left triangle commutes by the universal property of $\mathsf{GrayCat}[\mathcal{W}^{-1}]$. Note that for every Gray-functor $F\colon \mathcal{A} \to \mathcal{B}$,

$$\begin{aligned}(\Psi \circ Q)(F) = \Psi([F]) &= A \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \widehat{\mathcal{A}} \xrightarrow{\widehat{F}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \qquad (\widehat{2}) \\ &= \mathcal{A} \xleftarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A} \xrightarrow{F} \mathcal{B} = \mathcal{A} \xrightarrow{F} \mathcal{B} = \pi(F).\end{aligned}$$

Thus $\Psi \circ \Phi \circ \pi = \Psi \circ Q = \pi$. By the uniqueness of factorizations through $\mathsf{GrayCat}[\mathcal{W}^{-1}]$, it is easy to see that π is epic. From this we conclude $\Psi \circ \Phi = \mathrm{id}$. □

2.4 Corollaries

We obtain the following two corollaries which correspond to the surjectivity and injectivity /well-definedness respectively of the above bijection induced by Φ.

Corollary 2.4.0.1. *If $\mathcal{A}$ is a cofibrant Gray-category and $F\colon \mathcal{A} \to \mathcal{B}$ is a weak 3-functor, then F is pseudonaturally equivalent to a Gray-functor.*

Proof. Since $\mathrm{ev}_{\mathcal{A}}\colon \widehat{\mathcal{A}} \to \mathcal{A}$ is a trivial fibration, if $\varnothing \to \mathcal{A}$ is a cofibration, then there exists a Gray-functor lift $\ell\colon \mathcal{A} \to \widehat{\mathcal{A}}$ which makes the following triangles commute

$$\begin{tikzcd}
\varnothing \arrow[r, "!"] \arrow[d, "!"'] & \widehat{\mathcal{A}} \arrow[d, "\mathrm{ev}_{\mathcal{A}}"] \\
\mathcal{A} \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted, "\ell"] & \mathcal{A}
\end{tikzcd}$$

Thus ℓ is a section of $\mathrm{ev}_{\mathcal{A}}$ and $F = F \circ \mathrm{ev}_{\mathcal{A}} \circ \ell \underset{(\widehat{1})}{\cong} \mathrm{ev}_{\mathcal{B}} \circ \widehat{F} \circ \ell$, a Gray-functor. □

Corollary 2.4.0.2. *Suppose $\mathcal{A}$ is a cofibrant Gray-category and $F, G\colon \mathcal{A} \to \mathcal{B}$ are Gray-functors. Then F is pseudonaturally equivalent to G if and only if F and G are homotopic in Lack's model structure.*

Proof. First suppose F is pseudo-natually equivalent to G, i.e. $\Phi(F) = [F] = [G] = \Phi(G)$. By the injectivity of Φ, $F = G$ in $\mathsf{GrayCat}[\mathcal{W}^{-1}]$. Since $\mathrm{ev}_{\mathcal{A}}, \mathrm{ev}_{\mathcal{B}} \in \mathcal{W}$, it follows by $(\widehat{2})$

that $\widehat{F} = \widehat{G}$ in $\mathsf{GrayCat}[\mathcal{W}^{-1}]$. By the general theory of model categories, $\mathsf{GrayCat}[\mathcal{W}^{-1}]$ is equivalent to the category of fibrant-cofibrant Gray-categories together with homotopy classes of Gray-functors (see [Hir03, Thm. 8.3.9]). Through this equivalence we obtain that $\widehat{F}$ is homotopic to $\widehat{G}$ since $\widehat{A}$ and $\widehat{B}$ are fibrant-cofibrant. More specifically, there exists a Gray-functor $H\colon \widehat{A} \to \widehat{B}^I$ such that $S \circ H = \widehat{F}$ and $T \circ H = \widehat{G}$.

By (P2), there exists a map $\mathrm{ev}^I_{\mathcal{B}}\colon \widehat{B}^I \to \mathcal{B}^I$ such that the following diagrams commute.

$$\begin{tikzcd}
\widehat{\mathcal{B}}^I \arrow[r, "S"] \arrow[d, "\mathrm{ev}^I_{\mathcal{B}}"'] & \widehat{\mathcal{B}} \arrow[d, "\mathrm{ev}_{\mathcal{B}}"] \\
\mathcal{B}^I \arrow[r, "S"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\widehat{\mathcal{B}}^I \arrow[r, "T"] \arrow[d, "\mathrm{ev}^I_{\mathcal{B}}"'] & \widehat{\mathcal{B}} \arrow[d, "\mathrm{ev}_{\mathcal{B}}"] \\
\mathcal{B}^I \arrow[r, "T"'] & B
\end{tikzcd}$$

As before, since $\mathcal{A}$ is cofibrant, there exists a Gray-functor section $\ell\colon \mathcal{A} \to \widehat{\mathcal{A}}$ for $\mathrm{ev}_{\mathcal{A}}$. By defining the Gray-functor $H'\colon \mathcal{A} \to \mathcal{B}^I$ by $H' \coloneqq \mathrm{ev}^I_{\mathcal{B}} \circ H \circ \ell$ we see that

$$\begin{aligned}
S \circ H' &= \mathcal{A} \xrightarrow{\ell} \widehat{\mathcal{A}} \xrightarrow{H} \widehat{\mathcal{B}}^I \xrightarrow{\mathrm{ev}^I_{\mathcal{B}}} \mathcal{B}^I \xrightarrow{S} \mathcal{B} \\
&= \mathcal{A} \xrightarrow{\ell} \widehat{\mathcal{A}} \xrightarrow{H} \widehat{\mathcal{B}}^I \xrightarrow{S} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \\
&= \mathcal{A} \xrightarrow{\ell} \widehat{\mathcal{A}} \xrightarrow{\widehat{F}} \widehat{\mathcal{B}} \xrightarrow{\mathrm{ev}_{\mathcal{B}}} \mathcal{B} \\
&\underset{②}{=} \mathcal{A} \xrightarrow{\ell} \widehat{\mathcal{A}} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{A} \xrightarrow{F} \mathcal{B} = F
\end{aligned}$$

and similarly $T \circ H' = G$. Thus F, G are homotopic. Conversely, suppose F, G are homotopic so there exists a Gray-functor $H\colon \mathcal{A} \to \mathcal{B}^I$ with $S \circ H = F$ and $T \circ H = G$. By (P1), $(C\colon \mathcal{B} \to \mathcal{B}^I) \in \mathcal{W}$ with $S \circ C = T \circ C$, so $S = T$ in $\mathsf{GrayCat}[\mathcal{W}^{-1}]$. This implies $F = S \circ H = T \circ H = G$ in $\mathsf{GrayCat}[\mathcal{W}^{-1}]$ and by the well-definedness of Φ we have that $[F] = \Phi(F) = \Phi(G) = [G]$. Thus F is pseudo-naturally equivalent to G. $\square$

Chapter 3: Foundations for operator algebraic tricategories

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An operator algebraic tricategory is a higher categorical analogue of an operator algebra. For algebraic tricategories, Gordon, Power, and Street proved that every algebraic tricategory is equivalent to a Gray-category, a result later refined by Gurski. In this chapter, we adapt this result to the context of functional analysis, showing that every operator algebraic tricategory is equivalent to an operator Gray-category. We then categorify the Gelfand-Naimark theorem for operator algebras, inductively proving that every (small) operator algebraic tricategory is equivalent to a concrete operator Gray-category. We also provide several examples of interest for operator algebraic tricategories.

3.1 Introduction

Operator algebras were first conceived in the context of quantum mechanics emerging as concretely C*-algebras and von Neumann algebras of operators acting on a Hilbert space [GN43] [Seg47] [MvN36]. We have since seen a movement throughout mathematics of categorifying structures and their results. Higher operator algebraic categories arise in the study of subfactor theory where, in particular, operator algebras with their bimodules and intertwiners form an operator 2-category. Picking a particular operator algebra A, i.e a C*/W*-algebra, functors from a unitary tensor category $\mathcal{C}$ into $\mathrm{End}(A)$ in this operator 2-category are then quantum categorical/non-invertible symmetries.

Recently, even higher (braided) operator algebraic categories have emerged as invariants of operator algebraic objects, like that of the categorified Connes' $\chi(M)$ [CJP21]. Moreover, operator algebraic 2-categories are expected to form an operator algebraic 3-category [CP22]. Higher operator algebraic categories have also seen important applications in topological order. Starting in a 2+1D topological order, that is a unitary modular tensor category $\mathcal{C}$, the topological domain walls for $\mathcal{C}$ are classified by the fusion 2-category $\mathrm{Mod}(\mathcal{C})$ [KK12]. Condensed matter physics is manifestly unitary, where time reversal symmetry is taking the Hermitian adjoint. We thus desire a definition of unitarity for a fusion 2-category, which can also be viewed as a 3-category with 1 object. Furthermore, physicists want to go even higher — 3+1D topological order is expected to be described by a unitary modular fusion 2-category, for which there is not yet a formal definition in the literature [JF22].

Algebraic models for higher categories do, however, pose difficulties due to their large amount of constraint data. Coherence theorems allow us to consider semi-strictified models of algebraic higher categories while retaining full generality. These semistrictified models are then easier to work with as they require less data. Notably, Gordon, Power, and Street proved that every algebraic tricategory is equivalent to a Gray-category, the strictest model of a general algebraic tricategory [GPS95]. The definition of an algebraic category together with this theorem were later refined by Gurski [Gur13].

Operator algebraic tricategories are tricategorical analogues of operator algebras, for which one would desire an analoguous coherence theorem. However, constructions from ordinary category theory may fail the principle of equivalence [nLa25] for †-categories, as noted in [Pen20] for dual functors on a unitary multifusion category. Moreover, classical results may run into complications when equipping algebraic objects with topologies. For example, the algebraic tensor product of C*-algebras may be equipped with many possible norms, including the maximal and minimal tensor product norms. Out of these only the maximal tensor product satisfies hom-tensor adjunction, while the minimal tensor product is more concrete and proves to be useful in practice.

In this article, we present a definition for operator algebraic tricategories and operator Gray-categories, as well as provide examples of interest. This requires us to construct an operator algebraic Gray tensor product, which uses substantial functional analysis, e.g., the notion of biduals for C*-algebras, and its categorification to C*-categories and C*-2-categories. We then adapt the coherence theorem for algebraic tricategories to the context of functional analysis, obtaining our following main result.

Theorem A. *Every operator algebraic tricategory is equivalent to an operator Gray-category.*

Operator categories, and in particular operator algebras, are said to be concrete when they are made up of Hilbert spaces and operators between them. We inductively extend this notion to higher operator categories, saying that an operator n-category is concrete when it is made up of concrete operator $(n-1)$-categories and higher operators between them. As a corollary of our main result, we provide the following categorification of the Gelfand-Naimark theorem.

Theorem B. *Every (small) operator algebtraic tricategory is equivalent to a concrete operator Gray-category.*

This articles also provides a first stepping stone for future projects in 3-categorical unitary linear algebra. In particular, there is relevant interest in a theory of higher Hilbert spaces and manifestly unitary higher condensation, as well as a definition and classification of unitary dual 2-functors on a unitary multifusion 2-category.

The structure of this article is as follows. In Section §3.2, we construct the operator algebraic versions of the Gray-tensor product. In Section §3.3, we define operator 3-categories, operator Gray-categories, and state our main result. We provide background needed for Section §3.2 in §1.2, as well as defer some details from this section to §1.3. We then construct in detail the operator algebraic Morita 3-categories $\mathsf{AbC^*Alg}$ and $\mathsf{AbW^*Alg}$ in §3.5. Finally, we provide a detailed account of the Yoneda embedding for so-called cubical operator algebraic 3-categories in §3.4.

3.2 2-categorical results

3.2.1 Local tensor product for operator algebraic bicategories

In this section, we use the tensor product of operator categories to obtain a local tensor product of operator 2-categories.

Definition 3.2.1.1. Given C*-2-categories $\mathcal{A}_1$ and $\mathcal{A}_2$, we define the C*-local tensor product $\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2$ by:

(0) Objects are tuples of objects $(A_1, A_2) \in \mathcal{A}_1 \times \mathcal{A}_2$;

(hom) Hom-spaces are given by:

$$\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2\big((A_1, A_2) \to (A_1, B_2)\big) \coloneqq \mathcal{A}_1(A_1 \to B_1) \underset{\max}{\otimes} \mathcal{A}_2(A_2 \to B_2),$$

where we are using the maximal tensor product of C*-categories (see §1.2.5).

($\odot$) Composition $\odot$ of 1-morphisms is given pointwise and 1-composition $\odot$ of 2-morphisms is determined using the universal property of $\underset{\max}{\otimes}$. In particular

$$(x \otimes y) \odot (x' \otimes y') = (x \odot x') \otimes (y \odot y'),$$

whenever it makes sense.

($\circ$) Composition $\circ$ of 2-morphism is defined similarly.

(coh) Constraint data is given by simple tensors of constraint data in $\mathcal{A}_1$ and $\mathcal{A}_2$. These are indeed natural by the bilinearity and bicontinuity of $\odot$ and $\circ$.

Similarly, when $\mathcal{A}_1$ and $\mathcal{A}_2$ are W*-2-categories, we define the W*-2-category $\mathcal{A}_1 \underset{\max}{\overline{\otimes}} \mathcal{A}_2$ by replacing $\underset{\max}{\otimes}$ with W*-tensor product $\underset{\max}{\overline{\otimes}}$ (see Appendix 1.2.7).

Remark 3.2.1.2. When $\mathcal{A}_1$ and $\mathcal{A}_2$ are strict operator 2-categories, their local tensor product is also strict.

3.2.2 C*-Gray tensor product

In this section, we adapt [Gur13, §3] to the operator algebraic setting.

Definition 3.2.2.1. Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{B}$ be strict C*-2-categories. A †-2-functor $F\colon \mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ is *cubical* if it is strictly identity-preserving and the following condition holds:

(⧉) If (X_1, X_2) and (Y_1, Y_2) are composable 1-morphisms in $\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2$ such that Y_2 or X_1 is an identity, then the compositor 2-morphism

$$F(X_1, X_2) \odot F(Y_1, Y_2) \Rightarrow F\big((X_1, X_2) \odot (Y_1, Y_2)\big).$$

is an identity.

When $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}$ are W*, we say that a †-2-functor $F\colon \mathcal{A}_1 \underset{\max}{\overline{\otimes}} \mathcal{A}_2 \to \mathcal{B}$ is a *separately normal* cubical functor if it is both normal and cubical. In particular, a cubical functor $F\colon \mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ extends to a separately normal cubical functor only when $\alpha_i^\lambda \to \alpha_i$ weak* in $\mathcal{A}_i$ for $i = 1, 2$ implies

$$F(\alpha_1^\lambda \otimes \alpha_2) \to F(\alpha_1 \otimes \alpha_2) \text{ and } F(\alpha_1 \otimes \alpha_2^\lambda) \to F(\alpha_1 \otimes \alpha_2) \text{ weak* in } \mathcal{B}.$$

Proposition 3.2.2.2. *The data of a (separately normal) cubical* †*-2-functor* $F\colon \mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ *is determined uniquely by:*

(1) *For each object* $A_1 \in \mathcal{A}_1$*, a strict (normal)* †*-2-functor* $F_{A_1}\colon \mathcal{A}_2 \to \mathcal{B}$*;*

(2) *For each object* $A_2 \in \mathcal{A}_2$*, a strict (normal)* †*-2-functor* $F_{A_2}\colon \mathcal{A}_1 \to \mathcal{B}$*;*

such that

$$F_{A_1}(A_2) = F_{A_2}(A_1) \coloneqq F(A_1, A_2);$$

(Σ) *For each pair of 1-morphisms* $X_i\colon A_i \to A'_i$ *in* $\mathcal{A}_i$ *for* $i = 1, 2$*, an "interchanger" unitary 2-morphism*

$$\begin{tikzcd}
F(A_1, A_2) \arrow[r, "F_{A_1}(X_2)"] \arrow[d, "F_{A_2}(X_1)"'] & F(A_1, A'_2) \arrow[d, "F_{A'_2}(X_1)"] \arrow[dl, Rightarrow, "\Sigma_{X_1,X_2}" description] \\
F(A'_1, A_2) \arrow[r, "F_{A'_1}(X_2)"'] & F(A'_1, A'_2)
\end{tikzcd}$$

which is an identity 2-morphism whenever X_1 *or* X_2 *is an identity 1-morphism;*

satisfying the following three axioms:

(Σ1) *Naturality: For each pair of 2-morphisms* α_1, α_2 *in* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *respectively*

$$\begin{tikzcd}
F(A_1, A_2) \arrow[r] \arrow[d] \arrow[d, bend right=80, "\Leftarrow F_{A_2}\alpha_1 ="] & F(A_1, A'_2) \arrow[d] \arrow[dl, Rightarrow, "\Sigma" description] \\
F(A'_1, A_2) \arrow[r, "\shortparallel\ F_{A'_1}\alpha_2\ \Downarrow"'] \arrow[r, bend right=60] & F(A'_1, A'_2)
\end{tikzcd}
=
\begin{tikzcd}
F(A_1, A_2) \arrow[r] \arrow[r, bend left=60] \arrow[d] & F(A_1, A'_2) \arrow[d] \arrow[d, bend left=80, "\Leftarrow F_{A'_2}\alpha_1 ="'] \arrow[dl, Rightarrow, "\Sigma" description] \\
F(A'_1, A_2) \arrow[r] & F(A'_1, A'_2)
\end{tikzcd}$$

($\Sigma 2$) *Composability in the first entry:*

$$\begin{tikzcd}
F(A_1, A_2) \arrow[r] \arrow[d] & F(A_1, A_2') \arrow[d] \arrow[dl, Rightarrow, "\Sigma" description] & & F(A_1, A_2) \arrow[r] \arrow[d] & F(A_1, A_2') \arrow[d] \arrow[ddl, Rightarrow, "\Sigma" description] \\
F(A_1', A_2) \arrow[r] \arrow[d] & F(A_1', A_2') \arrow[d] \arrow[dl, Rightarrow, "\Sigma" description] & = & F(A_1', A_2) \arrow[d] & F(A_1', A_2') \arrow[d] \\
F(A_1'', A_2') \arrow[r] & F(A_1'', A_2') & & F(A_1'', A_2') \arrow[r] & F(A_1'', A_2')
\end{tikzcd}$$

($\Sigma 3$) *Composability in the second entry:*

$$\begin{tikzcd}
F(A_1, A_2) \arrow[r] \arrow[d] & F(A_1, A_2') \arrow[r] \arrow[d] \arrow[dl, Rightarrow, "\Sigma" description] & F(A_1, A_2'') \arrow[d] \arrow[dl, Rightarrow, "\Sigma" description] & & F(A_1, A_2) \arrow[r] \arrow[d] & F(A_1, A_2') \arrow[r] & F(A_1, A_2'') \arrow[d] \arrow[dll, Rightarrow, "\Sigma" description] \\
F(A_1', A_2) \arrow[r] & F(A_1', A_2') \arrow[r] & F(A_1', A_2'') & = & F(A_1', A_2) \arrow[r] & F(A_1', A_2') \arrow[r] & F(A_1', A_2'')
\end{tikzcd}$$

Proof. First suppose we have a (separately normal) cubical †-2-functor F. For an object $A_1 \in \mathcal{A}_1$ we define

$$F_{A_1}(A_2) \coloneqq F(A_1, A_2),$$

$$F_{A_1}(X) \coloneqq F(\mathrm{id}_{A_1}, X),$$

$$F_{A_1}(\alpha) \coloneqq F(\mathrm{id}_{\mathrm{id}_{A_1}} \otimes \alpha).$$

Then $F_{A_1} \colon \mathcal{A}_2 \to \mathcal{B}$ strictly preserves identity 1-morphisms since F does. Moreover,

$$F_{A_1}(Y_1 \odot Y_2) = F(\mathrm{id}_{A_1}, Y_1 \odot Y_2) \underset{(\square)}{=} F(\mathrm{id}_{A_1}, Y_1) \odot F(\mathrm{id}_{A_1}, Y_2) = F_{A_1}(Y_1) \odot F_{A_1}(Y_2),$$

so F_{A_1} is a (normal) strict †-2-functor. We define $F_{A_2}\colon \mathcal{A}_1 \to \mathcal{B}$ similarly for an object $A_2 \in \mathcal{A}_2$. We define the unitary Σ as follows:

$$\begin{tikzcd}
F(A_1,A_2) \arrow[r, "F_{A_1}(X_2)"] \arrow[d, "F_{A_2}(X_1)"'] & F(A_1,A_2') \arrow[d, "F_{A_2'}(X_1)"] \arrow[dl, Rightarrow, "\Sigma_{X_1,X_2}" description] \\
F(A_1',A_2) \arrow[r, "F_{A_1'}(X_2)"'] & F(A_1',A_2')
\end{tikzcd}
=
\begin{tikzcd}
F(A_1,A_2) \arrow[r, "F(\mathrm{id},X_2)"] \arrow[d, "F(X_1,\mathrm{id})"'] \arrow[dr, "F(X_1,X_2)" description] & F(A_1,A_2') \arrow[d, "F(X_1,\mathrm{id})"] \arrow[dl, Rightarrow, shorten >=40pt, "F^2"'] \\
F(A_1',A_2) \arrow[r, "F(\mathrm{id},X_2)"'] \arrow[ur, Rightarrow, shorten >=40pt, "F^2"'] & F(A_1',A_2')
\end{tikzcd}$$

$$\underset{(⧉)}{=}
\begin{tikzcd}
F(A_1,A_2) \arrow[r, "F(\mathrm{id},X_2)"] \arrow[dr, "F(X_1,X_2)"'] & F(A_1,A_2') \arrow[d, "F(X_1,\mathrm{id})"] \arrow[dl, Rightarrow, shorten >=30pt, "F^2"'] \\
 & F(A_1',A_2')
\end{tikzcd}$$

which is an identity 2-morphism when X_1 or X_2 is an identity by the cubicality (⧉) of F again. Axiom ($\Sigma 1$) follows by the naturality of the compositor F^2, and axioms ($\Sigma 2$) and ($\Sigma 3$) follow by the associativity axiom F^2 satisfies.

Conversely, suppose we are given such collections of (normal) †-2-functors and unitaries Σ. We construct $F\colon \mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ as follows.

(0) For objects, $F(A_1,A_2) := F_{A_1}(A_2) = F_{A_2}(A_1)$;

(hom) We first define $F_{\mathsf{hom}} : \mathcal{A}_1(A_1 \to A_1') \times \mathcal{A}_2(A_2 \to A_2') \to \mathcal{B}\big((F(A_1,A_2) \to F(A_1',A_2')\big)$ by

$$F(X_1,X_2) := F_{A_2}(X_1) \odot F_{A_1'}(X_2)$$
$$F(\alpha_1 \otimes \alpha_2) := F_{A_2}(\alpha_1) \odot F_{A_1'}(\alpha_2).$$

Since F_{hom} is (separately normal) †-bilinear and $\mathcal{B}\big((F(A_1,A_2) \to F(A_1',A_2')\big)$ is C*/W*, we may uniquely extend F_{hom} to

$$\mathcal{A}_1(A_1 \to A_1') \underset{\max}{\otimes} \mathcal{A}_2(A_2 \to A_2') = \mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2\big((A_1,A_2) \to (A_1',A_2')\big)$$

by the universal property of the maximal C*-tensor product $\underset{\max}{\otimes}$ (see §1.2.5).

We set the unitor F^0 to be identity and define the compositor component

$$F^2\colon F(X_1, X_2) \odot F(X_1', X_2') \Rightarrow F(X_1 \odot X_1', X_2 \odot X_2')$$

for $A_i \overset{X_i}{\Longrightarrow} A_i' \overset{X_i'}{\Longrightarrow} A_i''$ in $\mathcal{A}_i$ to be

$$F_{A_2}(X_1) \odot F_{A_1'}(X_2) \odot F_{A_2'}(X_1') \odot F_{A_2''}(X_2') \xRightarrow{\mathrm{id} \odot \Sigma_{X_2, X_1'} \odot \mathrm{id}} F_{A_2}(X_1) \odot F_{A_2}(X_1') \odot F_{A_1''}(X_2) \odot F_{A_2''}(X_2').$$

The naturality of F^2 is guaranteed by (Σ1) and the associativity axiom is given by (Σ2) and (Σ3). Finally, F is cubical since $\Sigma_{X_1, X_2'}$ is trivial whenever X_1 or X_2' is the identity. $\square$

We now explicitly construct the Gray tensor product for strict C*-2-categories.

Definition 3.2.2.3. The algebraic Gray tensor product $\mathcal{A} \boxtimes \mathcal{B}$ of strict C*-2-categories $\mathcal{A}$ and $\mathcal{B}$ is composed of tuples of objects $(A, B) \in \mathcal{A} \times \mathcal{B}$. The 1-morphisms of $\mathcal{A} \boxtimes \mathcal{B}$ are produced by two kinds of generators:

- $(X, \mathrm{id}_B) : (A, B) \to (A', B)$ for $X\colon A \to A'$ in $\mathcal{A}$,
- $(\mathrm{id}_A, Y) : (A, B') \to (A, B')$ for $Y : B \to B'$ in $\mathcal{B}$.

The 1-morphisms in $\mathcal{A} \boxtimes \mathcal{B}$ are equivalence classes of composable strings of generators. The equivalence relation is the smallest such that:

- $(X, \mathrm{id}_B)(X', \mathrm{id}_B) \sim (X \odot X', \mathrm{id}_B)$ for $A \xrightarrow{X} A' \xrightarrow{X'} A''$ in X and $B \in \mathcal{B}$,
- $(\mathrm{id}_A, Y)(\mathrm{id}_A, Y') \sim (\mathrm{id}_A, Y \odot Y')$ for $A \in X$ and $B \xrightarrow{Y} B' \xrightarrow{Y'} B''$ in Y,
- If $W \sim W'$, then $WV \sim W'V$ and $VW \sim VW'$ whenever they make sense.

We define the composition $\odot$ of 1-morphisms to be string concatenation. Notice $W \sim W'$ only when W, W' have the same source and target, and $(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{(A,B)}$.

The 2-morphisms of $\mathcal{A} \boxtimes \mathcal{B}$ are generated by three kinds of 2-morphisms:

- $x \otimes \mathrm{id}_{\mathrm{id}_B} : (X, \mathrm{id}_B) \Rightarrow (X', \mathrm{id}_B)$ for $x : X \Rightarrow X'$ in $\mathcal{A}$ and $B \in \mathcal{B}$,
- $\mathrm{id}_{\mathrm{id}_A} \otimes y : (\mathrm{id}_A, Y) \Rightarrow (\mathrm{id}_A, Y')$ for $A \in \mathcal{A}$ and $y : Y \Rightarrow Y'$ in $\mathcal{B}$,
- $\Sigma_{X,Y} : (X, \mathrm{id}_B) \odot (\mathrm{id}_A, Y) \overset{\sim}{\Rightarrow} (\mathrm{id}_A, Y) \odot (X, \mathrm{id}_B)$ for 1-morphisms $X \colon A \to A'$ in $\mathcal{A}$ and $Y \colon B \to B'$ in Y, such that $\Sigma_{\mathrm{id}_A, Y} = \mathrm{id}_{(\mathrm{id}_A, Y)}$ and $\Sigma_{X, \mathrm{id}_B} = \mathrm{id}_{(X, \mathrm{id}_B)}$.

We will omit the subscripts on id's for simplicity. The 2-morphisms in $\mathcal{A} \boxtimes \mathcal{B}$ include equivalence classes of strings of formal $\odot$-composites of generators. The equivalence relation is first defined horizontally (in terms of $\odot$-composites), then vertically (as strings). We first define $\sim$ as the smallest equivalence relation such that

- $(x \otimes \mathrm{id}) \odot (x' \otimes \mathrm{id}) \sim (x \odot x') \otimes \mathrm{id}$ and $(\mathrm{id} \otimes y) \odot (\mathrm{id} \otimes y') \sim \mathrm{id} \otimes (y \odot y')$,
- If $s \sim s'$ then $s \odot t \sim s' \odot t$ and $t \odot s \sim t \odot s'$ whenever they make sense.

Notice $s \sim s'$ only when s and s' have the same source and target morphisms. In what follows, we will denote the equivalence class of s by $[s]$. A 2-morphism in $\mathcal{A} \boxtimes \mathcal{B}$ is then an equivalence class of formal linear combinations of vertically composable strings $\sum_k \lambda_k [w_{1,k}] \cdots [w_{n,k}]$, where the equivalence relation is the smallest such that:

- $\lambda[x \otimes \mathrm{id}] + [\widetilde{x} \otimes \mathrm{id}] \sim [(\lambda x + \widetilde{x}) \otimes \mathrm{id}]$ and $\lambda[\mathrm{id} \otimes y] + [\mathrm{id} \otimes \widetilde{y}] \sim [\mathrm{id} \otimes (\lambda y + \widetilde{y})]$ for $\lambda \in \mathbb{C}$,
- $[x \otimes \mathrm{id}][x' \otimes \mathrm{id}] \sim [(x \circ x') \otimes \mathrm{id}]$, and $[\mathrm{id} \otimes y][\mathrm{id} \otimes y'] \sim [\mathrm{id} \otimes (y \circ y')]$

(Σ1) $[(\mathrm{id} \otimes y) \odot (x \otimes \mathrm{id})][\Sigma_{X,Y}] \sim [\Sigma_{X',Y'}][(x \otimes \mathrm{id}) \odot (\mathrm{id} \otimes y)]$

(Σ2) $[(\mathrm{id}_{X'} \otimes \mathrm{id}) \odot \Sigma_{X,Y}][\Sigma_{X',Y} \odot (\mathrm{id}_X \otimes \mathrm{id})] \sim [\Sigma_{X \odot X', Y}]$

(Σ3) $[\Sigma_{X,Y'} \odot (\mathrm{id} \otimes \mathrm{id}_Y)][(\mathrm{id} \otimes \mathrm{id}_{Y'}) \odot \Sigma_{X,Y}] \sim [\Sigma_{X, Y \odot Y'}]$

- For general 2-morphisms w, v, $(w+\widetilde{w})v \sim wv + \widetilde{w}v$, and $v(w+\widetilde{w}) \sim vw + v\widetilde{w}$ whenever they make sense, and similar relations for $\odot$; and
- if $w \sim w'$, then $wv \sim w'v$, $vw \sim vw'$, and $\lambda w + v \sim \lambda w' + v$ for $\lambda \in \mathbb{C}$ whenever they make sense.

Vertical composition $\circ$ of 2-morphisms is given by the bilinear extension of concatenation of strings. For horizontal composition of strings w, v, we can always express w and w' by the sum of strings $[w_{1,k}] \cdots [w_{n,k}]$ and $[v_{1,k}] \cdots [v_{n,k}]$ of the same length (by adding identities) and then define

$$w \odot v = \sum_{k,\ell} [w_{1,k} \odot v_{1,\ell}] \cdots [w_{n,k} \odot v_{n,\ell}].$$

We define $\dagger$ on generator 2-morphisms as follows:

- $(x \otimes \mathrm{id})^\dagger = x^\dagger \otimes \mathrm{id}$,
- $(\mathrm{id} \otimes y)^\dagger = \mathrm{id} \otimes y^\dagger$,
- $\Sigma_{X,Y}^\dagger = \Sigma_{X,Y}^{-1}$.

We then extend $\dagger$ to sums, tensors, and composites by:

- $(\sum_k \lambda_k [w_k])^\dagger = \sum \overline{\lambda}_k [w_k]^\dagger$
- $(w \circ v)^\dagger = v^\dagger \circ w^\dagger$
- $(w \odot v)^\dagger = w^\dagger \odot v^\dagger$.

We then define the C*-Gray tensor product $\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$ to be the completion of $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ on each hom-space with respect to the following maximal C*-norm

$$\|w\|_\mu \coloneqq \sup \left\{ \|F(w)\| \mid F\colon \mathcal{A} \boxtimes \mathcal{B} \to \mathrm{C}^*\mathsf{Cat} \text{ is a } \dagger\text{-2-functor} \right\},$$

after quotienting out by 2-morphisms w such that $\|w\|_\mu = 0$.

Remark 3.2.2.4. There indeed exists a †-2-functor

$$F\colon \mathcal{A} \boxtimes \mathcal{B} \to \mathrm{C}^*\mathsf{Cat}.$$

For example, we may use the universal representations $\text{よ}^{\mathrm{II}}_{\mathcal{A}}\colon \mathcal{A} \to \mathrm{C}^*\mathsf{Cat}$ and $\text{よ}^{\mathrm{II}}_{\mathcal{B}}\colon \mathcal{B} \to \mathrm{C}^*\mathsf{Cat}$ in Theorem 1.3.4.4 to determine a strict †-2-functor

(0) on objects by

$$(A, B) \mapsto \text{よ}^{\mathrm{II}}_{\mathcal{A}}(A) \underset{\max}{\otimes} \text{よ}^{\mathrm{II}}_{\mathcal{B}}(B);$$

(1) on 1-morphisms by

$$\begin{aligned} (X, \mathrm{id}) &\mapsto \text{よ}^{\mathrm{II}}_{\mathcal{A}}(X) \underset{\max}{\otimes} \mathrm{id}, \\ (\mathrm{id}, Y) &\mapsto \mathrm{id} \underset{\max}{\otimes} \text{よ}^{\mathrm{II}}_{\mathcal{B}}(Y); \end{aligned}$$

(2) on 2-morphisms by

$$\begin{aligned} x \otimes \mathrm{id} &\mapsto \text{よ}^{\mathrm{II}}_{\mathcal{A}}(x) \underset{\max}{\otimes} \mathrm{id}, \\ \mathrm{id} \otimes y &\mapsto \mathrm{id} \underset{\max}{\otimes} \text{よ}^{\mathrm{II}}_{B}(y), \\ \Sigma_{X,Y} &\mapsto \mathrm{id}_{\text{よ}^{\mathrm{II}}_{\mathcal{A}}(X) \underset{\max}{\otimes} \text{よ}^{\mathrm{II}}_{\mathcal{B}}(Y)}. \end{aligned}$$

Remark 3.2.2.5. We expect there to exist a "minimal" Gray-tensor product $\mathcal{A} \underset{\min}{\boxtimes} \mathcal{B}$ of strict C*-2-categories $\mathcal{A}$ and $\mathcal{B}$, given by completing $\mathcal{A} \boxtimes \mathcal{B}$ via a monic †-2-functor

$$\mathcal{A} \boxtimes \mathcal{B} \to \mathrm{C}^*\mathsf{Cat}.$$

The existence of such a monic representation would imply that there do not exist negligible 2-morphisms in $\mathcal{A} \boxtimes \mathcal{B}$. However, this will not be necessary for our main coherence and concreteness results.

Remark 3.2.2.6. For each object $B \in \mathcal{B}$, there is an organic strict †-2-functor

$$- \boxtimes B : \mathcal{A} \to \mathcal{A} \boxtimes \mathcal{B}$$

given by

$$\begin{aligned} A &\mapsto (A, B), \\ X &\mapsto (X, \mathrm{id}_B), \\ x &\mapsto x \otimes \mathrm{id}_{\mathrm{id}_B}, \end{aligned}$$

on objects, 1-morphisms, and 2-morphisms in $\mathcal{C}$ respectively. Similarly, each object $A \in \mathcal{A}$ induces a strict †-2-functor

$$A \boxtimes - : \mathcal{B} \to \mathcal{A} \boxtimes \mathcal{B}.$$

Proposition 3.2.2.7. *For strict C*-2-categories $\mathcal{A}$ and $\mathcal{B}$, $\| \cdot \|_\mu$ is a norm on each hom-space of $\mathcal{A} \boxtimes \mathcal{B}$. Furthermore, $\mathcal{A} \underset{max}{\boxtimes} \mathcal{B}$ is a strict C*-2-category.*

Proof. A simple verification reveals that $\| \cdot \|_\mu$ is a C*-norm, possibly taking value ∞. For example,

$$\|w\|_\mu = \sup_F \|F(w)\| = \sup_F \|F(w)F(w)^\dagger\|^{1/2} = \sup_F \|F(ww^\dagger)\|^{1/2} = \|ww^\dagger\|_\mu^{1/2}$$

for every 2-morphism w in $\mathcal{A} \boxtimes \mathcal{B}$. Furthermore, $\| \cdot \|_\mu$ is sub-cross by the naturality and unitality of compositors. Indeed, if $\sigma \colon {}_A X_B \to {}_A Y_B$ and $\sigma' \colon {}_B X'_C \to {}_B Y'_C$, then

$$\begin{aligned} \|\sigma \odot \sigma'\|_\mu = \sup_F \|F(\sigma \odot \sigma')\| = \sup_F \|F^2(F\sigma \odot F\sigma')F^{-2}\| \leq \sup_F \|F\sigma \odot F\sigma'\| \\ \leq \sup_F \|F\sigma\| \|F\sigma'\| \leq \|\sigma\|_\mu \|\sigma'\|_\mu. \end{aligned}$$

Thus, to prove $\| \cdot \|_\mu < \infty$ on each hom-set, it suffices to verify that this is the case for generator 2-morphisms $\alpha \otimes \mathrm{id}$, $\mathrm{id} \otimes \beta$, and $\Sigma_{f,g}$. First observe that, for every object $B \in \mathcal{B}$

and 2-morphism α in $\mathcal{A}$, we have

$$\|F(\alpha \otimes \mathrm{id}_{\mathrm{id}_B})\| = \|F \circ (- \boxtimes B)(\alpha)\| \leq \|\alpha\| \quad \text{for every } F : \mathcal{A} \boxtimes \mathcal{B} \to \mathrm{C}^*\mathsf{Cat}.$$

Therefore $\|\alpha \otimes \mathrm{id}\|_\mu \leq \|\alpha\|$. Similarly, one shows $\|\mathrm{id} \otimes \beta\|_\mu \leq \|\beta\|$. Finally, each $\|\Sigma_{f,g}\|_\mu = 1$ since $\Sigma_{f,g}$ is unitary and †-2-functors preserve unitaries. Therefore $\|\cdot\|_\mu$ is a norm on each hom-space. It remains to show that each hom-category

$$\mathfrak{H} := \mathcal{A} \underset{\max}{\boxtimes} \mathcal{B}\big((A_1, A_2) \to (B_1, B_2)\big)$$

satisfies the positivity condition required of C*-categories (see §1.2.2). We already know that each endomorphism algebra in $\mathfrak{H}$ is a C*-algebra. So for any morphism τ in $\mathfrak{H}$ (which is a 2-morphism in $\mathcal{A} \underset{\max}{\boxtimes} \mathcal{B}$), $\tau^* \circ \tau$ is contained in such an endomorphism C*-algebra $\mathcal{E}$. So there exists a positive $\sigma \in \mathcal{E}$ such that $\sigma^4 = \tau^* \circ \tau \circ \tau^* \circ \tau$. We claim that $\sigma^2 = \tau^* \circ \tau$. Let $F \colon \mathcal{A}_1 \boxtimes \mathcal{A}_2 \to \mathrm{C}^*\mathsf{Cat}$ be a †-2-functor, which we may extend to a representation of $\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$ by continuity. The fact that $\sigma \geq 0$ implies that $F(\sigma^2) \geq 0$. Now notice $F(\sigma^2)^2 = F(\tau^*\tau)^2$. Since $\mathrm{C}^*\mathsf{Cat}$ is a C*-2-category, $F(\tau^*\tau) = F(\tau)^*F(\tau) \geq 0$. Since $F\mathcal{E}$ is a C*-algebra, the uniqueness of positive square roots yields $F(\sigma^2) = F(\tau^*\tau)$. Since F was arbitrary, we conclude $\sigma^2 = \tau^*\tau$. Therefore $\mathfrak{H}$ is a C*-category and we conclude that $\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$ is a C*-2-category. □

Universal Property 3.2.2.8. *For strict C*-2-categories $\mathcal{A}_1$ and $\mathcal{A}_2$, there is a natural cubical †-2-functor*

$$\mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2 \to \mathcal{A}_1 \underset{max}{\boxtimes} \mathcal{A}_2.$$

For any cubical functor $F : \mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ into a strict C-2-category $\mathcal{B}$, there exists a unique strict †-2-functor*

$$\begin{tikzcd}
\mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2 \arrow[r, "F"] \arrow[d] & \mathcal{B} \\
\mathcal{A}_1 \underset{max}{\boxtimes} \mathcal{A}_2 \arrow[ur, dashed, "F"'] &
\end{tikzcd}$$

Verification. We provide the data of the cubical functor $C : \mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \to \mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$ using Proposition 3.2.2.2. For objects $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, we define the strict †-2-functors

$$C_{A_2} \colon \mathcal{A}_1 \to \mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$$
$$C_{A_1} \colon \mathcal{A}_2 \to \mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$$

to be $- \boxtimes A_2$ and $A_1 \boxtimes -$ as in Remark 3.2.2.6. For a pair 1-morphisms X_1 and X_2 in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, the unitary 2-morphism Σ_{X_1,X_2} in $\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$ serves the role of Σ_{X_1,X_2} in Proposition 3.2.2.2. This data satisfies all of the desired axioms by construction of the algebraic Gray tensor product. So we have successfully defined $C \colon \mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \to \mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$.

Let $F \colon \mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ be a cubical †-2-functor, with data of F as in Proposition 3.2.2.2. In what follows, we denote the interchanger of F by Σ^F. We first determine a strict †-2-functor $F \colon \mathcal{A}_1 \boxtimes \mathcal{A}_2 \to \mathcal{B}$:

(0) on objects,

$$F(A_1, A_2) := F_{A_1}(A_2) = F_{A_2}(A_1);$$

(1) on generator 1-morphisms,

$$F(X, \mathrm{id}_{A_2}) := F_{A_2}(X),$$
$$F(\mathrm{id}_{A_1}, Y) := F_{A_1}(Y);$$

(2) on generator 2-morphisms by

$$F(x \otimes \mathrm{id}_{\mathrm{id}_{A_2}}) := F_{A_2}(x)$$
$$F(\mathrm{id}_{\mathrm{id}_{A_1}} \otimes y) := F_{A_1}(y)$$
$$F(\Sigma_{X,Y}) := \Sigma^F_{X,Y}.$$

We extend F to all of $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ as a strict †-2-functor, which is well-defined by construction.

Now consider some arbitrary 2-morphism w in $\mathcal{A}_1 \boxtimes \mathcal{A}_2$. Let $\text{よ}^{\mathrm{II}} \colon \mathcal{B} \to \mathrm{C}^*\mathsf{Cat}$ be the universal representation on $\mathcal{B}$, which is faithful on all levels. Notice that よ^{II} is isometric on 2-morphisms by the C*-identity and the fact that injective maps between (endomorphism) C*-algebras are isometric. Since $\text{よ}^{\mathrm{II}} \circ F \colon \mathcal{A}_1 \boxtimes \mathcal{A}_2 \to \mathrm{C}^*\mathsf{Cat}$ is a †-2-functor, we obtain

$$\|F(w)\| = \|(\text{よ}^{\mathrm{II}} \circ F)(w)\| \leq \|w\|_\mu.$$

Therefore, we may uniquely extend F to $\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$ by continuity. □

Proposition 3.2.2.9 (Unitary hom-tensor Adjunction)**.** *For strict C*-2-categories $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{B}$, we have that the following strict C*-2-categories are unitarily naturally equivalent:*

$$\mathrm{C}^*2\mathsf{Cat}_{\mathsf{st}}(\mathbb{C} \to \mathcal{B}) \cong \mathcal{B},$$

$$\mathrm{C}^*2\mathsf{Cat}_{\mathsf{st}}(\mathcal{A}_1 \underset{max}{\boxtimes} \mathcal{A}_2 \to \mathcal{B}) \cong \mathrm{C}^*2\mathsf{Cat}_{\mathsf{st}}(\mathcal{A}_1 \to \mathrm{C}^*2\mathsf{Cat}_{\mathsf{st}}(\mathcal{A}_2 \to \mathcal{B})).$$

Proof. We will merely sketch the assignments at the level of objects for both isomorphisms. For the first isomorphism, note that we are viewing $\mathbb{C} = B^2\mathbb{C}$ as a C*-2-category with one object $\bullet$, a single 1-morphism $\mathrm{id}_\bullet$, with

$$\mathbb{C}(\mathrm{id}_\bullet \Rightarrow \mathrm{id}_\bullet) \coloneqq \mathbb{C}.$$

Composition is given by multiplication and † is given by conjugation. From this it is easy to see that the assignment

$$(F \colon \mathbb{C} \to \mathcal{B}) \mapsto F(\bullet)$$

extends to a strict †-2-functor which is bijective on all levels. For the latter isomorphism, one uses Proposition 3.2.2.2 to produce a bijective correspondence

$$(F \colon \mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2 \to \mathcal{B}) \mapsto (A_1 \in \mathcal{A}_1 \mapsto (F_{A_1} \colon \mathcal{A}_2 \to \mathcal{B})).$$

Here we have omitted the assignment of 1-morphisms and 2-morphisms. We do however note that the interchanger Σ is used to form part of the naturator unitary for the †-2-natural transformation corresponding to a 1-morphism X_1 in $\mathcal{A}_1$. The naturality axiom for this †-2-natural transformation follows by (Σ1) and (Σ3), while the strictness of this †-2-functor follows by cubicality and (Σ2). For 2-morphisms in $\mathcal{A}_1$, the obviously assigned uniformly bounded modification in $\mathrm{C}^*\mathsf{2Cat}_{\mathsf{st}}(\mathcal{A}_2 \to \mathcal{B})$ satisfies the modification axiom by (Σ1) as well. $\square$

We obtain the following result as a corollary of [EK66, §II.3,4].

Corollary 3.2.2.10. $\mathrm{C}^*\mathsf{Gray} \coloneqq (\mathrm{C}^*\mathsf{2Cat}_{\mathsf{st}}, \underset{max}{\boxtimes})$ *forms a closed symmetric monoidal category.*

3.2.3 W*-completion of a C*-2-category

For a C*-2-category $\mathcal{A}$, we wish to construct the enveloping W*-2-category $\mathrm{W}^*(\mathcal{A})$ together with a monic †-2-functor $\mathcal{A} \hookrightarrow \mathrm{W}^*(\mathcal{A})$, which satisfies the following universal property:

- For every †-2-functor $F\colon \mathcal{A} \to \mathcal{B}$ into a W*-2-category $\mathcal{B}$, there exists a unique normal extension making the following diagram commute:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, hook] & \mathcal{B} \\
\mathrm{W}^*(\mathcal{A}) \arrow[ur, dashed, "\exists! \widetilde{F}"'] &
\end{tikzcd}$$

Consider the C*-category enriched graph $\mathcal{A}^{**}$ with vertices $\mathsf{Ob}\,\mathcal{A}$ and edges

$$\mathcal{A}^{**}(A \to B) \coloneqq \mathcal{A}(A \to B)^{**} \qquad \text{for } A, B \in \mathcal{A}.$$

Here $\mathcal{A}(A \to B)^{**}$ is the enveloping W*-category described in §1.2.6. We define two Arens 1-compositions on $\mathcal{A}^{**}$, which serve to equip $\mathcal{A}^{**}$ with the structure of a †-2-category.

Definition 3.2.3.1. For 1-morphisms in $\mathcal{A}^{**}$, we define the left and right Arens 1-compositions $\odot_\ell$ and $\odot_r$ to act as 1-composition $\odot$ on $\mathcal{A}$.

For 1-composable 2-morphisms $\Phi \in \mathcal{A}^{**}({}_AX_B \Rightarrow {}_AY_B)$ and $\Psi \in \mathcal{A}^{**}({}_BX'_C \Rightarrow {}_BY'_C)$, we define

$$\Phi \odot_\ell \Psi,\ \Phi \odot_r \Psi \in \mathcal{A}^{**}({}_AX \odot_B X'_C \Rightarrow {}_AY \odot_B Y'_C)$$

(ℓ) For $\varphi \in \mathcal{A}({}_AX \odot X'_C \Rightarrow {}_AY \odot Y'_C)^*$, we set $(\Phi \odot_\ell \Psi)(\varphi) := \Phi(\varphi \triangleleft \Psi)$ where

$$\varphi \triangleleft \Psi \in \mathcal{A}({}_AX_B \Rightarrow {}_AY_B)^*$$

is given by:

($\triangleleft$) For $a \in \mathcal{A}({}_AX_B \Rightarrow {}_AY_B)$, we set $(\varphi \triangleleft \Psi)(a) := \Psi(a \triangleright \varphi)$ where

$$a \triangleright \varphi \in \mathcal{A}({}_BX'_C \Rightarrow {}_BY'_C)^*$$

is given by:

($\triangleright$) For $b \in \mathcal{A}({}_BX'_C \Rightarrow {}_BY'_C)$, we set $(a \triangleright \varphi)(b) := \varphi(a \odot b)$.

More succinctly, $\Phi \odot_\ell \Psi$ is given by the following formula:

$$\begin{aligned} \Phi \odot_\ell \Psi &= \varphi \mapsto \Phi\Big(\varphi \triangleleft \Psi\Big), \\ &= \varphi \mapsto \Phi\Big(a \mapsto \Psi\big(a \triangleright \varphi\big)\Big), \\ &= \varphi \mapsto \Phi\Big(a \mapsto \Psi\big(b \mapsto \varphi(a \odot b)\big)\Big). \end{aligned}$$

(r) For $\varphi \in \mathcal{A}({}_AX \odot X'_C \Rightarrow {}_AY \odot Y'_C)^*$, we set $(\Phi \odot_r \Psi)(\varphi) := \Psi(\Phi \triangleright \varphi)$ where

$$\Phi \triangleright \varphi \in \mathcal{A}({}_BX'_C \Rightarrow {}_BY'_C)^*$$

is given by:

($\triangleright$) For $b \in \mathcal{A}({}_BX'_C \Rightarrow {}_BY'_C)$, we set $(\Phi \triangleright \varphi)(b) := \Phi(\varphi \triangleleft b)$ where

$$\varphi \triangleleft b \in \mathcal{A}({}_AX_B \Rightarrow {}_AY_B)^*$$

is given by:

($\triangleleft$) For $a \in \mathcal{A}({}_AX_B \Rightarrow {}_AY_B)$, we set $(\varphi \triangleleft b)(a) := \varphi(a \odot b)$.

More succinctly, $\Phi \odot_r \Psi$ is given by the following formula:

$$\begin{aligned} \Phi \odot_r \Psi &= \varphi \mapsto \Psi\Big(\Phi \triangleright \varphi\Big), \\ &= \varphi \mapsto \Psi\Big(b \mapsto \Phi\big(\varphi \triangleleft b\big)\Big), \\ &= \varphi \mapsto \Psi\Big(b \mapsto \Phi\big(a \mapsto \varphi(a \odot b)\big)\Big). \end{aligned}$$

Proposition 3.2.3.2. *For a C*-2-category $\mathcal{A}$, the left and right Arens 1-compositions on $\mathcal{A}^{**}$ coincide and serve to equip $\mathcal{A}^{**}$ with the structure of a W*-2-category.*

Proof. Notice the following facts about $\mathcal{A}^{**}_\ell := (\mathcal{A}^{**}, \odot_\ell)$ and $\mathcal{A}^{**}_r := (\mathcal{A}^{**}, \odot_r)$.

- $\mathrm{ev}_a \odot_\ell \mathrm{ev}_b = \mathrm{ev}_{a \odot b} = \mathrm{ev}_a \odot_r \mathrm{ev}_b$ for 1-composable 2-morphisms a, b in $\mathcal{A}$. Therefore, we may upgrade ev into †-2-functors $\mathcal{A} \hookrightarrow \mathcal{A}^{**}_\ell$ and $\mathcal{A} \hookrightarrow \mathcal{A}^{**}_r$ which act as the identity on objects, and as the †-functor ev on hom W*-categories.

- $\mathcal{A}^{**}_\ell$ and $\mathcal{A}^{**}_r$ inherit associators $\mathrm{ev}_{\alpha_{XYZ}}$ which are natural since $\operatorname{Im}\mathrm{ev}$ is dense in $\mathcal{A}^{**}$ and 2-composition is weak*-continuous in each hom W*-category of $\mathcal{A}^{**}$. Similarly, units id_A and unitors ev_{λ_X}, ev_{ρ_X} are inherited from $\mathcal{A}$.

Therefore $\mathcal{A}^{**}_\ell$ and $\mathcal{A}^{**}_r$ are C*-2-categories. It remains to show that $\odot_\ell$ and $\odot_r$ are separately normal. It is clear that $- \odot_\ell \Psi$ and $\Phi \odot_r -$ are normal for 2-morphisms Φ, Ψ in $\mathcal{A}^{**}$.

We will now show condition (W*2$'$), that is $-\odot_\ell \mathrm{ev}_{\mathrm{id}_{X'}}$ and $\mathrm{ev}_{\mathrm{id}_X} \odot_\ell -$ are separately normal for 1-morphisms X, X' in $\mathcal{A}^{**}$. For $\Phi_\lambda \to \Phi$ in the weak topology, we have

$$(\Phi_\lambda \odot_\ell \mathrm{ev}_{\mathrm{id}_{X'}})(\varphi) = \Phi_\lambda(\varphi \triangleleft \mathrm{ev}_{\mathrm{id}_{X'}}) \to \Phi(\varphi \triangleleft \mathrm{ev}_{\mathrm{id}_{X'}}) = (\Phi \odot_\ell \mathrm{ev}_{\mathrm{id}_{X'}})(\varphi).$$

Hence $\Phi_\lambda \odot_\ell \mathrm{ev}_{\mathrm{id}_{X'}} \to \Phi \odot_\ell \mathrm{ev}_{\mathrm{id}_{X'}}$ weakly. Moreover, for $\Psi_\lambda \to \Psi$ in the weak topology, observe

$$\begin{aligned}(\mathrm{ev}_{\mathrm{id}_X} \odot \Psi_\lambda)(\varphi) &= \mathrm{ev}_{\mathrm{id}_X}(\varphi \triangleleft \Psi_\lambda)\\ &= \Psi_\lambda(b \mapsto \varphi(\mathrm{id}_X \odot b))\\ &\to \Psi(b \mapsto \varphi(\mathrm{id}_X \odot b))\\ &= (\mathrm{ev}_{\mathrm{id}_X} \odot \Psi_\lambda)(\varphi).\end{aligned}$$

Hence $\mathrm{ev}_{\mathrm{id}_X} \odot \Psi_\lambda \to \mathrm{ev}_{\mathrm{id}_X} \odot \Psi$ weakly. Therefore $\odot_\ell$ is separately normal, and one similarly shows $\odot_r$ is separately normal. Finally, since $\operatorname{Im}\mathrm{ev}$ is dense at the level of two morphisms, $\odot_\ell$ and $\odot_r$ are separately normal, and they agree on $\operatorname{Im}\mathrm{ev}$, we conclude that $\odot_\ell = \odot_r$ on $\mathcal{A}^{**}$. □

Universal Property 3.2.3.3. *For every †-2-functor $F\colon \mathcal{A} \to \mathcal{B}$ into a W*-2-category $\mathcal{B}$, there exists a unique normal †-2-functor $\widetilde{F}\colon \mathcal{A}^{**} \to \mathcal{B}$ making the following diagram commute.*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, hook, "\mathrm{ev}"'] & \mathcal{B} \\
\mathcal{A}^{**} \arrow[ur, dashed, "\exists!\widetilde{F}"'] &
\end{tikzcd}$$

Verification. First note that $\widetilde{F}$ must acts on objects as F. Then, for every hom C*-category $\mathcal{A}(A \to B)$, we may extend F to $\mathcal{A}^{**}(A \to B)$ by the universal property of W*-envelopes or C*-categories. This yields a map $\widetilde{F} : \mathcal{A}^{**} \to \mathcal{B}^{**}$ which is locally a normal †-functor. Then observe that the tensorator and unitor of F equips $\widetilde{F}$ with the structure of a †-2-functor since $\odot$ and $\circ$ are separately weak*-continuous and $\operatorname{Im}\mathrm{ev}$ is weak*-dense in $\mathcal{A}^{**}$. □

As a corollary, we obtain another proof of the concreteness theorem for C*-2-categories.

Theorem 3.2.3.4 (Gelfand-Naimark for C*-2-categories)**.** *For a small C*-2-category $\mathcal{A}$, the universal representation*

$$\Upsilon_2 \colon \mathcal{A} \to \mathsf{Hilb}^{\mathsf{W}^*}$$

given by $\mathcal{A} \xrightarrow{\mathrm{ev}} \mathcal{A}^{**} \xrightarrow{\text{よ}^{\mathrm{II}}} \mathrm{W}^*\mathsf{Cat} \xrightarrow{\mathsf{GNS}} \mathsf{Hilb}^{\mathsf{W}^*}$ *is monic. Thus, every C*-2-category $\mathcal{A}$ can be realized as a norm-closed strict †-2-category $\mathsf{GNS}_2(\mathcal{A}) := \operatorname{Im}\Upsilon_2$ of W*-categories of Hilbert spaces and operators. Moreover, if $\mathcal{A}$ is strict, Υ_2 is a strict †-2-functor.*

Remark 3.2.3.5. We expect a Sherman-Takeda theorem to hold for C*-2-categories. Namely, for a small C*-2-category $\mathcal{A}$, the †-2-functor

$$\widetilde{\Upsilon}_2 \colon \mathcal{A}^{**} \to \mathsf{GNS}_2(\mathcal{A})''$$

given by the universal property of the W*-envelope of a C*-2-category is an equivalence of W*-2-categories extending $\Upsilon_2 \colon \mathcal{A} \to \mathsf{GNS}_2(\mathcal{A})$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Upsilon_2"] \arrow[d, hook, "\mathrm{ev}"'] & \mathsf{GNS}_2(\mathcal{A}) \arrow[d, hook] \\
\mathcal{A}^{**} \arrow[r, "\widetilde{\Upsilon}_2"'] & \mathsf{GNS}_2(\mathcal{A})''
\end{tikzcd}$$

3.2.4 W*-Gray tensor product

For strict W*-2-categories $\mathcal{A}_1$ and $\mathcal{A}_2$, we wish to construct a W*-2-category $\mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2$ together with a separately normal cubical †-2-functor $\mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2$ satisfying the following universal property:

- For every separately normal cubical †-2-functor $H \colon \mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ into a strict W*-2-category $\mathcal{B}$, there exists a unique normal †-2-functor $H \colon \mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2 \to \mathcal{B}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \arrow[r, "H"] \arrow[d] & \mathcal{B} \\
\mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2 \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

Definition 3.2.4.1. For strict W*-2-categories $\mathcal{A}_1$, $\mathcal{A}_2$, and composible 1-morphisms $X_i, Y_i \in \mathcal{A}_i$ for $i = 1, 2$, we define

$$(\mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2)\big((X_1, X_2) \to (Y_1, Y_2)\big)_* \subset (\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2)\big((X_1, X_2) \to (X_1, X_2)\big)^*$$

as follows. First, let $\mathsf{SN}\left((X_1, X_2) \to (Y_1, Y_2)\right)$ be the closed subspace consisting of all functionals which are separately normal, i.e. the functionals $\varphi \in (\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2)\big((X_1, X_2) \to (Y_1, Y_2)\big)^*$ such that

$$\varphi \circ (\mathrm{id}_{\mathrm{id}_{A_1}} \otimes -) \in \mathcal{A}_2(X_2 \to Y_2)_*, \quad \text{for every } A_1 \in \mathcal{A}_1, \text{ and}$$
$$\varphi \circ (- \otimes \mathrm{id}_{\mathrm{id}_{A_2}}) \in \mathcal{A}_1(X_1 \to X_1)_*, \quad \text{for every } A_2 \in \mathcal{A}_2.$$

We then define $(\mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2)_*$ to be the largest closed subspace of SN invariant under the four actions of $\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2$ on $(\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2)^*$ given by precomposing and postcomposing the argument of $\varphi \in \mathsf{SN}$ horizontally (1-composition $\otimes$) and vertically (2-composition $\circ$).

Remark 3.2.4.2. We are using the suggestive notation $(\mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2)\big((X_1, X_2) \to (Y_1, Y_2)\big)_*$ even though we have not yet defined the W*-category $\mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2$. However, once we do construct this W*-Gray tensor product, we will see that this Banach space is indeed the predual of the hom space $(\mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2)\big((X_1, X_2) \to (Y_1, Y_2)\big)$.

The following result will serve as motivation for the definition of the W*-Gray tensor product $\underset{\max}{\overline{\boxtimes}}$.

Proposition 3.2.4.3. *For objects $X, Y \in \mathcal{B}$ in a strict W*-2-category $\mathcal{B}$, consider the polar*

$$\mathcal{B}(X \Rightarrow Y)_*^\perp := \{\Phi \in \mathcal{B}^{**}(X \Rightarrow Y) \mid \Phi(\varphi) = 0 \text{ for all } \varphi \in \mathcal{B}(X \Rightarrow Y)_*\}.$$

Then $\mathcal{B}_*^\perp \subseteq \mathcal{B}^{**}$ is closed under pre- and post- 1-composition and 2-composition of 2-morphisms in $\mathcal{B}^{**}$, so that $\mathcal{B}^{**}/\mathcal{B}_*^\perp$ is a (strict) W-2-category. Moreover, if $\pi \colon \mathcal{B}^{**} \to \mathcal{B}^{**}/\mathcal{B}_*^\perp$ is the natural quotient †-2-functor, then the composite*

$$\mathcal{B} \overset{\text{ev}}{\hookrightarrow} \mathcal{B}^{**} \xrightarrow{\pi} \mathcal{B}^{**}/\mathcal{B}_*^\perp$$

is an equivalence of (strict) W-2-categories.*

Proof. Let us first show that $\mathcal{B}_*^\perp$ is closed under both pre-compositions and post-compositions with 2-morphisms in $\mathcal{B}^{**}$. First note that normal functionals on hom spaces of $\mathcal{B}$ are closed under the four actions of $\mathcal{B}$ since 1-composition and 2-composition of 2-morphisms are separately normal in a W*-2-category. One can then use the fact that left and right Arens 1-compositions and 2-compositions agree respectively to show that $\mathcal{B}_*^\perp$ is closed under pre- and post- 1-composing and 2-composing 2-morphisms in $\mathcal{B}^{**}$. Indeed, if $\Phi \in \mathcal{B}(X \Rightarrow Y)_*^\perp$ and $\Psi \in \mathcal{B}^{**}(Y \Rightarrow Z)$ for parallel 1-morphisms X, Y, Z in $\mathcal{B}$, there exists some $\text{ev}_{x_\lambda} \to \Psi$ weak* in $\mathcal{B}^{**}$. So for $\varphi \in \mathcal{B}(X \Rightarrow Z)_*$ we have $\{\varphi \triangleleft x_\lambda\} \subset \mathcal{B}(X \Rightarrow Y)_*$ and hence

$$(\Psi \circ \Phi)(\varphi) = \lim_\lambda (\text{ev}_{x_\lambda} \circ_r \Phi)(\varphi) = \lim_\lambda \Phi(\varphi \triangleleft x_\lambda) = 0.$$

Therefore $\Psi \circ \Phi \in \mathcal{B}(X \Rightarrow Z)_*^\perp$, and one similarly shows our three other claims. Therefore $\mathcal{B}^{**}/\mathcal{B}_*^\perp$ is a C*-2-category. For Banach spaces $E \subseteq F$, it is an elementary fact that

$$F^*/E^\perp \cong E^*$$

via the map $[\Phi] \to \Phi|_E$ for $\Phi \in F^*$. In particular, we have

$$(\mathcal{B}^{**}/\mathcal{B}_*^\perp)(X \Rightarrow Y)_* = \mathcal{B}(X \Rightarrow Y)_*,$$

i.e. $(\mathcal{B}^{**}/\mathcal{B}_*^{\perp})(X \Rightarrow Y) \cong \mathcal{B}(X \Rightarrow Y)_*^* \cong \mathcal{B}(X \Rightarrow Y)$. We then conclude $\mathcal{B}^{**}/\mathcal{B}_*^{\perp}$ is a W*-2-category.

It is now easy to see that $\pi \circ \mathrm{ev} \colon \mathcal{B} \to \mathcal{B}^{**}/\mathcal{B}_*^{\perp}$ is an isomorphism of W*-2-categories. Indeed, when viewing $\mathcal{B}(X \Rightarrow Y)_* \subseteq \mathcal{B}(X \Rightarrow Y)^*$ as the subspace of normal functionals on $\mathcal{B}(X \Rightarrow Y)$, the isomorphism

$$\mathcal{B}(X \Rightarrow Y) \to \mathcal{B}(X \Rightarrow Y)_*^*$$

is given by $x \mapsto \mathrm{ev}_x|_{\mathcal{B}(X \Rightarrow Y)_*}$. On the other hand, the isomorphism

$$(\mathcal{B}^{**}/\mathcal{B}_*^{\perp})(X \Rightarrow Y) \to \mathcal{B}(X \Rightarrow Y)_*^*$$

is given by $[\Phi] \to \Phi|_{\mathcal{B}(X \Rightarrow Y)_*}$. Hence, the map $\pi \circ \mathrm{ev}$ given by $x \mapsto [\mathrm{ev}_x]$ is an isomorphism on each hom-space. Since both π and ev act as the identity on objects and 1-morphisms, we conclude that $\pi \circ \mathrm{ev}$ is an (automatically normal) equivalence of W*-2-categories. $\square$

Definition 3.2.4.4. For strict W*-2-categories $\mathcal{A}_1, \mathcal{A}_2$, we define their W*-Gray tensor product to be

$$\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2 \coloneqq (\mathcal{A}_1 \underset{\text{max}}{\boxtimes} \mathcal{A}_2)^{**}/(\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*^{\perp}.$$

Lemma 3.2.4.5. *The W*-Gray tensor product* $\mathcal{A}_1 \underset{max}{\overline{\boxtimes}} \mathcal{A}_2$ *is a (strict) W*-2-category.*

Proof. Since $(\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*^{\perp} \subseteq (\mathcal{A}_1 \underset{\text{max}}{\boxtimes} \mathcal{A}_2)^{**}$ is closed under pre- and post- 1-composition and 2-composition of 2-morphisms in $\mathcal{A}_1 \underset{\text{max}}{\overline{\otimes}} \mathcal{A}_2$ by definition, it follows that $\mathcal{A}_1 \underset{\text{max}}{\overline{\otimes}} \mathcal{A}_2$ is indeed a well-defined C*-2-category. As before, it is an elementary fact that the predual of each hom-space is the previously-defined $(\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*$. Hence $\odot$ is separately normal on $\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2$ as it is separately normal on $(\mathcal{A}_1 \underset{\text{max}}{\boxtimes} \mathcal{A}_2)^{**}$, and we conclude that $\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2$ is indeed a W*-2-category. $\square$

For the remaining portion of this section, we shall fix W*-2-categories $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ and a separately normal cubical †-2-functor $H \colon \mathcal{A}_1 \underset{\text{max}}{\otimes} \mathcal{A}_2 \to \mathcal{B}$.

Lemma 3.2.4.6. *Consider the induced* $\dagger$*-2-functor* $\widetilde{H}\colon \mathcal{A}_1 \underset{max}{\boxtimes} \mathcal{A}_2 \to \mathcal{B}$ *on the C*-Gray tensor product. For parallel 1-morphisms* X_i, Y_i *in* $\mathcal{A}_i$ *where* $i = 1, 2$ *and* $\psi \in \mathcal{B}\big(\widetilde{H}(X_1, X_2) \to \widetilde{H}(Y_1, Y_2)\big)_*$*, we may define*

$$\widetilde{H}^*\psi \in (\mathcal{A}_1 \underset{max}{\boxtimes} \mathcal{A}_2)\big((X_1, X_2) \to (Y_1, Y_2)\big)^* \quad \text{by } (\widetilde{H}^*\psi)(t) := \psi(\widetilde{H}t).$$

Then $\widetilde{H}^*\psi \in (\mathcal{A}_1 \underset{max}{\overline{\boxtimes}} \mathcal{A}_2)\big((X_1, X_2) \to (X_1, X_2)\big)_*$*. We will denote this by*

$$\widetilde{H}^*\mathcal{B}_* \subseteq (\mathcal{A}_1 \underset{max}{\overline{\boxtimes}} \mathcal{A}_2)_*$$

for simplicity.

Proof. Let $A_1 \in \mathcal{A}_1$ and suppose $a_2^\lambda \to a_2$ weak* in $\mathcal{A}_2(A_2 \to B_2)$. Since H is separately normal, $H(\mathrm{id}_{\mathrm{id}_{A_1}} \otimes a_2^\lambda) \to H(\mathrm{id}_{\mathrm{id}_{A_1}} \otimes a_2)$ σ-WOT. Since ψ is normal, we conclude

$$(\widetilde{H}^*\psi)(\mathrm{id}_{\mathrm{id}_{A_1}} \otimes a_2^\lambda) = \psi(H(\mathrm{id}_{\mathrm{id}_{A_1}} \otimes a_2^\lambda)) \to \psi(H(\mathrm{id}_{\mathrm{id}_{A_1}} \otimes a_2)) = (\widetilde{H}^*\psi)(\mathrm{id}_{\mathrm{id}_{A_1}} \otimes a_2).$$

A similar argument reveals $(\widetilde{H}^*\psi)(a_1^\lambda \otimes \mathrm{id}_{\mathrm{id}_{A_2}}) \to (\widetilde{H}^*\psi)(a_1 \otimes \mathrm{id}_{\mathrm{id}_{A_1}})$ when $a_1^\lambda \to a_1$ weak* in $\mathcal{A}_1$. Therefore $\widetilde{H}^*\psi$ is a separately normal functional. Moreover, since H is a $\dagger$-2-functor and both 1-composition and 2-composition of 2-morphisms in $\mathcal{B}$ is separately normal, it is easy to see that $\widetilde{H}^*\psi \in (\mathcal{A}_1 \underset{\mathrm{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*$. $\square$

Note that the cubical $\dagger$-2-functor $\mathcal{A}_1 \underset{\mathrm{max}}{\otimes} \mathcal{A}_2 \to \mathcal{A}_1 \underset{\mathrm{max}}{\overline{\boxtimes}} \mathcal{A}_2$ given by

$$\mathcal{A}_1 \underset{\mathrm{max}}{\otimes} \mathcal{A}_2 \to \mathcal{A}_1 \underset{\mathrm{max}}{\boxtimes} \mathcal{A}_2 \xrightarrow{\mathrm{ev}} (\mathcal{A}_1 \underset{\mathrm{max}}{\boxtimes} \mathcal{A}_2)^{**} \xrightarrow{\pi} \mathcal{A}_1 \underset{\mathrm{max}}{\overline{\boxtimes}} \mathcal{A}_2$$

is separately normal since $(\mathcal{A}_1 \underset{\mathrm{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*$ consists of functionals on $\mathcal{A}_1 \underset{\mathrm{max}}{\otimes} \mathcal{A}_2$ which are separately normal.

Universal Property 3.2.4.7. *For strict W*-2-categories* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, if* $H\colon \mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2 \to \mathcal{B}$ *is a separately normal cubical* $\dagger$*-2-functor into a strict W*-2-category* $\mathcal{B}$*, then there exists a* *unique normal †-2-functor* $\overline{H}\colon \mathcal{A}_1 \underset{max}{\overline{\boxtimes}} \mathcal{A}_2 \to \mathcal{B}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2 \arrow[r, "H"] \arrow[d] & \mathcal{B} \\
\mathcal{A}_1 \underset{max}{\overline{\boxtimes}} \mathcal{A}_2 \arrow[ur, dashed, "\overline{H}"'] &
\end{tikzcd}$$

Proof. By Lemma 3.2.4.6, $\widetilde{H}^*\mathcal{B}_* \subseteq (\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*$, so $(\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*^\perp \subseteq (\widetilde{H}^*\mathcal{B})_*^\perp$ and $\widetilde{H}^{**}(\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*^\perp \subseteq \mathcal{B}_*^\perp$. We then obtain a morphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2)_*^\perp \arrow[r] \arrow[d] & (\mathcal{A}_1 \underset{\text{max}}{\boxtimes} \mathcal{A}_2)^{**} \arrow[r, "\pi"] \arrow[d, "\widetilde{H}^{**}"'] & \mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2 \arrow[r] \arrow[d, dashed, "{[\widetilde{H}^{**}]}"] & 0 \\
0 \arrow[r] & \mathcal{B}_*^\perp \arrow[r] & \mathcal{B}^{**} \arrow[r, "\pi"'] & \mathcal{B}^{**}/\mathcal{B}_*^\perp \arrow[r] & 0
\end{tikzcd}$$

We now define the normal functor †-2-functor $\overline{H}\colon \mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2 \to \mathcal{B}$ to be

$$\overline{H} \coloneqq \mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2 \xrightarrow{[\widetilde{H}^{**}]} \mathcal{B}/\mathcal{B}^\perp \xrightarrow{(\pi\circ \text{ev})^{-1}} \mathcal{B}.$$

The desired triangle commutes by the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{A}_1 \underset{\text{max}}{\otimes} \mathcal{A}_2 \arrow[r, "H"] \arrow[d] & \mathcal{B} \arrow[d, equal] \\
\mathcal{A}_1 \underset{\text{max}}{\boxtimes} \mathcal{A}_2 \arrow[r, "\widetilde{H}" description] \arrow[d, "\text{ev}"'] & \mathcal{B} \arrow[d, "\text{ev}"] \\
(\mathcal{A}_1 \underset{\text{max}}{\boxtimes} \mathcal{A}_2)^{**} \arrow[r, "\widetilde{H}^{**}" description] \arrow[d, "\pi"'] & \mathcal{B}^{**} \arrow[d, "\pi"] \\
\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2 \arrow[r, "{[\widetilde{H}^{**}]}"'] & \mathcal{B}/\mathcal{B}_*^\perp
\end{tikzcd}$$

To see that $\overline{H}$ is unique, suppose $K\colon \mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2 \to \mathcal{B}$ is a normal †-2-functor making the following diagram commute:

$$\begin{tikzcd}
\mathcal{A}_1 \underset{\text{max}}{\otimes} \mathcal{A}_2 \arrow[r, "H"] \arrow[d] & \mathcal{B} \\
\mathcal{A}_1 \underset{\text{max}}{\overline{\boxtimes}} \mathcal{A}_2 \arrow[ur, "K"'] &
\end{tikzcd}$$

By the universal property of the C*-Gray tensor product, we have

$$\widetilde{H} = \mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2 \xrightarrow{\mathrm{ev}} (\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2)^{**} \xrightarrow{\pi} \mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2 \xrightarrow{K} \mathcal{B}.$$

We also know $\mathrm{ev} \circ \widetilde{H} = \widetilde{H}^{**} \circ \mathrm{ev}$, so by the universal property of W*-completion, we have

$$\widetilde{H}^{**} = (\mathcal{A}_1 \underset{\max}{\boxtimes} \mathcal{A}_2)^{**} \xrightarrow{\pi} \mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2 \xrightarrow{K} \mathcal{B} \xrightarrow{\mathrm{ev}} \mathcal{B}^{**}.$$

Finally, we know $\pi \circ \widetilde{H}^{**} = [\widetilde{H}^{**}] \circ \pi$, so the surjectivity of π implies

$$[\widetilde{H}^{**}] = \mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2 \xrightarrow{K} \mathcal{B} \xrightarrow{\mathrm{ev}} \mathcal{B}^{**} \xrightarrow{\pi} \mathcal{B}^{**}/\mathcal{B}_*^{\perp}.$$

From this we conclude that $K = \overline{H}$ and hence $\overline{H}$ is unique. □

Definition 3.2.4.8. Let $F_i \colon \mathcal{A}_i \to \mathcal{A}_i'$ be normal †-2-functors between strict W*-2-categories $\mathcal{A}_i$ for $i = 1, 2$. We define $F_1 \underset{\max}{\overline{\boxtimes}} F_2$ to be the normal †-2-functor induced by the universal property of $\underset{\max}{\overline{\boxtimes}}$ as follows:

$$\begin{tikzcd}
\mathcal{A}_1 \underset{\max}{\otimes} \mathcal{A}_2 \arrow[r, "F_1 \underset{\max}{\otimes} F_2"] \arrow[d] & \mathcal{A}_1' \underset{\max}{\otimes} \mathcal{A}_2' \arrow[d] \\
\mathcal{A}_1 \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2 \arrow[r, dashed, "F_1 \underset{\max}{\overline{\otimes}} F_2"'] & \mathcal{A}_1' \underset{\max}{\overline{\boxtimes}} \mathcal{A}_2'
\end{tikzcd}$$

Remark 3.2.4.9. Due to the uniqueness of normal †-2-functors induced by the universal property of $\underset{\max}{\overline{\boxtimes}}$, it is easy to see that $\underset{\max}{\overline{\boxtimes}} \colon \mathrm{W}^*\mathsf{2Cat}_{\mathsf{st}} \times \mathrm{W}^*\mathsf{2Cat}_{\mathsf{st}} \to \mathrm{W}^*\mathsf{2Cat}_{\mathsf{st}}$ is a functor when viewing $\underset{\max}{\overline{\boxtimes}}$ as a 1-category of strict W*-2-categories and normal †-2-functors.

Theorem 3.2.4.10 (Unitary hom-Tensor Adjunction)**.** *For strict W*-2-categories $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{B}$, we have that the following W*-2-categories are unitarily naturally isomorphic:*

$$\mathrm{W}^*\mathsf{2Cat}_{\mathsf{st}}(\mathbb{C} \to \mathcal{B}) \cong \mathcal{B},$$

$$\mathrm{W}^*\mathsf{2Cat}_{\mathsf{st}}(\mathcal{A}_1 \underset{max}{\overline{\boxtimes}} \mathcal{A}_2 \to \mathcal{B}) \cong \mathrm{W}^*\mathsf{2Cat}_{\mathsf{st}}(\mathcal{A}_1 \to \mathrm{W}^*\mathsf{2Cat}_{\mathsf{st}}(\mathcal{A}_2 \to \mathcal{B})).$$

Proof. This follows from an identical proof to that of Proposition 3.2.2.9, using the fact that $\underset{\mathrm{max}}{\overline{\boxtimes}}$ represents separately normal cubical functors to obtain normal strict †-2-functors as expected. □

As in the C* case, we obtain the following as a corollary of [EK66, §II.3,4].

Corollary 3.2.4.11. $\mathrm{W}^*\mathsf{Gray} \coloneqq (\mathrm{W}^*\mathsf{2Cat}_{\mathsf{st}}, \underset{max}{\overline{\boxtimes}})$ *forms a closed symmetric monoidal category.*

3.2.5 Cofibrant replacement for operator 2-categories

In the following section, we provide cofibrant replacement results for operator 2-categories, (normal) †-functors, as well as a tensorator for this replacement. This section follows the treatment in [Gur13, §2.2] closely, so we have provided the details in §1.3.5.

Proposition 3.2.5.1. *For every C*-2-category* $\mathcal{A}$*, there exists a strict C*-2-category* $\widehat{\mathcal{A}}$ *together with an epic 2-equivalence* $\mathrm{ev}_{\mathcal{A}} \colon \widehat{\mathcal{A}} \to \mathcal{A}$*. Moreover, when* $\mathcal{A}$ *is a W*-2-category we have that* $\widehat{\mathcal{A}}$ *is a W*-2-category and hence* $\mathrm{ev}_{\mathcal{A}}$ *is automatically normal.*

Proposition 3.2.5.2. *For each †-2-functor* $F \colon \mathcal{A} \to \mathcal{B}$ *between C*-2-categories, there exists a strict †-2-functor* $\widehat{F} \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$ *and a unitary icon*[8] u^F *as follows:*

$$\begin{tikzcd}
\widehat{\mathcal{A}} \arrow[r, "\widehat{F}"] \arrow[d, "\mathrm{ev}_{\mathcal{A}}"'] & \widehat{\mathcal{B}} \arrow[d, "\mathrm{ev}_{\mathcal{B}}"] \arrow[dl, Rightarrow, "u^F"', shorten=12pt] \\
\mathcal{A}_1 \arrow[r, "F"'] & \mathcal{A}_2
\end{tikzcd}$$

Moreover, when F *is a normal †-2-functor between W*-2-categories, then* $\widehat{F}$ *is also normal.*

Proposition 3.2.5.3. *For C*-2-categories* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, there exists a cubical †-2-functor*

$$C \colon \widehat{\mathcal{A}}_1 \underset{max}{\otimes} \widehat{\mathcal{A}}_2 \to \widehat{\mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2}$$

---

[8] By an icon, we mean a strictified version of a 2-natural transformation between 2-functors that agree on objects [Lac08].

which is the identity on objects, and a unitary icon u as follows:

$$\begin{tikzcd}
& \widehat{\mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2} \arrow[dr, "\mathrm{ev}"] \arrow[d, phantom, "\begin{array}{c} \| \\ u^{\mathrm{ev}} \\ \Downarrow \end{array}"] & \\
\widehat{\mathcal{A}}_1 \underset{max}{\otimes} \widehat{\mathcal{A}}_2 \arrow[ur, "C"] \arrow[rr, "\mathrm{ev} \underset{max}{\otimes} \mathrm{ev}"'] & & \mathcal{A}_1 \underset{max}{\otimes} \mathcal{A}_2
\end{tikzcd}$$

Moreover, when $\mathcal{A}_1$ and $\mathcal{A}_2$ are W-2-categories, we may upgrade C to a separately normal cubical †-2-functor*

$$\overline{C} \colon \widehat{\mathcal{A}}_1 \underset{max}{\overline{\otimes}} \widehat{\mathcal{A}}_2 \to \widehat{\mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2}$$

and u to a unitary icon $\overline{u}$ as follows:

$$\begin{tikzcd}
& \widehat{\mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2} \arrow[dr, "\mathrm{ev}"] \arrow[d, phantom, "\begin{array}{c} \| \\ \overline{u}^{\mathrm{ev}} \\ \Downarrow \end{array}"] & \\
\widehat{\mathcal{A}}_1 \underset{max}{\overline{\otimes}} \widehat{\mathcal{A}}_2 \arrow[ur, "\overline{C}"] \arrow[rr, "\mathrm{ev} \underset{max}{\overline{\otimes}} \mathrm{ev}"'] & & \mathcal{A}_1 \underset{max}{\overline{\otimes}} \mathcal{A}_2
\end{tikzcd}$$

3.3 3-categorical results

3.3.1 Operator 3-categories

We now provide the functional analytic versions of algebraic 3-categories, 3-functors, 3-natural transformations, 3-modifications, and perturbations as seen in Section §1.4.1.

Definition 3.3.1.1. A C*-3-category $\mathcal{A}$ consists of an algebraic tricategory in the sense of [Gur13], equipped with a C*-2-category structure on each hom 2-category $\mathcal{A}(A \to B)$, such that the underlying coherence 2-functors, 2-natural transformations, and modifications are †-2-functors, †-2-natural transformations, and unitary modifications respectively. More specifically, we have

(0) A collection T of *objects*, or *0-cells*, of $\mathcal{A}$;

(hom) For $A, B \in T$, a C*-2-category $\mathcal{A}(A \to B)$ where:

- The objects of $\mathcal{A}(A \to B)$ are called *1-cells* of $\mathcal{A}$ with source A and target B,
- The arrows of $\mathcal{A}(A \to B)$ will be referred to as *2-cells* of $\mathcal{A}$ with their same source and target, and
- The 2-morphisms of $\mathcal{A}(A \to B)$ are called *3-cells* of $\mathcal{A}$, also with their same source and target.

As [Gur13], we will denote $\mathcal{A}(A_0 \to A_1) \underset{\max}{\otimes} \cdots \underset{\max}{\otimes} \mathcal{A}(A_{n-1} \to A_n)$ by the abbreviation

$$\mathcal{A}^n(A_0 \to \cdots \to A_n).$$

($\circledcirc$) For $A, B, C \in \mathcal{A}$, a †-2-functor $\circledcirc \colon \mathcal{A}^2(A \to B \to C) \to \mathcal{A}(A \to C)$ called *1-composition.*

(I) For $A \in \mathcal{A}$, a †-2-functor $I_A \colon \mathbb{C} \to \mathcal{A}(A \to A)$

(a) For $A, B, C, D \in \mathcal{A}$, an *"associator"* unitary adjoint equivalence $\mathbf{a} = (a, a^{\bullet}, \epsilon^a, \eta^a)$ where a is a †-2-natural transformation in $\mathrm{C}^*\mathsf{2Cat}(\mathcal{A}^3(A \to B \to C \to D) \to \mathcal{A}(A \to D))$;

(u) For $A, B \in \mathcal{A}$, *left and right "unitor"* unitary adjoint equivalences $\mathbf{l}$ and $\mathbf{r}$ where l and r are †-2-natural transformations in $\mathrm{C}^*\mathsf{2Cat}(\mathcal{A}(A \to B) \to \mathcal{A}(A \to B))$;

(π) For $A, B, C, D, E \in \mathcal{A}$, a *"pentagonator"* unitary modification $\pi \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}^4(A \to B \to C \to D \to E) \to \mathcal{A}(A \to E))$.

(coh) For $A, B, C \in \mathcal{A}$, *middle, left, and right unity coheretor* unitary modifications $\mu, \lambda, \rho \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}^2(A \to B \to C) \to \mathcal{A}(A \to C))$.

Definition 3.3.1.2. A W*-3-category $\mathcal{A}$ is a C*-3-category such that:

- Each $\mathcal{A}(A \to B)$ is a W*-2-category;

- The 1-composition †-2-functor $\circledcirc$ is separately normal, hence extending to a normal †-2-functor

$$\circledcirc\colon \mathcal{A}(A \to B) \underset{\max}{\overline{\otimes}} \mathcal{A}(B \to C) \to \mathcal{A}(A \to C).$$

Note that the unit †-2-functor $I_A\colon \mathcal{A}(A \to A)$ is automatically normal. Using the fact that 2-composition $\odot$ and 3-composition $\circ$ are separately normal on each hom-W*-2-category, we may extend the constraint unitary adjoint equivalences so that they all occur in $(\mathrm{W}^*\mathsf{2Cat}, \underset{\max}{\overline{\otimes}})$.

Definition 3.3.1.3. A †-3-functor $F\colon \mathcal{A} \to \mathcal{A}'$ between C*-3-categories consists of an underlying 3-functor in the sense of [Gur13], such that F is locally a †-2-functor and the underlying coherence 2-natural transformations and modifications are †-natural transformations and unitary modifications respectively. More specifically, we have

(0) A function from objects of $\mathcal{A}$ to objects of $\mathcal{A}'$;

(hom) For $A, B \in \mathcal{A}$, a †-2-functor $F\colon \mathcal{A}(A, B) \to \mathcal{A}'(FA, FB)$

($\circledcirc$) A unitary adjoint equivalence $\mathbf{F^2}$ where F^2 is a †-2-natural transformation in $\mathrm{C}^*\mathsf{2Cat}(\mathbb{C} \to \mathcal{A}'(FA \to FA))$.

(a) For $A, B, C, D \in \mathcal{A}$, a unitary modification $F^a \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}(A \to B \to C \to D) \to \mathcal{A}'(FA \to FD))$.

(u) For $A, B \in \mathcal{A}$, unitary modifications $F^l, F^r \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}(A \to B) \to \mathcal{A}'(FA \to FB))$.

Definition 3.3.1.4. A †-3-natural transformation $\theta\colon F \Rightarrow F'$ between †-3-functors $F, F'\colon \mathcal{A} \to \mathcal{A}'$ consists of an underlying 3-natural transformation in the sense of [Gur13] such that the underlying coherence 2-natural transformations and modifications are †-2-natural transformations and unitary modifications respectively. More specifically, we have

(0) A family of 1-morphisms $\theta_A \in \mathcal{A}'(FA \to F'A)$ indexed by the objects of $\mathcal{A}$

(hom) For $A, B \in \mathcal{A}$, a unitary adjoint equivalence $\boldsymbol{\theta}$ where θ is a †-2-natural transformation in $\mathrm{C}^*\mathsf{2Cat}(\mathcal{A}(A, B) \to \mathcal{A}'(FA, F'B))$.

(⊚) For $A, B, C \in \mathcal{A}$, a unitary modification $\theta^2 \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}(A \to B \to C) \to \mathcal{A}'(FA \to F'B))$.

(I) For $A \in \mathcal{A}$, a unitary modification $\theta^0 \in \mathrm{C}^*\mathsf{2Cat}(\mathbb{C} \to \mathcal{A}'(FA \to F'A))$.

Definition 3.3.1.5. A †-3-modification $m\colon \theta \Rrightarrow \theta'$ between †-3-natural transformations $\theta, \theta'\colon F \Rightarrow F'$ consists of an underlying 3-modification in the sense of [Gur13] such that the underlying coherence modifications are unitary. More specifically, we have

(0) A family of 2-morphisms $m_A \in \mathcal{A}'(\theta_A \Rightarrow \theta'_A)$ indexed by the objects of $\mathcal{A}$

(hom) For $A, B \in \mathcal{A}$, a unitary modification $m \in \mathrm{C}^*\mathsf{2Cat}(\mathcal{A}(A \to B) \to \mathcal{A}'(FA \to F'B))$.

Definition 3.3.1.6. A uniformly bounded perturbation $\sigma\colon m \nRrightarrow m'$ between †-3-modifications consists of a family of 3-morphisms $\sigma_A \in \mathcal{A}'(m_a \Rrightarrow n_a)$ indexed by objects in $\mathcal{A}$, satisfying the obvious compatibility axiom with the higher data for m and n, and

$$\|\sigma\| \coloneqq \sup_{A \in \mathcal{A}} \|\sigma_A\| < \infty.$$

We refer the curious reader to [Gur13, Definition 4.21].

Example 3.3.1.7. From [CP22] we see that $\mathrm{C}^*\mathsf{2Cat}$ forms a C*-3-category. Hence, the sub-†-3-category $\mathrm{C}^*\mathsf{2Cat}_{\mathsf{strict}}$ of strict C*-2-categories, strict †-2-functors, all †-2-natural transformations, and uniformly bounded modifications forms a C*-3-category.

Analogously, $\mathrm{W}^*\mathsf{2Cat}$ forms a W*-3-category and there is a sub-W*-3-category $\mathrm{W}^*\mathsf{2Cat}_{\mathsf{strict}}$

of strict W*-2-categories, strict normal †-2-functors, all †-2-natural transformations, and uniformly bounded modifications.

Moreover, there are full sub-W*-3-categories of $\mathrm{W}^*\mathsf{2Cat}$, denoted by $\mathsf{Hilb}^{\mathsf{W}^*2}$ and $\mathsf{FSSU2C}$, whose objects are concrete W*-2-categories and finitely semisimple unitary 2-categories respectively.

Example 3.3.1.8. A monoidal C*-2-category can be viewed as a C*-3-category with a single object. Similarly, a braided C*-category can be viewed as a C*-3-category with a single object, and a single (identity) 1-morphism. In particular, every unitary braided multifusion category can be viewed as a W*-3-category.

The following examples are appropriate operator-algebraic analogues for Morita 2-categories. We refer the reader to [CHPJP22] for a detailed exposition.

Example 3.3.1.9. There is a C*-3-category $\mathsf{AbC^*Alg}$ whose:

(0) objects are compact Hausdorff spaces;

(1) 1-morphisms $A\colon X \to Y$ are C*-algebras A equipped with $*$-homomorphisms $C(X) \to Z(A)$ and $C(Y) \to Z(A)$;

(2) 2-morphisms $H\colon A \Rightarrow B$ are A-B C*-correspondences compatible with the $C(X)$ and $C(Y)$ actions,

(3) 3-morphisms are adjointable intertwiners.

One similarly defines an analogous W*-3-category $\mathsf{AbW^*Alg}$ whose:

- objects are commutative W*-algebras ;
- 1-morphisms are W*-algebras A equipped with normal $*$-homomorphisms into $Z(A)$;

- 2-morphisms $H\colon A \Rightarrow B$ are A-B W*-correspondences compatible with the central actions,
- 3-morphisms are adjointable normal intertwiners.

We verify the details of this example in Section §3.5 as it will be our main object of study in Chapter 4.

Remark 3.3.1. We note that if $X\colon A \Rightarrow B$ is an invertible 2-morphism in $\mathsf{AbC^*Alg}$, then X is actually an A-B imprimitivity bimodule (and so is its inverse) by Lemma 2.4 in [EKQR06].

In the following examples, we make reference to the theory of higher Hilbert spaces established in joint work with Chen, Hungar, Penneys, and Sanford. As we will not dive deeper into the relevant results, we omit the details and refer the interested reader to [CFH$^+$26].

Example 3.3.1.10. There is a W*-3-category $\mathsf{H^*mFC}$ whose:

(0) objects are H*-multifusion categories $(\mathcal{C}, \vee, \psi)$;

(1) 1-morphisms $M\colon \mathcal{C} \to \mathcal{D}$ are finitely semisimple unitary $\mathcal{C}$-$\mathcal{D}$ bimodule categories $({}_{\mathcal{C}}M_{\mathcal{D}}, \mathrm{Tr})$ equipped with bimodule traces Tr;

(2) 2-morphisms $F\colon M \Rightarrow N$ are $\mathcal{C}$-$\mathcal{D}$ bimodule †-functors;

(3) 3-morphisms $\theta\colon F \Rrightarrow G$ are $\mathcal{C}$-$\mathcal{D}$ bimodule †-natural transformations (which are automatically uniformly bounded).

Example 3.3.1.11. There is a W*-3-category $\mathsf{3Hilb}$ whose:

(0) objects are 3-Hilbert spaces $(\mathfrak{X}, \vee, \Psi)$;

(1) 1-morphisms $F\colon \mathfrak{X} \to \mathfrak{Y}$ are $\vee$-preserving †-2-functors;

(2) 2-morphisms $\theta\colon F \Rightarrow G$ are †-2-natural transformations;

(3) 3-morphisms $m\colon \theta \Rrightarrow \sigma$ are modifications (which are automatically uniformly bounded).

Theorem 3.3.1.12 ([CFH$^+$26]). *There exists an equivalence of W*-3-categories*

$$\mathsf{Mod}^\dagger\colon \mathsf{H^*mFC} \to \mathsf{3Hilb}$$

given by sending an H-multifusion category $\mathcal{C}$ to its (W*-2-)category of modules $\mathsf{Mod}^\dagger(\mathcal{C})$.*

The following example is an operator-algebraic analogue of a hom 3-category in the speculated Morita 4-category UBmFC of unitary braided multifusion categories.

Example 3.3.1.13. For unitary braided multifusion categories $\mathcal{A}$ and $\mathcal{B}$, there is a W*-3-category $\mathsf{UBmFC}(\mathcal{A} \to \mathcal{B})$ whose:

(0) Objects are unitary multifusion categories $\mathcal{C}$ equipped with unitary braided functors $\mathcal{A} \to Z(\mathcal{C}) \leftarrow \mathcal{B}^{\mathsf{rev}}$;

(1) 1-morphisms $M\colon \mathcal{C} \to \mathcal{D}$ are unitary $\mathcal{C}$-$\mathcal{D}$ bimodule categories together with compatibility data;

(2) 2-morphisms $F\colon M \Rightarrow N$ are unitary $\mathcal{C}$-$\mathcal{D}$ bimodule functors satisfying compatibility conditions;

(3) 3-morphisms $\eta\colon F \Rrightarrow G$ are $\mathcal{C}$-$\mathcal{D}$ bimodule natural transformations.

Notice $\mathsf{UBFC}(\mathbb{C} \to \mathbb{C}) = \mathsf{UmFC}$.

3.3.2 Transport and change of structure

In this section, we provide three results which will allow us to locally strictify an operator 3-category, into what will be known as a "cubical" operator 3-category. In particular, these three results will allow us to strictify hom-2-categories, composition of objects, and units respectively in a compatible way.

Lemma 3.3.2.1 (Transport of structure)**.** *Let $\mathcal{A}$ be a C*-3-category and let*

$$\{\widetilde{\mathcal{A}}(A,B) \in \mathrm{C}^*\mathsf{2Cat}\}_{A,B\in\mathcal{A}}$$

be a collection of C-2-categories indexed by pairs of objects in $\mathcal{A}$ with †-2-equivalences*

$$F_{A,B}\colon \mathcal{A}(A,B) \to \widetilde{\mathcal{A}}(A,B) \text{ for every } A, B \in \mathcal{A}.$$

Then we may upgrade this data to a C-3-category $\widetilde{\mathcal{A}}$ and a †-3-functor $F\colon \mathcal{A} \to \widetilde{\mathcal{A}}$ such that*

- $\widetilde{\mathcal{A}} = \mathcal{A}$,
- $\widetilde{\mathcal{A}}(A,B)$ *is the hom-C*-2-category from A to B in $\widetilde{\mathcal{A}}$,*
- *Choosing biadjoint †-2-equivalences $(F_{AB}, F^{\bullet}_{AB}, \eta^{F_{AB}}, \epsilon^{F_{AB}})$ for each F_{AB}, 1-composition $\widetilde{\circledcirc}$ in $\widetilde{\mathcal{A}}$ is then defined by*

$$\begin{tikzcd}
\widetilde{\mathcal{A}}^2(A \to B \to C) \arrow[r, "F_{AB} \underset{max}{\otimes} F_{BC}"] \arrow[d, dashed, "\widetilde{\circledcirc}"'] & \mathcal{A}^2(A \to B \to C) \arrow[d, "\circledcirc"] \\
\widetilde{\mathcal{A}}(A \to C) & \mathcal{A}(A \to C) \arrow[l, "F^{\bullet}_{AC}"]
\end{tikzcd}$$

- *F is a †-3-equivalence which acts as the identity on objects and as $F_{A,B}$ on hom-C*-2-categories.*

Moreover, if $\mathcal{A}$ is a W-3-category, each $\widetilde{\mathcal{A}}(A,B)$ is a W*-2-category, and each $F_{A,B}$ a normal †-2-equivalence, then $\widetilde{\mathcal{A}}$ is a W*-3-category.*

Proof. The structure equipped on the 3-category $\widetilde{\mathcal{A}}$ and the 3-functor F are built using the units and counits of the biadjoint †-2-equivalences (which are unitaries) and the constraint data on $\mathcal{A}$. From this it is quite easy to see that $\widetilde{\mathcal{A}}$ is a C*-3-category and F is a †-3-functor. Furthermore, from the definition of $\widetilde{\circledcirc}$ we see that 1-composition is separately normal in $\widetilde{\mathcal{A}}$ the W*-case, and F is automatically normal as a †-3-equivalence. We refer the reader to [Gur13, §7.4, Theorem 7.22] for the details of this construction for non-linear 3-categories. $\square$

We will state the next two lemmas without proof, as they are similar in spirit to Lemma 3.3.2.1, and refer the interested reader to [Gur13, §7.4, Theorems 7.23, 7.24] for details.

Lemma 3.3.2.2 (Change of composition)**.** *Let $\mathcal{A}$ be a C*-3-category and let*

$$\{\widetilde{\circledcirc}\colon \mathcal{A}^2(A \to B \to C) \to \mathcal{A}(A \to C)\}_{A,B,C\in\mathcal{A}}$$

be a collection of †-2-functors indexed by triples of objects in $\mathcal{A}$ with †-2-natural equivalences

$$F^2\colon\ \circledcirc \Rightarrow \widetilde{\circledcirc} \qquad \text{for every } A, B, C \in \mathcal{A}.$$

Then we may upgrade this data to a C-3-category $\widetilde{\mathcal{A}}$ and a †-3-functor $F\colon \mathcal{A} \to \widetilde{\mathcal{A}}$ such that*

- $\widetilde{\mathcal{A}} = \mathcal{A}$*,*
- $\widetilde{\mathcal{A}}(A \to B) = \mathcal{A}(A \to B)$ *as C*-2-categories,*
- $\widetilde{\mathcal{A}}$ *has the same units* I_A *as* $\mathcal{A}$*,*
- *1-composition in* $\widetilde{\mathcal{A}}$ *is given by* $\widetilde{\circledcirc}$*,*
- F *is a †-3-equivalence which as a map acts as the identity, and*
- *Choosing unitary adjoint equivalences for* F^2*, we obtain the 1-tensorator unitary adjoint equivalences for* F*.*

Furthermore, when $\mathcal{A}$ is a W-3-category and each $\widetilde{\circledcirc}$ is separately normal, it is immediate that $\widetilde{\mathcal{A}}$ is a W*-3-category.*

Lemma 3.3.2.3 (Change of units)**.** *Let $\mathcal{A}$ be a C*-3-category and let*

$$\{\widetilde{I}_A \colon \mathbb{C} \to \mathcal{A}(A, A)\}_{A \in \mathcal{A}}$$

be a collection of †-2-functors indexed by the objects in $\mathcal{A}$ with †-2-natural equivalences

$$F^1 \colon I_A \Rightarrow \widetilde{I}_A \quad \text{for every } A \in \mathcal{A}.$$

Then we may upgrade this data to a C-3-category $\widetilde{\mathcal{A}}$ and a †-3-functor $F \colon \mathcal{A} \to \widetilde{\mathcal{A}}$ such that*

- $\widetilde{\mathcal{A}} = \mathcal{A}$,
- $\widetilde{\mathcal{A}}(A \to B) = \mathcal{A}(A \to B)$ *as C*-2-categories,*
- *Units in $\widetilde{\mathcal{A}}$ are given by $\widetilde{I}_A$,*
- *$\widetilde{\mathcal{A}}$ has the same 1-composition $\circledcirc$ as $\mathcal{A}$,*
- *F is a †-3-equivalence which as a map acts as the identity, and*
- *Choosing unitary adjoint equivalences for F^1, we obtain the 1-unitor unitary adjoint equivalences for F.*

When $\mathcal{A}$ is a W-3-category, it is automatic that $\widetilde{\mathcal{A}}$ is a W*-3-category.*

3.3.3 Operator cubical categories

In this section, we provide an intermediate strictification step for our main coherence result.

Definition 3.3.3.1. We say that a C*-3-category $\mathcal{A}$ (resp. W*-3-category) if its underlying 3-category is cubical, i.e. for all objects $A, B, C \in \mathcal{A}$ we have

(hom) each $\mathcal{A}(A \to B)$ is *strict* C*-2-category (resp. W*-2-category);

($\circledcirc$) composition $\circledcirc \colon \mathcal{A}^2(A \to B \to C) \to \mathcal{A}(A \to C)$ is a (resp. separately normal) *cubical* †-2-functor;

(I) the unit †-2-functor $I_A \colon \mathcal{A}(A \to A)$ is *strict.*

Theorem 3.3.3.2. *Every C*-3-category $\mathcal{A}$ is equivalent to a cubical C*-3-category $\mathcal{A}^{⧉}$. Furthermore, if $\mathcal{A}$ is a W*-3-category, then $\mathcal{A}^{⧉}$ is a cubical W*-3-category.*

Proof. For a C*-3-category $\mathcal{A}$, we define $\mathcal{A}^{⧉}$ as follows:

- $\mathcal{A}^{⧉} = \mathcal{A}$,
- $\mathcal{A}^{⧉}(A \to B) = \mathcal{A}(\widehat{A \to B})$ for every $A, B \in \mathcal{A}$.

Using transport of structure, we obtain a C*-3-category equivalent to $\mathcal{A}$ with strict hom-C*-2-categories and 1-composition given by

$$\mathcal{A}(\widehat{A \to B}) \underset{\max}{\otimes} \mathcal{A}(\widehat{B \to C}) \xrightarrow{\mathrm{ev} \underset{\max}{\otimes} \mathrm{ev}} \mathcal{A}^2(A \to B \to C) \xrightarrow{\circledcirc} \mathcal{A}(A \to C) \xrightarrow{\mathrm{ev}^{\bullet}} \mathcal{A}(\widehat{A \to C}),$$

where we are choosing unitary adjoint equivalences $(\mathrm{ev}, \mathrm{ev}^{\bullet}, \eta^{\mathrm{ev}}, \epsilon^{\mathrm{ev}})$ for ev. Observe, by Propositions 3.2.5.2 and 3.2.5.3 we have the following †-2-natural equivalence

$$\begin{tikzcd}
 & \widehat{\mathcal{A}^2} \arrow[r, "\widehat{\circledcirc}"] \arrow[d, "\mathrm{ev}"] \arrow[dl, phantom, "\overset{u^{\mathrm{ev}}}{\Swarrow}"] \arrow[dr, phantom, "\overset{u^{\circledcirc}}{\Swarrow}"] & \widehat{\mathcal{A}} \arrow[d, "\mathrm{ev}"] \arrow[dr, "\mathrm{id}"] \arrow[dr, phantom, "\overset{\eta^{\mathrm{ev}}}{\Swarrow}"'] & \\
(\widehat{\mathcal{A}})^2 \arrow[ur, "C"] \arrow[r, "\mathrm{ev} \underset{\max}{\otimes} \mathrm{ev}"'] & \mathcal{A}^2 \arrow[r, "\circledcirc"'] & \mathcal{A} \arrow[r, "\mathrm{ev}^{\bullet}"'] & \widehat{\mathcal{A}}
\end{tikzcd}$$

between the cubical †-2-functor $\widehat{\circledcirc} \circ C$ and the previously mentioned 1-composition. Using change of composition, we obtain a C*-3-category with strict hom-C*-2-categories and cubical 1-composition. Finally, by using change of units to replace each I_A with its strictification, we obtain the desired equivalent cubical C*-3-category $\mathcal{A}^{\cube}$. In the case when $\mathcal{A}$ is a W*-3-category, we use the corresponding results to conclude that $\mathcal{A}^{\cube}$ is a W*-3-category. □

3.3.4 Coherence and concreteness for operator algebraic tricategories

Definition 3.3.4.1. A $\mathrm{C}^*\mathsf{Gray}$-category is a category enriched in $\mathrm{C}^*\mathsf{Gray}$ in the sense of [Kel05]. A functor of $\mathrm{C}^*\mathsf{Gray}$-categories is then just a strict †-3-functor. We denote the category of $\mathrm{C}^*\mathsf{Gray}$-categories and strict †-3-functors by $\mathrm{C}^*\mathsf{GrayCat}$.
Similarly, a $\mathrm{W}^*\mathsf{Gray}$-category is a category enriched in $\mathrm{W}^*\mathsf{Gray}$. A functor of $\mathrm{W}^*\mathsf{Gray}$-categories is then just a strict normal †-3-functor. We denote the category of $\mathrm{W}^*\mathsf{Gray}$-categories and strict normal †-3-functors by $\mathrm{W}^*\mathsf{GrayCat}$.

In what remains, we consider the Yoneda embedding for cubical operator 3-categories, referring the interested reader to §3.4 for the details of this construction.

Proposition 3.3.4.2. *For $\mathcal{A}$ a C*-3-category and $\mathcal{B}$ a* $\mathrm{C}^*\mathsf{Gray}$*-category,* $\mathrm{C}^*\mathsf{3Cat}(\mathcal{A} \to \mathcal{B})$ *forms a* $\mathrm{C}^*\mathsf{Gray}$*-category of*

(0) *†-3-functors from $\mathcal{A}$ to $\mathcal{B}$,*

(hom) *Hom-C*-2-categories* $\mathrm{C}^*\mathsf{3Cat}({}_{\mathcal{A}}F_{\mathcal{B}} \Rightarrow {}_{\mathcal{A}}G_{\mathcal{B}})$ *as in §3.4 Lemma 3.4.0.1,*

($\circledcirc$) *Cubical 1-composition $\circledcirc$ as in §3.4 Lemma 3.4.0.2*

Moreover, when $\mathcal{A}$ is a W-3-category and $\mathcal{B}$ is a* $\mathrm{W}^*\mathsf{Gray}$*-category, we have that* $\mathrm{W}^*\mathsf{3Cat}(\mathcal{A} \to \mathcal{B})$ *forms a* $\mathrm{W}^*\mathsf{Gray}$*-category of normal †-3-functors from $\mathcal{A}$ to $\mathcal{B}$.*

Theorem 3.3.4.3. *The Yoneda embedding for a cubical C*-3-category $\mathcal{B}$, given on objects by*

$$B \mapsto \mathcal{B}(- \to B),$$

can be equipped with the structure of monic †-3-functor

$$\text{よ} \colon \mathcal{B} \to \mathrm{C}^*\mathsf{3Cat}(\mathcal{B}^{\circledcirc\mathrm{p}} \to \mathrm{C}^*\mathsf{2Cat}_{\mathsf{strict}}),$$

which is locally a 2-equivalence. When $\mathcal{B}$ is a W-3-category, we obtain a monic normal †-3-functor*

$$\text{よ} \colon \mathcal{B} \to \mathrm{W}^*\mathsf{3Cat}(\mathcal{B}^{\circledcirc\mathrm{p}} \to \mathrm{W}^*\mathsf{2Cat}_{\mathsf{strict}}).$$

Theorem 3.3.4.4 (Coherence for operator 3-categories)**.** *Every C*-3-category is 3-equivalent to a* $\mathrm{C}^*\mathsf{Gray}$*-category, and every W*-3-category is 3-equivalent to a* $\mathrm{W}^*\mathsf{Gray}$*-category.*

Proof. By Theorem 3.3.3.2 and Theorem 3.3.4.3

$$\mathcal{A} \xrightarrow{\sim} \mathcal{A}^{\cube} \xhookrightarrow{\text{よ}} \mathrm{C}^*\mathsf{3Cat}\big((\mathcal{A}^{\cube})^{\circledcirc\mathrm{p}} \to \mathrm{C}^*\mathsf{2Cat}_{\mathsf{strict}}\big)$$

where the latter is a $\mathrm{C}^*\mathsf{Gray}$-category by Proposition 3.3.4.2. □

Theorem 3.3.4.5 (Gelfand-Naimark for operator 3-categories)**.** *Every small C*-3-category is 3-equivalent to a sub-*$\mathrm{C}^*\mathsf{Gray}$*-category of* $\mathsf{Hilb}^{\mathsf{W}^*2}$.

Proof. Notice there is a monic †-3-functor $\text{よ}^{\amalg} \colon \mathcal{A} \to \mathrm{C}^*\mathsf{2Cat}_{\mathsf{small}}$ given on objects by

$$\text{よ}^{\amalg}(B) = \coprod_{A \in \mathcal{A}} \mathcal{A}(A \to B).$$

By extending the GNS construction for C*-2-categories into a monic †-3-functor

$$\mathsf{GNS} \colon \mathrm{C}^*\mathsf{2Cat}_{\mathsf{small}} \to \mathsf{Hilb}^{\mathsf{W}^*2},$$

the image of the composition

$$\mathcal{A} \xhookrightarrow{\text{よ}^{\amalg}} \mathrm{C}^*\mathsf{2Cat}_{\mathsf{small}} \xhookrightarrow{\mathsf{GNS}} \mathsf{Hilb}^{\mathsf{W}^*2}$$

is equivalent to $\mathcal{A}$. □

3.4 Yoneda for operator algebraic tricategories

We will now construct the Yoneda embedding for a *cubical* C*-3-category $\mathcal{B}$

$$よ\colon \mathcal{B} \to \mathrm{C}^*\mathsf{3Cat}(\mathcal{B}^{\circledcirc\mathrm{p}} \to \mathrm{C}^*\mathsf{Gray}),$$

and a corresponding normal version

$$よ\colon \mathcal{B} \to \mathrm{W}^*\mathsf{3Cat}(\mathcal{B}^{\circledcirc\mathrm{p}} \to \mathrm{W}^*\mathsf{Gray}),$$

when $\mathcal{B}$ is a W*-3-category. We begin by constructing the targets of these †-3-functors.

Lemma 3.4.0.1. *For C*-3-categories $\mathcal{A}$ and $\mathcal{B}$, and †-3-functors $F, G\colon \mathcal{A} \to \mathcal{B}$, there exists an organic C*-2-category structure on $\mathrm{C}^*\mathsf{3Cat}({}_{\mathcal{A}}F_{\mathcal{B}} \Rightarrow {}_{\mathcal{A}}G_{\mathcal{B}})$, the 2-category of*

(0) †-3-natural transformations from F to G,

(1) †-3-modifications, and

(2) uniformly bounded perturbations.

which is strict whenever $\mathcal{B}$ is locally strict.

Moreover, when $\mathcal{A}$ and $\mathcal{B}$ are W-3-categories and F, G are normal †-3-functors, we have that $\mathrm{W}^*\mathsf{3Cat}(F \Rightarrow G)$ forms a W*-2-category.*

Proof. By [Gur13, §9.1, Theorem 9.1] we know that all 3-natural transformations from F to G, together with all 3-modifications and perturbations between them form a 2-category $\mathsf{3Cat}(F \Rightarrow G)$ where the components for the associators and unitors are given by the associators and unitors of the components in $\mathcal{B}$ respectively. First note that composites of unitary constraints for 3-natural transformations and 3-modifications are also unitary, so that †-3-natural transformations, †-3-modifications, and all perturbations form a 2-subcategory of

$\mathsf{3Cat}(F \Rightarrow G)$. Since linear combinations and compositions occur componentwise, it is clear that $\mathrm{C}^*\mathsf{3Cat}(F \Rightarrow G)$ (resp. $\mathrm{W}^*\mathsf{3Cat}(F \Rightarrow G)$) forms a (linear) 2-subcategory of $\mathsf{3Cat}(F \Rightarrow G)$. By defining † to act componentwise on uniformly bounded perturbations, this yields a C*-2-category structure on $\mathrm{C}^*\mathsf{3Cat}(F \Rightarrow G)$ (resp. $\mathrm{W}^*\mathsf{3Cat}(F \Rightarrow G)$).

In the W* case, one then uses the Kaplansky density theorem together with [CP22, Lemma 2.13] to show that $\mathrm{W}^*\mathsf{3Cat}(F \Rightarrow G)$ is locally a W*-category and that $\odot$ is separately normal. $\square$

Lemma 3.4.0.2. *Let $\mathcal{A}$ be a C*-3-category, $\mathcal{B}$ a $\mathrm{C}^*\mathsf{Gray}$-category, and $F, G, H \colon \mathcal{A} \to \mathcal{B}$ †-3-functors. Then 1-composition forms cubical †-2-functor*

$$\circledcirc \colon \mathrm{C}^*\mathsf{3Cat}(F \Rightarrow G) \underset{max}{\otimes} \mathrm{C}^*\mathsf{3Cat}(G \Rightarrow H) \to \mathrm{C}^*\mathsf{3Cat}(F \Rightarrow H).$$

Moreover, when $\mathcal{A}$ is a W-3-category, $\mathcal{B}$ a $\mathrm{W}^*\mathsf{Gray}$-category, and F, G, H are normal †-3-functors, we have that $\circledcirc$ is separately normal.*

Proof. One easily verifies that the underlying composite of †-3-natural transformations, †-3-modifications, and uniformly bounded perturbations is again of the respective type. By [Gur13, Thm. 9.3], we then know that $\circledcirc$ assembles into a cubical 2-functor of the underlying 2-categories. It is also easy to see that $\circledcirc$ is †-bilinear since linear combinations of perturbations are obtained componentwise, and that the interchanger Σ for $\circledcirc$ is unitary as it arises from the $\mathrm{C}^*\mathsf{Gray}$-category $\mathcal{B}$ on each component of the relevant †-3-modifications. Therefore $\circledcirc$ forms a cubical †-2-functor.

In the W* case, since $\circledcirc$ is separately normal in $\mathcal{B}$, the Kaplansky density theorem together with [CP22, Lemma 2.13] yields that $\circledcirc$ is separately normal. $\square$

Corollary 3.4.0.3. *For $\mathcal{A}$ a C*-3-category and $\mathcal{B}$ a $\mathrm{C}^*\mathsf{Gray}$-category, $\mathrm{C}^*\mathsf{3Cat}(\mathcal{A} \to \mathcal{B})$ forms a $\mathrm{C}^*\mathsf{Gray}$-category of*

(0) *†-3-functors from $\mathcal{A}$ to $\mathcal{B}$,*

(hom) *Hom-C*-2-categories* $\mathrm{C}^*\mathsf{3Cat}({}_{\mathcal{A}}F_{\mathcal{B}} \Rightarrow {}_{\mathcal{A}}G_{\mathcal{B}})$ *as in Lemma 3.4.0.1,*

(⊚) *Cubical 1-composition ⊚ as in Lemma 3.4.0.2*

Moreover, when $\mathcal{A}$ is a W-3-category and $\mathcal{B}$ is a* $\mathrm{W}^*\mathsf{Gray}$*-category,* $\mathrm{W}^*\mathsf{3Cat}(\mathcal{A} \to \mathcal{B})$ *forms a* $\mathrm{W}^*\mathsf{Gray}$*-category of normal †-3-functors from $\mathcal{A}$ to $\mathcal{B}$.*

We now begin constructing the actual embedding.

Lemma 3.4.0.4. *For a cubical C*-3-category $\mathcal{B}$ and an object $B \in \mathcal{B}$, there is an organic contravariant hom-†-3-functor*

$$\text{よ}_B \colon \mathcal{B}^{\circledcirc\mathrm{p}} \to \mathrm{C}^*\mathsf{2Cat}_{\mathsf{strict}}.$$

When $\mathcal{B}$ is a cubical W-3-category, this construction yields a normal †-3-functor*

$$\text{よ}_B \colon \mathcal{B}^{\circledcirc\mathrm{p}} \to \mathrm{W}^*\mathsf{2Cat}_{\mathsf{strict}}.$$

Proof. Let $\mathcal{B}$ be a C*-3-category (resp. W*-3-category). We will present the definition for the underlying Yoneda embedding for 3-categories seen in [Gur13], noting that all the relevant data is compatible with dagger structures.

(0) On an object $B' \in \mathcal{B}$, we set

$$\text{よ}_B(B') \coloneqq \mathcal{B}(B' \to B),$$

which is a strict C*-2-category (resp. W*-2-category) since $\mathcal{B}$ is cubical.

(1) On a 1-morphism $Y \in \mathcal{B}(B'' \to B')$, we define the strict (normal) †-2-functor

$$\text{よ}_B(Y) \colon \text{よ}_B(B') \to \text{よ}_B(B'')$$

as follows:

(1.0) For an object $X \in \text{よ}_B(B') = \mathcal{B}(B' \to B)$, which is a 1-morphism in $\mathcal{B}$, we set

$$\text{よ}_B(Y)(X) \coloneqq {}_{B''}Y \circledcirc_{B'} X_B.$$

(1.1) For a 1-morphism b in $\text{よ}_B(B')$, which is a 2-morphism in $\mathcal{B}$, we set

$$\text{よ}_B(Y)(b) \coloneqq \mathrm{id}_Y \circledcirc_{B'} b.$$

(1.2) Similarly, for a 2-morphism β in $\text{よ}_B(B')$, which is a 3-morphism in $\mathcal{B}$, we set

$$\text{よ}_B(Y)(\beta) \coloneqq \mathrm{id}_{\mathrm{id}_Y} \circledcirc_{B'} \beta.$$

Notice $\text{よ}_B(Y)$ is linear, †-preserving, and preserves 3-composition $\circ$ and identities at the level of 2-morphisms in $\text{よ}_B(B')$ (which are 3-morphisms in $\mathcal{B}$). Moreover, $\text{よ}_B(Y)$ is strict since $\circledcirc$ is cubical.

In the W* case, since $\circledcirc$ is separately normal in $\mathcal{B}$, the Kaplansky density theorem together with [CP22, Lemma 2.13] yields that $\text{よ}_B(Y)$ is normal.

(2) For a 2-morphism $b \in \mathcal{B}({}_{B''}Y_{B'} \Rightarrow {}_{B''}Y'_{B'})$, we define the †-2-natural transformation

$$\text{よ}_B(b) \colon \text{よ}_B(Y) \Rightarrow \text{よ}_B(Y').$$

as follows:

(2.0) For an object $X \in \text{よ}_B(B')$, we define the component

$$\text{よ}_B(b)_X \colon \underbrace{\text{よ}_B(Y)(X)}_{Y \circledcirc X} \to \underbrace{\text{よ}_B(Y')(X)}_{Y' \circledcirc X}$$

to be $\text{よ}_B(b)_X \coloneqq b \circledcirc \mathrm{id}_X$.

(2.1) For a 1-morphism a in $\text{よ}_B(B')$, we define the component unitary $\text{よ}_B(b)_a$ to be $\Sigma^{\dagger}_{a,b}$, which comes from the unitary interchanger for the cubical 1-composition $\circledcirc$ in $\mathcal{B}$.

(3) On a 3-morphism $\beta \in \mathcal{B}(b \Rrightarrow b')$, we define the uniformly bounded modification

$$\text{よ}_B(\beta) \colon \text{よ}_B(b) \Rrightarrow \text{よ}_B(b')$$

as follows:

(3.0) For an object $X \in \text{よ}_B(B')$, we define the component

$$\text{よ}_B(\beta)_X \colon \underbrace{\text{よ}_B(b)_X}_{b \circledcirc \mathrm{id}_X} \Rrightarrow \underbrace{\text{よ}_B(b')_X}_{b' \circledcirc \mathrm{id}_X}$$

to be $\text{よ}_B(\beta)_X \coloneqq \beta \circledcirc \mathrm{id}_{\mathrm{id}_X}$.

Notice $\|\text{よ}_B(\beta)\| = \sup_X \|\beta \circledcirc \mathrm{id}_{\mathrm{id}_X}\| \leq \|\beta\|$, so $\text{よ}_B(\beta)$ is indeed uniformly bounded.

From this definition it is clear that よ_B is linear, †-preserving, and preserves 3-composition $\circ$ and identities at the level of 3-morphisms.

In the W* case, since $\circledcirc$ is separately normal in $\mathcal{B}$, the Kaplansky density theorem together with [CP22, Lemma 2.13] yields that よ_B is normal.

Using cubicality, one can further check that よ_B is locally a strict (normal) †-2-functor between strict C*-2-categories (resp. W*-2-categories). We now recall the construction for the constraint unitary adjoint equivalences よ_B^2 and よ_B^0 for the †-3-functor よ_B:

(よ_B^2) For composable 1-morphisms ${}_{B'''}Y'_{B''}$, ${}_{B''}Y_{B'}$ in $\mathcal{B}$, we provide a tensorator †-2-natural transformation

$$(\text{よ}_B^2)_{Y',Y} \colon \text{よ}_B(Y') \circledcirc \text{よ}_B(Y) \Rightarrow \text{よ}_B(Y' \circledcirc_{B''} Y').$$

as follows:

(よ_B^2.0) On an object $X \in \text{よ}_B(B')$, we define the component

$$((\text{よ}_B^2)_{Y',Y})_X \colon \underbrace{\big(\text{よ}_B(Y') \circledcirc \text{よ}_B(Y)\big)(X)}_{Y' \circledcirc_{B''} (Y \circledcirc_{B'} X)} \Rightarrow \underbrace{\big(\text{よ}_B(Y' \circledcirc_{B''} Y')\big)(X)}_{(Y' \circledcirc_{B''} Y) \circledcirc_{B'} X},$$

to be $((\text{よ}^2_B)_{Y',Y})_X \coloneqq a_{Y',Y,X}$.

(よ^2_B.1) For a 1-morphism b in $\text{よ}_B(B')$, we define the unitary

$$((\text{よ}^2_B)_{Y',Y})_b \coloneqq a_{\mathrm{id}_{Y'},\mathrm{id}_Y,b}.$$

One continues by defining the unitary adjoint equivalence for よ^2_B to be the unitary adjoint equivalence for a with the first two variables held constant.

(よ^0_B) For an object $B' \in \mathcal{B}$, we provide a unitor †-2-natural transformation

$$(\text{よ}^0_B)_{B'} \colon \mathrm{id}_{\text{よ}_B(B')} \Rightarrow \text{よ}_B(\mathrm{id}_{B'}),$$

as follows:

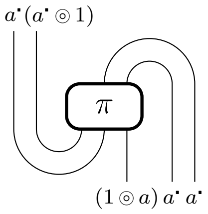
(よ^0_B.0) On an object ${}_{B'}X_B \in \text{よ}_B(B')$, we define the component

$$((\text{よ}^0_B)_{B'})_X \colon X \Rightarrow \mathrm{id}_{B'} \circledcirc X,$$

to be $((\text{よ}^0_B)_{B'})_X \coloneqq r^{\bullet}_X$. One continues defining the unitary adjoint equivalence for よ^0_B similarly to よ^2_B.

We now provide the associativity and unitality constraint unitaries よ^α_B, よ^λ_B, and よ^ρ_B for the †-3-functor よ_B:

(よ^α_B) We define the associator unitary よ^α_B to be the following mate of π, the pentagonator for $\mathcal{B}$.



Indeed, $よ_B^\alpha$ is unitary as π is unitary, and the unit and counit for the unitary adjoint equivalence α are unitaries.

($よ_B^\lambda$) The left unitor unitary $よ_B^\lambda$ is given by a mate of ρ.

($よ_B^\rho$) The right unitor unitary $よ_B^\lambda$ is given by a mate of μ.

We refer the interested reader to [Gur13] for the proof that $よ_B$ satisfies both constraint axioms for a 3-functor. □

Lemma 3.4.0.5. *For a cubical C*-3-category $\mathcal{B}$ and a 1-morphism $X \in \mathcal{B}(B \to B')$, there is an organic †-3-natural transformation*

$$よ_X \colon よ_B \Rightarrow よ_{B'}.$$

Proof. We will present the definition found in [Gur13, Lemma 9.8], noting that all the relevant data is compatible with dagger structures.

(0) For an object $A \in \mathcal{B}$, we define the component

$$(よ_X)_A \colon \underbrace{よ_B(A)}_{\mathcal{B}(A \to B)} \to \underbrace{よ_{B'}(A)}_{\mathcal{B}(A \to B')},$$

to be the (normal) strict †-2-functor given by:

(0.0) For an object ${}_AY_B \in よ_B(A)$,

$$(よ_X)_A(Y) := {}_AY \circledcirc_B X_{B'}.$$

(0.1) For a 1-morphism b in $よ_B(A)$,

$$(よ_X)_A(b) := b \circledcirc \mathrm{id}_X.$$

(0.2) For a 2-morphism β in $\text{よ}_B(A)$,

$$(\text{よ}_X)_A(b) \coloneqq \beta \circledcirc \mathrm{id}_{\mathrm{id}_X}.$$

(1) For a 1-morphism $Y \in \mathcal{B}(A \to A')$, we provide a unitary adjoint equivalence for $(\text{よ}_X)_Y$, where $(\text{よ}_X)_Y$ is a †-2-natural transformation given by:

(1.0) On an object ${}_AZ_B \in \text{よ}_B(A)$, we define the component

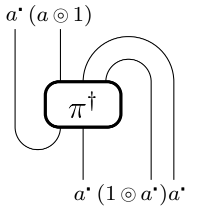
$$(((\text{よ}_X)_Y)_Z) \colon \underbrace{(\text{よ}_X)_{A'} \circledcirc \text{よ}_B(Y)(Z)}_{({}_{A'}Y \circledcirc_A Z) \circledcirc_B X_{B'}} \Rightarrow \underbrace{\text{よ}_{B'}(Y) \circledcirc (\text{よ}_X)_A(Z)}_{{}_{A'}Y \circledcirc_A (Z \circledcirc_B X_{B'})}$$

to be $(((\text{よ}_X)_Y)_Z) \coloneqq a^{\bullet}_{YZX}$.

(1.1) On a 1-morphism c in $\text{よ}_B(A)$, we set $(((\text{よ}_X)_Y)_c \coloneqq a^{\bullet}_{\mathrm{id}_Y, c, \mathrm{id}_X}$.

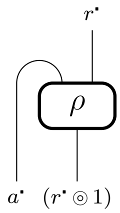
One continues by defining the unitary adjoint equivalence for $(\text{よ}_X)_Y$ using the unitary adjoint equivalence for a with the first and last variables held constant. We now provide the tensorator and unitality constraint unitaries $\overset{2}{\text{よ}}_X$ and $\overset{0}{\text{よ}}_X$.

($\overset{2}{\text{よ}}_X$) We define the unitary tensorator modification $\overset{2}{\text{よ}}_X$ to be the following mate of $\pi^\dagger$, coming from the pentagonator of $\mathcal{B}$.



Indeed, $\overset{2}{\text{よ}}_X$ is unitary since π is unitary and the unit and counit for the adjoint equivalence a are unitaries.

($\overset{0}{\text{よ}}_X$) We define the unitary unitor modification $\overset{0}{\text{よ}}_X$ to be the following mate of ρ.



Indeed, よ^2_X is unitary since ρ is unitary and the unit and counit for the adjoint equivalence r are unitaries

We refer the interested reader to [Gur13] for the proof that よ_X satisfies the three constraint axioms for a 3-natural transformation. □

Lemma 3.4.0.6. *For a cubical C*-3-category $\mathcal{B}$ and a 2-morphism $b \in \mathcal{B}(X \Rightarrow X')$, there is an organic †-3-modification*

$$\text{よ}_b \colon \text{よ}_X \Rrightarrow \text{よ}_{X'}.$$

Proof. We will present the definition found in [Gur13, Lemma 9.9], noting that all the relevant data is compatible with dagger structures.

(0) For an object $A \in \mathcal{B}$, we define the component

$$(\text{よ}_b)_A \colon (\text{よ}_X)_A \Rightarrow (\text{よ}_{X'})_A$$

to be the †-2-natural transformation given by:

(0.0) On an object ${}_AY_B$, we define the component

$$(((\text{よ}_b)_A)_Y \colon \underbrace{(\text{よ}_X)_A(Y)}_{Y \circledcirc X} \Rightarrow \underbrace{(\text{よ}_{X'})_A(Y)}_{Y \circledcirc X'}$$

to be $(((\text{よ}_b)_A)_Y \coloneqq \mathrm{id}_Y \circledcirc b$.

(0.1) On a 1-morphism a, we define the naturality unitary constraint $((\text{よ}_b)_A)_a$ to be $((\text{よ}_b)_A)_a \coloneqq \Sigma^{\dagger}_{b,a}$, which comes from the unitary interchanger for the cubical 1-composition $\circledcirc$ in $\mathcal{B}$.

(1) For a 1-morphism $Z \in \mathcal{B}(A \to A')$, we define the naturality constraint unitary $(\text{よ}_b)_Z$ to be the unitary modification given by:

(1.0) On an object $Y \in \mathcal{B}$, we define the component $((\text{よ}_b)_Z)_Y \coloneqq a^{\cdot}_{\mathrm{id}_Z,\mathrm{id}_Y,b}$.

We refer the interested reader to [Gur13] for the proof that よ_b satisfies the two constraint axioms for a 3-modification. $\square$

Lemma 3.4.0.7. *For a cubical C*-3-category $\mathcal{B}$ and a 3-morphism $\beta \in \mathcal{B}(b \Rrightarrow b')$, there is an organic uniformly bounded perturbation*

$$\text{よ}_\beta \colon \text{よ}_b \Rrightarrow \text{よ}_{b'}.$$

This assignment is linear and †-preserving in β. Furthermore, when $\mathcal{B}$ is a cubical W-3-category, we have that this assignment is normal.*

Proof. We will present the definition found in [Gur13, Lemma 9.10], noting that all the relevant data is compatible with dagger structures.

(0) For an object $A \in \mathcal{B}$, we define the component

$$(\text{よ}_\beta)_A \colon (\text{よ}_b)_A \Rrightarrow (\text{よ}_{b'})_A$$

to be the uniformly bounded modification given by:

(0.0) For an object ${}_AY_B$, we define the component

$$((\text{よ}_\beta)_A)_Y \colon \underbrace{((\text{よ}_b)_A)_Y}_{\mathrm{id}_Y \circledcirc b} \Rrightarrow \underbrace{((\text{よ}_{b'})_A)_Y}_{\mathrm{id}_Y \circledcirc b'}$$

to be $((\text{よ}_\beta)_A)_Y \coloneqq \mathrm{id}_{\mathrm{id}_Y} \circledcirc \beta$.

From this definition it is clear that the assignment $\beta \mapsto \text{よ}_\beta$ is linear, †-preserving, and preserves 3-composition $\circ$ and identities at the level of 3-morphisms. Furthermore, this assignment is normal when $\mathcal{B}$ is W* by our usual argument using [CP22, Lemma 2.13]. We refer the interested reader to [Gur13] for the proof that よ_β satisfies the constraint axiom for perturbations. □

Theorem 3.4.0.8. *We may upgrade the previous data to a monic †-3-functor*

$$\text{よ} : \mathcal{B} \to \mathrm{C}^*\mathsf{3Cat}(\mathcal{B}^{\circledcirc\mathrm{p}} \to \mathrm{C}^*\mathsf{2Cat}_{\mathsf{strict}}),$$

which is locally a 2-equivalence. When $\mathcal{B}$ is a W-3-category, we obtain a monic normal †-3-functor*

$$\text{よ} : \mathcal{B} \to \mathrm{W}^*\mathsf{3Cat}(\mathcal{B}^{\circledcirc\mathrm{p}} \to \mathrm{W}^*\mathsf{2Cat}_{\mathsf{strict}}).$$

Proof. We provide the constraint unitary adjoint equivalences よ^2 and よ^0 for the †-3-functor よ_B found in [Gur13, Theorem 9.12].

(よ^2) For composable 1-morphisms ${}_BX_{B'}$ and ${}_{B'}Y_{B''}$ in cB, we define the component

$$\text{よ}^2_{X,Y} \colon \text{よ}_X \circledcirc \text{よ}_Y \Rrightarrow \text{よ}_{X \circledcirc Y}$$

to be the †-3-modification given by:

(よ^2.0) On an object $A \in \mathcal{A}$, we define the component

$$(\text{よ}^2_{X,Y})_A \colon (\text{よ}_X \circledcirc \text{よ}_Y)_A \Rightarrow (\text{よ}_{X \circledcirc Y})_A$$

to be the †-2-natural transformation given by:

(よ^2.0.0) For an object ${}_AZ_B \in \text{よ}_B(A)$,

$$((\text{よ}^2_{X,Y})_A)_Z \colon \underbrace{(\text{よ}_X \circledcirc \text{よ}_Y)_A(Z)}_{(Z \circledcirc X) \circledcirc Y} \Rightarrow \underbrace{(\text{よ}_{X \circledcirc Y})_A(Z)}_{Z \circledcirc (X \circledcirc Y)},$$

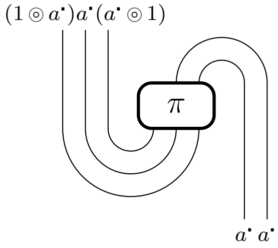
to be $((\text{よ}^2_{X,Y})_A)_Z \coloneqq a^{\bullet}_{ZXY}$.

($よ^2$.0.1) For a 1-morphism z in $よ_B(A)$, we then define

$$((よ^2_{X,Y})_A)_Z := a^{\cdot}_{z,\mathrm{id}_X,\mathrm{id}_Y}.$$

($よ^2$.1) On a 1-morphism ${}_AV_{A'}$, we define the naturator $(よ^2_{X,Y})_V$ to be the unitary modification given by:

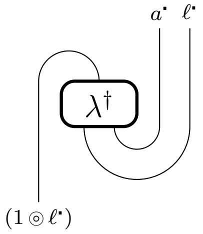
($よ^2$.1.0) On an object ${}_{A'}W_B$, we define the component $((よ^2_{X,Y})_V)_W$ to be the following mate of π, the unitary pentagonator of $\mathcal{B}$.



One continues by defining the unitary adjoint equivalence for $よ^2_{X,Y}$ using the unitary adjoint equivalence for a with the last two entries fixed, and a mate of $\pi^{\dagger}$.

Then one defines the naturality constraints for $よ^2$ using the unitary adjoint equivalence for a, with the first entry fixed.

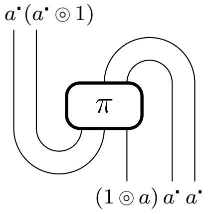
($よ^0$) We define the unitor adjoint equivalence $よ^0$ similarly to $よ^2$, using $\ell^{\cdot}$ instead of $a^{\cdot}$, and the following mate of $\lambda^{\dagger}$.



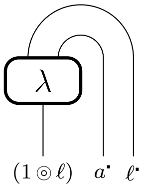
We also define unitary perturbation $(よ^0)_{\bullet}$ to be the identity.

We now provide the constraint unitaries $よ^{\alpha}$, $よ^{\lambda}$, and $よ^{\rho}$ for the †-3-functor $よ_B$.

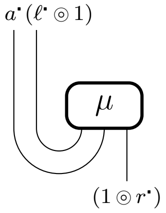
(よ^{α}) For composable 1-morphisms ${}_{B}X_{B'}$, ${}_{B'}Y_{B''}$, ${}_{B''}Z_{B'''}$ in $\mathcal{B}$, we define the components $((\text{よ}^{\alpha}_{XYZ})_A)_{{}_AW_B}$ of the unitary modification $(\text{よ}^{\alpha}_{XYZ})_A$ for the unitary perturbation よ^{α}_{XYZ} using the following mate of the pentagonator π_{WXYZ}.



(よ^{λ}) Similarly, for a 1-morphism X in $\mathcal{B}$ we use the following mate of the left unitor λ to define the unitary perturbation $(\text{よ}^{\lambda})_X$.



(よ^{ρ}) Finally, for a 1-morphism X in $\mathcal{B}$ we use the following mate of the middle unitor μ to define the unitary perturbation $(\text{よ}^{\rho})_X$.



We refer the interested reader to [Gur13] for the proof that よ satisfies both constraint axioms for 3-functors and locally a biequivalence. Quite pedantically, it is also clear that よ is injective on every level. □

Theorem 3.4.0.9 (Gelfand-Naimark for operator 3-categories)**.** *Every small C*-3-category $\mathcal{B}$ is 3-equivalent to a sub-*$\mathrm{C}^*\mathsf{Gray}$*-category of* $\mathsf{Hilb}^{\mathsf{W}^*2}$.

Proof. When $\mathcal{B}$ is a small C*Gray-category, notice

$$
\begin{aligned}
よ^{\amalg}\colon \mathcal{B} &\to \mathrm{C}^*2\mathsf{Cat}_{\mathsf{strict,small}} \\
B &\mapsto \coprod_{B'\in\mathcal{B}} \mathcal{B}(B' \to B)
\end{aligned}
$$

is a monic †-3-functor. Hence, we need only extend the Gelfand-Naimark-Segal construction for C*-2-categories into a monic †-3-functor

$$
\begin{aligned}
\mathsf{GNS}_2''\colon \mathrm{C}^*2\mathsf{Cat}_{\mathsf{strict,small}} &\to \mathsf{Hilb}^{\mathsf{W}^*2}. \\
\mathcal{C} &\mapsto \Upsilon_2(\mathcal{C}).
\end{aligned}
$$

This is done analogously to how one extends the Gelfand-Naimark-Segal construction for C*-1-categories into a monic †-2-functor

$$
\begin{aligned}
\mathsf{GNS}''\colon \mathrm{C}^*\mathsf{Cat}_{\mathsf{small}} &\to \mathsf{Hilb}^{\mathsf{W}^*}. \\
\mathcal{D} &\mapsto \Upsilon(\mathcal{D}).
\end{aligned}
$$

We will provide the details for how to define $\mathsf{GNS}_2''(F)\colon \Upsilon_2(\mathcal{C}) \to \Upsilon_2(\mathcal{C}')$ for a strict †-2-functor $F\colon \mathcal{C} \to \mathcal{C}'$ between strict, small C*-2-categories, and leave the remaining details to the reader. The strict †-2-functor $\mathsf{GNS}_2''(F)$ maps an object in $\Upsilon_2(\mathcal{C})$ given by

$$
\mathsf{GNS}\left(\coprod_{C'\in\mathcal{C}} \mathcal{C}(C' \to C)\right)'' \cong \coprod_{C'\in\mathcal{C}} \mathsf{GNS}(\mathcal{C}(C' \to C))''
$$

to the following object in $\Upsilon_2(\mathcal{C}')$

$$
\mathsf{GNS}\left(\coprod_{C'\in\mathcal{C}'} \mathcal{C}'(C' \to FC)\right)'' \cong \coprod_{C'\in\mathcal{C}'} \mathsf{GNS}(\mathcal{C}'(C' \to FC))''.
$$

On 1-morphisms and 2-morphisms in $\Upsilon_2(\mathcal{C})$, the action of $\mathsf{GNS}_2''(F)$ is then given by the universal completion of post-composition in each coproduct component. Notice this †-2-functor is indeed strict since $\mathcal{C}'$ is a strict C*-2-category and F is a strict †-2-functor. □

3.5 The Morita operator 3-category of abelian C*-algebras

In this section, we will revisit correspondences between operator algebras (see Section §1.1.8). To ease our notation throughout the following computations, we will drop the subscript for the operator algebra valued inner products and simplify both left and right actions $\triangleright$, $\triangleleft$ to $\cdot$.

Example 3.5.0.1. We define $\mathsf{AbC^*Alg}$ to be the following C*-3-category:

(0) Objects are compact Hausdorff spaces T, or equivalently, abelian C*-algebras $C(T)$;

(1) A morphism $A\colon X \to Y$ is a C*-algebra equipped with $*$-homomorphisms $C(X) \to Z(A)$ and $C(Y) \to Z(A)$;

(2) A 2-morphism $H\colon A \Rightarrow B$ is an A-B C*-correspondence, i.e. a right B-Hilbert module H together with a left A-action $A \to \mathcal{L}_B(H)$, such that

$$ay \cdot h \cdot b = a \cdot h \cdot yb \quad \text{for } a \in A, y \in C(Y), h \in H, b \in B,$$

and a similar compatibility axiom for the $C(X)$-action. Here $\mathcal{L}_B(H)$ is the C*-algebra of adjointable B-intertwiner

(3) A 3-morphism $\varphi\colon H \to K$ is an adjointable intertwiner, and define $\varphi^\dagger$ to be its adjoint.

($\circ$) The composition $\circ$ of 3-morphisms is given by composition of maps, which is clearly linear and †-preserving.

($\odot$) The composition $\odot$ of 2-morphisms $H\colon A \Rightarrow B$ and $K\colon B \Rightarrow C$ is given by the right C-Hilbert module

$$H \otimes_B K := H \otimes K / \overline{\operatorname{span}}(hb \otimes k - h \otimes bk : h \in H, b \in B, k \in K)$$

with C-valued inner product determined by

$$\langle h \otimes k, h' \otimes k' \rangle \coloneqq \langle k, \langle h, h' \rangle \cdot k' \rangle \quad \text{for } h, h' \in H \text{ and } k, k' \in K.$$

We determine the left action of A on $H \otimes_B K$ by

$$a \cdot (h \otimes k) \coloneqq (a \cdot h) \otimes k,$$

which is well-defined since the action of A on H is given by right B-module homomorphisms. We define the right C-action on $H \otimes_B K$ similarly. Furthermore, notice

$$\begin{aligned} ay \cdot (h \otimes k) \cdot c &= (ay \cdot h) \otimes (k \cdot c) \\ &= (a \cdot h \cdot y) \otimes (k \cdot c) \\ &= (a \cdot h) \otimes (y \cdot k \cdot c) \\ &= (a \cdot h) \otimes (k \cdot yc) \\ &= a \cdot (h \otimes k) \cdot yc, \end{aligned}$$

and a similar argument shows the compatibility axiom for the $C(X)$-action. Hence $H \odot_B K$ satisfies the property required of 2-morphisms. The composition $\odot$ of 3-morphisms $\varphi\colon {}_AH_B \Rrightarrow {}_AH'_B$ and $\psi\colon {}_BK_C \Rrightarrow {}_BK'_B$ is determined by

$$(\varphi \odot_B \psi)(h \otimes k) = \varphi(h) \otimes \psi(k) \quad \text{for } h \in H, k \in K.$$

This is well-defined on $H \otimes_B K$ since

$$(\varphi \odot_B \psi)(hb \otimes k) = \varphi(h)b \otimes \psi(k) = \varphi(h) \otimes b\psi(k) = (\varphi \odot_B \psi)(h \otimes bk).$$

One similarly shows that $\varphi \odot_B \psi$ is intertwiner. To see that it is also adjointable, observe

$$\begin{aligned}\langle (\varphi \odot_B \psi)(h \otimes k), h' \otimes k' \rangle &= \langle \psi(k), \langle \varphi(h), h' \rangle \cdot k' \rangle \\ &= \langle k, \langle h, \varphi^\dagger(h') \rangle \cdot \psi^\dagger(k') \rangle \\ &= \langle h \otimes k, (\varphi^\dagger \otimes_B \psi^\dagger)(h' \otimes k') \rangle\end{aligned}$$

Hence $(\varphi \odot_B \psi)^\dagger = \varphi^\dagger \odot_B \psi^\dagger$. It is clear that $\odot$ is bilinear from this definition.

($\circledcirc$) We define the composition $\circledcirc$ of 1-morphisms $A\colon X \to Y$ and $B\colon Y \to Z$ to be

$$A \circledcirc_Y B \coloneqq (A \underset{\max}{\otimes} B)/\overline{\operatorname{span}}(ay \otimes b - a \otimes yb : a \in A, y \in C(Y), b \in B),$$

where we note that $\overline{\operatorname{span}}(ay \otimes b - a \otimes yb : a \in A, y \in C(Y), b \in B)$ is a (norm-closed) two-sided ideal in $A \underset{\max}{\otimes} B$ since

$$(a' \otimes b')(ay \otimes b - a \otimes yb) = a'ay \otimes b'b - a'a \otimes b'yb = a'ay \otimes b'b - a'a \otimes yb'b$$

$$(ay \otimes b - a \otimes yb)(a' \otimes b') = aya' \otimes bb' - aa' \otimes ybb' = aa'y \otimes bb' - aa' \otimes ybb'.$$

In fact, $A \circledcirc_Y B$ is the coequalizer of $C(Y) \to A \to A \underset{\max}{\otimes} B$ and $C(Y) \to B \to A \underset{\max}{\otimes} B$ given by

$$A \circledcirc_Y B = (A \underset{\max}{\otimes} B)/\langle y \otimes 1_B - 1_A \otimes y : y \in C(Y) \rangle.$$

We determine the maps $C(X) \to Z(A \odot_Y B)$ and $C(Y) \to Z(A \odot_Y B)$ by $x \mapsto x \otimes 1_B$ and $y \mapsto 1_A \otimes y$ respectively, which are clearly $*$-homomorphisms.

We define the composition $\circledcirc$ of 2-morphisms $H\colon {}_XA_Y \Rightarrow {}_XA'_Y$ and $K\colon {}_YB_Z \Rightarrow {}_YB'_Z$ to be the $(A \circledcirc_Y B)$-$(A' \circledcirc_Y B')$ C*-correspondence with right $(A' \circledcirc_Y B')$-Hilbert module

$$H \circledcirc_Y K \coloneqq H \otimes K/\overline{\operatorname{span}}(hy \otimes k - h \otimes yk : h \in H, y \in C(Y), k \in K).$$

Indeed, we equip $H \circledcirc_Y K$ with the $(A' \circledcirc_Y B')$-valued inner product determined by

$$\langle h \otimes k, h' \otimes k' \rangle = \langle h, h' \rangle \otimes \langle k, k' \rangle.$$

Notice this is well-defined since

$$\begin{aligned} \langle hy \otimes k, h' \otimes k' \rangle &= \langle hy, h' \rangle \otimes \langle k, k' \rangle = \langle h, h' \rangle y^* \otimes \langle k, k' \rangle \\ &= \langle h, h' \rangle \otimes y^* \langle k, k' \rangle = \langle h, h' \rangle \otimes \langle k, k' \rangle y^* \\ &= \langle h, h' \rangle \otimes \langle ky, k' \rangle = \langle h, h' \rangle \otimes \langle yk, k' \rangle \\ &= \langle h \otimes yk, h' \otimes k' \rangle. \end{aligned}$$

We then determine the right $(A' \circledcirc_Y B')$-action on $H \circledcirc_Y K$ by

$$(h \otimes k) \cdot (a' \otimes b') = ha' \otimes kb',$$

which is well-defined since

$$\begin{aligned} (h \otimes k) \cdot (a'y \otimes b') &= ha'y \otimes kb' = ha' \otimes ykb' = ha' \otimes kyb' = (h \otimes k) \cdot (a' \otimes yb') \\ (hy \otimes k) \cdot (a' \otimes b') &= hya' \otimes kb' = ha'y \otimes kb' = ha' \otimes ykb' = (h \otimes yk) \cdot (a' \otimes b'). \end{aligned}$$

One determines the left $(A \circledcirc_Y B)$-action similarly.

Finally, one determines the composition $\circledcirc$ of 3-morphisms akin to $\odot$ and it is easy to show $\circledcirc$ is †-preserving and bilinear.

We now define the tensorator (or interchanger) for $\circledcirc$. In particular, for $A, A', A'' \colon X \to Y$, $B, B', B'' \colon Y \to Z$ and $A \xrightarrow{H} A' \xrightarrow{H'} A''$, $B \xrightarrow{K} B' \xrightarrow{K'} B''$, we define a unitary

$$\Sigma \colon (H \circledcirc_Y K) \odot_{A' \circledcirc_Y B'} (H' \circledcirc_Y K') \Rrightarrow (H \odot_{A'} H') \circledcirc_Y (K \odot_{B'} K'),$$

determined by

$$(h \otimes k) \otimes (h' \otimes k') \mapsto (h \otimes h') \otimes (k \otimes k').$$

A simple computation reveals that Σ is a well-defined intertwiner, clearly admitting an inverse intertwiner. Hence, to show Σ is unitary, it suffices to show it is isometric. Observe

$$\begin{aligned}
\langle (h \otimes k) \otimes (h' \otimes k'), (\widetilde{h} \otimes \widetilde{k}) \otimes (\widetilde{h}' \otimes \widetilde{k}') \rangle &= \langle h' \otimes k', \langle h \otimes k, \widetilde{h} \otimes \widetilde{k} \rangle \cdot \widetilde{h}' \otimes \widetilde{k}' \rangle \\
&= \langle h' \otimes k', (\langle h, \widetilde{h} \rangle \otimes \langle k, \widetilde{k} \rangle) \cdot \widetilde{h}' \otimes \widetilde{k}' \rangle \\
&= \langle h' \otimes k', (\langle h, \widetilde{h} \rangle \cdot \widetilde{h}') \otimes (\langle k, \widetilde{k} \rangle \cdot \widetilde{k}') \rangle \\
&= \langle h', \langle h, \widetilde{h} \rangle \cdot \widetilde{h}' \rangle \otimes \langle k', \langle k, \widetilde{k} \rangle \cdot \widetilde{k}' \rangle \\
&= \langle h \otimes h', \widetilde{h} \otimes \widetilde{h}' \rangle \otimes \langle k \otimes k', \widetilde{k} \otimes \widetilde{k}' \rangle \\
&= \langle (h \otimes h') \otimes (k \otimes k'), (\widetilde{h} \otimes \widetilde{h}') \otimes (\widetilde{k} \otimes \widetilde{k}') \rangle.
\end{aligned}$$

On the other hand, the unitor for $\odot$ is defined on components $\mathrm{id}_{A \circledcirc_Y B} \Rightarrow \mathrm{id}_A \circledcirc_Y \mathrm{id}_B$ to be the identity unitary interchanger on $A \circledcirc_Y B$ viewed as a C*-correspondence over itself. Since the tensorator (interchanger) is morally a swap between the middle tensorands, it is quite easy to see that these satisfy the associativity and unitalily axioms for a †-2-functor.

It is well-known that C*-algebras, C*-correspondences, and adjointable intertwiners form a C*-2-category, which we will denote by $\mathsf{C^*Alg}$. [CHPJP22] [BLM04] [Pas73] [Rie74] One can slightly modify this argument to show that $\mathsf{AbC^*Alg}(X, Y)$ is a C*-2-category for every $X, Y \in \mathsf{AbC^*Alg}$.

(I) For an object $X \in \mathsf{AbC^*Alg}$, we set $I_A \coloneqq C(X)$ where the maps $C(X) \to Z(C(X))$ are given by the identity map on $C(X)$. We then extend $I_A \colon \mathbb{C} \to \mathsf{AbC^*Alg}(X \to X)$ trivially into a †-2-functor.

Recall that any (non-degenerate) $*$-homomorphism $\varphi\colon A \to B$ between C*-algebras induces an $A - B$ C*-correspondence H_φ given by:

- The underlying vector space is B with $\langle b, b' \rangle = b^* b'$ for $b, b' \in B$.
- The left A-action is given by

  $$a \cdot b \coloneqq \varphi(a) b,$$

  for $a \in A$ and $b \in B$.
- The right B-action is given by right multiplication in B.

When φ is unitary, there exists an induced unitary adjoint equivalence $(H_\varphi, H^{\bullet}_{\varphi}, \eta, \epsilon)$ given by:

- the B-A C*-correspondence $H^{\bullet}_{\varphi} \coloneqq H_{\varphi^*}$,
- a unitary intertwiner $\eta\colon \mathrm{id}_A \to H_\varphi \odot_B H^{\bullet}_{\varphi}$ defined as follows. Note that

  $$b \otimes a = 1 \cdot \varphi(\varphi^*(b)) \otimes a = 1 \otimes \varphi^*(b) \cdot a$$

  in $H_\varphi \odot_B H^{\bullet}_{\varphi}$. Hence every element $\sum b_i \otimes a_i$ in $H_\varphi \odot_B H^{\bullet}_{\varphi}$ is of the form $1 \otimes a$, and it is easy to show there is a unique such $a \in A$. We thus define η to be the bijection $a \mapsto 1 \otimes a$. Notice η is an intertwiner

  $$\begin{aligned} &\eta(a) \cdot \widetilde{a} = (1 \otimes a) \cdot \widetilde{a} = 1 \otimes (a\widetilde{a}) = \eta(a\widetilde{a}) \\ &\widetilde{a} \cdot \eta(a) = \widetilde{a}(1 \otimes a) = \varphi(\widetilde{a}) \otimes a = 1 \otimes \varphi^*(\varphi(\widetilde{a}))a = 1 \otimes \widetilde{a}a = \eta(\widetilde{a}a), \end{aligned}$$

  and an isometry

  $$\langle 1 \otimes a, 1 \otimes \widetilde{a} \rangle = \langle a, \langle 1, 1 \rangle \cdot \widetilde{a} \rangle = \langle a, (1^*1) \cdot \widetilde{a} \rangle = \langle a, \widetilde{a} \rangle.$$

  Therefore η is indeed a unitary intertwiner.

- One defines the unitary intertwiner $\epsilon\colon H_\varphi^\bullet \odot_B H_\varphi \to \mathrm{id}_B$ similarly by exchanging the roles of A, B and of $\varphi, \varphi^\bullet$.

A formal calculation reveals that ϵ and η satisfies the zig-zag equations.

(a) For composable 1-morphisms ${}_XA_Y$ and ${}_YB_Z$, The universal property of coequalizers yields an isomorphism $\varphi_a\colon A \circledcirc_Y (B \circledcirc_Z C) \to (A \circledcirc_Y B) \circledcirc_Z C$. This map is determined by $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$, and on easy verifies φ_a is unitary. We thus define the $A \circledcirc (B \circledcirc C) - (A \circledcirc B) \circledcirc C$-correspondence $a_{A,B,C}$ to be H_{φ_a} and extend it to a unitary adjoint equivalence as mentioned previously.

We now define the naturality constraint for a. In particular, for $X \xrightarrow{A} Y \xrightarrow{B} Z \xrightarrow{C} W$, $X \xrightarrow{A'} Y \xrightarrow{B'} Z \xrightarrow{C'} W$, and C*-correspondences ${}_AH_{A'}$, ${}_BK_{B'}$, ${}_CL_{C'}$, we define a unitary

$$a_{HKL}\colon (H \circledcirc (K \circledcirc L)) \odot a_{A'B'C'} \Rrightarrow a_{ABC} \odot ((H \circledcirc K) \circledcirc L).$$

We note that that every element in $(H \circledcirc (K \circledcirc L)) \odot a_{A'B'C'}$ can be written as a sum of elements of the form

$$(h \otimes (k \otimes l)) \otimes ((1_{A'} \otimes 1_{B'}) \otimes 1_{C'}).$$

We then determine a_{HKL} to be the map

$$(h \otimes (k \otimes l)) \otimes ((1_{A'} \otimes 1_{B'}) \otimes 1_{C'}) \mapsto ((1_A \otimes 1_B) \otimes 1_C) \otimes ((h \otimes k) \otimes l).$$

A simple computation reveals that a_{HKL} is a well-defined intertwiner, clearly admitting an inverse intertwiner. To see that a_{HKL} is unitary, we will show it is isometric. Observe,

$$\begin{aligned}
&\langle (h \otimes (k \otimes l)) \otimes ((1 \otimes 1) \otimes 1), (\widetilde{h} \otimes (\widetilde{k} \otimes \widetilde{l})) \otimes ((1 \otimes 1) \otimes 1) \rangle \\
&= \langle (1 \otimes 1) \otimes 1, \langle h \otimes (k \otimes l), \widetilde{h} \otimes (\widetilde{k} \otimes \widetilde{l}) \rangle \cdot ((1 \otimes 1) \otimes 1) \rangle \\
&= \langle (1 \otimes 1) \otimes 1, (\langle h, \widetilde{h} \rangle \otimes \langle k, \widetilde{k} \rangle) \otimes \langle l, \widetilde{l} \rangle \rangle \\
&= (\langle h, \widetilde{h} \rangle \otimes \langle k, \widetilde{k} \rangle) \otimes \langle l, \widetilde{l} \rangle \\
&= \langle (h \otimes k) \otimes l, \langle (1 \otimes 1) \otimes 1, (1 \otimes 1) \otimes 1 \rangle ((\widetilde{h} \otimes \widetilde{k}) \otimes \widetilde{l}) \rangle \\
&= \langle ((1 \otimes 1) \otimes 1) \otimes ((h \otimes k) \otimes l), ((1 \otimes 1) \otimes 1) \otimes ((\widetilde{h} \otimes \widetilde{k}) \otimes \widetilde{l}) \rangle.
\end{aligned}$$

It is also clear from construction that a_{HKL} is natural in H, K, and L.

(u) Notice $I_X \circledcirc_X A = C(X) \circledcirc_X A \cong A$ for a 1-morphism ${}_XA_Y$. Denoting this unitary by φ_ℓ, we define the $(I_X \circledcirc A) - A$ C*-correspondence ℓ_A to be H_{φ_ℓ}, as extend it to a unitary adjoint equivalence as mentioned previously. One defines the C*-correspondence $r_A = H_{\varphi_r}$ similarly through a map $\varphi_r \colon A \circledcirc_Y I_Y \to A$. For a C*-correspondence ${}_AH_B$ between C*-algebras $A, B \colon X \to Y$, one determines the naturality constraint unitary interchangers

$$\begin{aligned}
\ell_X &\colon (I_X \circledcirc H) \odot \ell_B \Rrightarrow \ell_A \odot H \\
r_X &\colon (H \circledcirc I_Y) \odot r_B \Rrightarrow r_A \odot H
\end{aligned}$$

respectively by

$$\begin{aligned}
(1_{C(X)} \otimes h) \otimes 1_B &\mapsto 1_A \otimes h, \\
(h \otimes 1_{C(Y)}) \otimes 1_B &\mapsto 1_A \otimes h.
\end{aligned}$$

(π) In what follows, we will supress the associators for $\odot$. For composable C*-algebras A, B, C, D, we determine the natural unitary interchanger

$$\pi_{ABCD} \colon (\mathrm{id}_A \circledcirc a_{BCD}) \odot a_{A,B \circledcirc C,D} \odot (a_{ABC} \circledcirc \mathrm{id}_D) \Rrightarrow a_{A,B,C \circledcirc D} \odot a_{A \circledcirc B,C,D},$$

to be the map which sends

$$(1_A \otimes ((1_B \otimes 1_C) \otimes 1_D)) \otimes ((1_A \otimes (1_B \otimes 1_C)) \otimes 1_D) \otimes (((a \otimes b) \otimes c) \otimes d),$$

to

$$((1_A \otimes 1_B) \otimes (1_C \otimes 1_D)) \otimes (((a \otimes b) \otimes c) \otimes d).$$

(coh) For composable C*-algebras ${}_XA_Y$ and ${}_YB_Z$, we define the middle, left, and right unity coheretor unitary intertwiners

$$\mu_{AB} \colon (\mathrm{id}_a \circledcirc \ell^{\scriptscriptstyle\blacksquare}_B) \odot a_{A,I_Y,B} \odot (r_A \circledcirc \mathrm{id}_B) \Rrightarrow \mathrm{id}_{A \circledcirc B}$$

$$\lambda_{AB} \colon \ell^{\scriptscriptstyle\blacksquare}_A \circledcirc \mathrm{id}_B \Rrightarrow \ell^{\scriptscriptstyle\blacksquare}_{A \circledcirc B} \odot a_{I_X,A,B}$$

$$\rho_{AB} \colon \mathrm{id}_A \circledcirc r_B \Rrightarrow a_{A,B,I_Z} \odot r_{A \circledcirc B}$$

to be the maps determined by

$$\begin{aligned} (1_A \otimes (1_{C(Y)} \otimes 1_B)) \otimes ((1_A \otimes 1_{C(Y)}) \otimes 1_B) \otimes (a \otimes b) &\mapsto a \otimes b, \\ (1_{C(X)} \otimes a) \otimes b &\mapsto (1_{C(X)} \otimes (1_A \otimes 1_B)) \otimes ((1_{C(X)} \otimes a) \otimes b), \\ a \otimes b &\mapsto ((1_A \otimes 1_B) \otimes 1_{C(Z)}) \otimes (a \otimes b). \end{aligned}$$

The associativity condition for $\circledcirc$ compares two 3-morphisms with source

$$(\mathrm{id}_A \circledcirc \mathrm{id}_B \circledcirc a_{CDE}) \odot (\mathrm{id}_A \circledcirc a_{B,CD,E}) \odot (\mathrm{id}_A \circledcirc (a_{BCD} \circledcirc \mathrm{id}_E)) \odot a_{A,(BC)D,E} \odot (a_{A,BC,D} \circledcirc \mathrm{id}_E) \odot ((a_{ABC} \circledcirc \mathrm{id}_D) \circledcirc \mathrm{id}_E),$$

and target

$$a_{A,B,C(DE)} \odot a_{AB,C,DE} \odot a_{(AB)C,D,E}.$$

A formal calculation reveals that both 3-morphisms are determined by mapping

$$1_{A(B((CD)E))} \otimes 1_{A((B(CD))E)} \otimes 1_{A(((BC)D)E} \otimes 1_{(A((BC)D))E} \otimes 1_{(((AB)C)D)E} \otimes ((((a \otimes b) \otimes c) \otimes d) \otimes e)$$

to

$$1_{(AB)(C(DE))} \otimes 1_{((AB)C)(DE)} \otimes ((((a \otimes b) \otimes c) \otimes d) \otimes e)$$

where we denote $1_A \otimes (1_B \otimes ((1_C \otimes 1_D) \otimes 1_E))$ by $1_{A(B((CD)E))}$ and so on.

The two axioms relating the unity coheretors with the associators and pentagonator compare two 3-morphisms with sources

$$(\mathrm{id}_A \circledcirc (\mathrm{id}_B \circledcirc \ell^{\centerdot}_C)) \odot (\mathrm{id}_A \circledcirc a_{B,I_Z,C}) \odot (\mathrm{id}_A \circledcirc (r_B \circledcirc B)) \odot a_{ABC},$$

$$a_{ABC} \odot ((\mathrm{id}_A \circledcirc \ell^{\centerdot}_B) \circledcirc \mathrm{id}_C) \odot (a_{A,I_Y,B} \circledcirc \mathrm{id}_C) \odot ((r_A \circledcirc \mathrm{id}_B) \circledcirc \mathrm{id}_C),$$

and targets

$$a_{ABC} \odot \mathrm{id}_{(AB)C},$$

$$\mathrm{id}_{A(BC)} \odot a_{ABC}$$

respectively. Another formal calculation reveals that both 3-morphisms are determined by mapping

$$1_{A(B(I_Z C)} \otimes 1_{A((B I_Z)C)} \otimes 1_{A(BC)} \otimes ((a \otimes b) \otimes c),$$

$$1_{(AB)C} \otimes 1_{(A(I_Y B))C} \otimes 1_{((A I_Y)B)C} \otimes ((a \otimes b) \otimes c),$$

into

$$1_{(AB)C} \otimes ((a \otimes b) \otimes c),$$

$$1_{A(BC)} \otimes ((a \otimes b) \otimes c),$$

respectively.

Chapter 4: The homotopy 3-type of abelian C*-algebras

The content of this chapter is based on joint work with Gregory Faurot [GF26]. The authors would like to thank Corey Jones for initially suggesting this problem. We also thank David Penneys and Nick Gurski for numerous helpful conversations about bimodules and homotopy types. The authors were partially supported by NSF grant DMS-2154389.

In this chapter, we compute the homotopy groups at each unital abelian C*-algebra $C(T)$ in the Morita 3-category of abelian C*-algebras, C*-algebras with central maps, C*-correspondences, and adjointable bimodule maps. We then describe these groups in terms of the topological data of the underlying compact Hausdorff space T. Finally, we compute the actions of the first homotopy group on the second and third homotopy groups in terms of these topological invariants of T.

4.1 Introduction

The study of C*-algebras is frequently referred to as "noncommutative topology" since Gelfand duality describes the correspondence between abelian C*-algebras and compact Hausdorff spaces. This allows topological properties to be reformulated in the context of a (generally noncommutative) C*-algebra. The most notable example of this transfer is the introduction of topological K-theory to the study of C^*-algebras, which led to the Elliott

Classification Program for simple, amenable C*-algebras [CGS+]. In a similar vein, the Serre–Swan Theorem ([Swa62, p. 267]) gives another correspondence between abelian C*-algebras and topology; in this case, however, it is between finitely generated projective $C(T)$-modules and vector bundles over a compact Hausdorff space T.

Categorically, Gelfand duality is a statement at the level of 1-categories, giving an equivalence between the C*-algebraic and topological categories. The Serre–Swan Theorem, on the other hand, can be interpreted as a 2-categorical statement. Because the C*-2-category $\mathsf{C^*Alg}$ has bimodules as its 1-morphisms, the Serre–Swan Theorem uses topology to describe the finitely generated projective 1-morphisms from $C(T)$ to itself. In this paper, we explore a 3-categorical analogue of Gelfand duality and the Serre–Swan Theorem.

To elaborate more on C*-categories, the category $\mathsf{C^*Alg}_1$, whose objects are C*-algebras and whose morphisms are $*$-homomorphisms, is a 1-category. C*-algebras also lie inside of a 2-category $\mathsf{C^*Alg}$, where the objects are C*-algebras, the 1-morphisms Hilbert C*-bimodules, and the 2-morphisms adjointable bimodule maps. Furthermore, $\mathsf{C^*Alg}$ is actually a C*-2-category, where the sets of 2-endomorphisms have the structure of C*-algebras. By considering abelian C*-algebras as E_2-algebras in the category of vector spaces Vect, abelian C*-algebras form a Morita 3-category of E_2-algebras called $\mathsf{AbC^*Alg}$. This category turns out to be an example of a C*-3-category and was first investigated by the second-named author in [Fer].

We therefore look to the 3-category $\mathsf{AbC^*Alg}$ for a 3-categorical correspondence between topology and abelian C*-algebras. To that end, the homotopy hypothesis of Grothendieck ([Gro]) states that there should be an equivalence between homotopy n-types and (weak) n-groupoids. Thus, one possible 3-categorical approach would be to compute the homotopy groups at abelian C*-algebras in the 3-category $\mathsf{AbC^*Alg}$, thus describing the homotopy 3-type. Corey Jones suggested that the first homotopy group at a unital abelian C*-algebra in $\mathsf{AbC^*Alg}$ should decompose as a short exact sequence involving $H^3(T;\mathbb{Z})$ and $\mathrm{Homeo}(T)$. Our first theorem proves this statement; using the Serre–Swan theorem, it also describes the other homotopy groups in terms of topological invariants of the compact Hausdorff space T. We emphasize that these are not the traditional homotopy groups of T from algebraic topology.

Theorem 4.1.0.1. *Let $C(T)$ be an abelian C^*-algebra. Then the homotopy groups at $C(T)$ in $\mathsf{AbC^*Alg}$ are as follows:*

$$\begin{aligned}\pi_0(\mathsf{AbC^*Alg}) &\cong \{\textit{Compact Hausdorff Spaces}\}/\cong,\\ \pi_1(\mathsf{AbC^*Alg}, C(T)) &\cong H^3_{\mathrm{tor}}(T;\mathbb{Z}) \rtimes \mathrm{Homeo}(T),\\ \pi_2(\mathsf{AbC^*Alg}, C(T)) &\cong \mathrm{Pic}(T), \ \textit{and}\\ \pi_3(\mathsf{AbC^*Alg}, C(T)) &\cong C(T)^\times.\end{aligned}$$

Furthermore, if T has the homotopy type of a CW-complex,

$$\pi_2(\mathsf{AbC^*Alg}, C(T)) \cong H^2(T;\mathbb{Z}).$$

Here, $H^3_{\mathrm{tor}}(T;\mathbb{Z})$ is the torsion subgroup of the third Čech cohomology group of T and $\mathrm{Pic}(T)$ is the Picard group of isomorphism classes of complex line bundles over T.

However, a homotopy 3-type is classified by more data than just the homotopy groups. We compute part of this additional information: the actions of π_1 on the higher homotopy groups π_2 and π_3.

Theorem 4.1.0.2. *Let $C(T)$ be an abelian C^*-algebra with the homotopy groups as described in Theorem 4.1.0.1. Then the actions of $\pi_1(\mathsf{AbC^*Alg}, C(T))$ on the homotopy groups $\pi_2(\mathsf{AbC^*Alg}, C(T))$ and $\pi_3(\mathsf{AbC^*Alg}, C(T))$ are described as follows: Given a 3-cocycle*

$\omega \in H^3_{\mathrm{tor}}(T;\mathbb{Z})$, a homeomorphism $\Phi \in \mathrm{Homeo}(T)$, and a line bundle E, we have

$$(\omega, \Phi) \curvearrowright E \cong (\Phi^{-1})^*(E),$$

where $(\Phi^{-1})^*(E)$ is isomorphic to the pullback bundle along Φ^{-1}. When T has the homotopy type of a CW-complex (and so $\mathrm{Pic}(T) \cong H^2(T;\mathbb{Z})$), this action corresponds to the pullback on $H^2(T;\mathbb{Z})$ by Φ^{-1}. If we furthermore have a 3-morphism $f \in C(T)^\times$, we have

$$(\omega, \Phi) \curvearrowright f = f \circ \Phi^{-1}.$$

Future applications of this work involve the interplay between quantum symmetries and C*-algebras. Classical symmetries of C*-algebras arise as group actions. The study of group actions on C*-algebras has been quite fruitful, such as through the construction of crossed-product C*-algebras ([Wil07]) or the classification of group actions on simple purely infinite C*-algebras ([GS24]). From a categorical viewpoint, this is because the $*$-automorphisms of a C*-algebra form a 1-category. As C*-algebras naturally lie inside the 2-category $\mathsf{C^*Alg}$, quantum symmetries of C*-algebras may be obtained from actions of tensor categories on C*-algebras, which is a promising active area of research (e.g. [Kit; AKK24; EGPJ]). By considering abelian C*-algebras inside a 3-category, we may construct actions by monoidal 2-categories to obtain higher quantum symmetries.

The paper is laid out as follows. In Section 1.1, we recount a wide variety of background, both on C*-algebras and C*-categories. In Section 4.2, we compute the homotopy groups at $C(T)$, proving Theorem 4.1.0.1. Theorem 4.1.0.2 is proven in Section 4.3. Finally, in order to compute π_2 in $\mathsf{AbC^*Alg}$, we needed to know that the Serre–Swan Theorem is a monoidal equivalence. As we were unable to locate a reference for this fact, we provide a proof in §1.3.3.

4.2 Computing the homotopy groups of AbC*Alg

In this section, we will compute the homotopy groups in $\mathsf{AbC^*Alg}$. Some of the arguments are easier to understand using $*$-isomorphisms rather than invertible bimodules. The following lemma justifies using appropriate $*$-isomorphisms in place of 2-morphisms in $\mathsf{AbC^*Alg}$.

Lemma 4.2.0.1. *Let ${}_\phi A_\psi$ and ${}_\mu B_\nu$ be 1-morphisms from $C(T) \to C(S)$. Suppose there is a $*$-isomorphism $\tau \colon A \to B$ such that $\tau \circ \phi = \mu$ and $\tau \circ \psi = \nu$. Then τ defines a 2-isomorphism $A \Rightarrow B$ in $\mathsf{AbC^*Alg}$.*

Proof. Consider the bimodule ${}_A B_B$ with the usual B-action ($b_1 \triangleleft b_2 = b_1 b_2$) and B-valued inner product ($\langle b_1 | b_2 \rangle = b_1^\dagger b_2$), and with the left action by A

$$a \triangleright b = \tau(a)b.$$

It is routine to verify that ${}_A B_B$ is an A-B imprimitivity bimodule, with A-valued inner product

$$\langle b_1, b_2 \rangle = \tau^{-1}(b_1 b_2^\dagger).$$

To see that ${}_A B_B$ is a 2-morphism in $\mathsf{AbC^*Alg}$ (and therefore a 2-equivalence), observe that for $f \in C(T)$ and $b \in B$, we have

$$\phi(f) \triangleright b = (\tau \circ \phi)(f)b = \mu(f)b = b \triangleleft \mu(f).$$

We similarly see that the left and right actions by continuous functions on S agree, and so ${}_A B_B$ is a 2-equivalence in $\mathsf{AbC^*Alg}$. □

Remark 4.2.0.2. In general, we cannot replace a 2-equivalence by a $*$-isomorphism. For example, $M_2(\mathbb{C})$ and $\mathbb{C}$ are not isomorphic C*-algebras, but are Morita equivalent, and so produce equivalent 1-morphisms in $\mathsf{AbC^*Alg}$ (assuming the central maps are chosen appropriately).

Our first major goal is the construction of $\pi_1(\mathsf{AbC^*Alg}, C(T))$. This can be broken into two parts: describing the structure of the central $*$-homomorphisms $\phi, \psi\colon C(T) \to Z(A)$ and the C*-algebraic structure of A. In the following subsection, we will work towards understanding the central $*$-homomorphisms. Along the way, we pick up a characterization of $\pi_0(\mathsf{AbC^*Alg})$.

4.2.1 Results for π_0 and π_1

Proving that the central $*$-homomorphisms are actually $*$-isomorphisms is proven using the faithfulness of the actions on imprimitivity bimodules.

Lemma 4.2.1.1. *If* ${}_{\phi}A_{\psi}$ *is a* 1*-isomorphism* $C(T) \to C(S)$*, both* $\phi\colon C(T) \to Z(A)$ *and* $\psi\colon C(S) \to Z(A)$ *are* $*$*-isomorphisms. In particular,* $T \cong S$*.*

Proof. We begin with the case where $T = S$ and we have a 1-morphism ${}_{\eta}C_{\kappa}$ that is equivalent to ${}_{\mathrm{id}}C(T)_{\mathrm{id}}$ via a 2-isomorphism ${}_{C}X_{C(T)}$. By Remark 3.3.1, ${}_{C}X_{C(T)}$ is an imprimitivity bimodule. Since ${}_{C}X_{C(T)}$ is full as a left Hilbert module, the left action of C is faithful by Remark 1.1.8.2.

We first show that η is a $*$-isomorphism onto $Z(C)$. We begin by showing injectivity. Suppose $f \in C(T)$ satisfies $\eta(f) = 0$. For $x \in X$, we have $x \triangleleft f = \eta(f) \triangleright x = 0$ since ${}_{C}X_{C(T)}$ is a 2-morphism. As the right action of $C(T)$ is faithful, we conclude that $f = 0$, so η is injective. To show surjectivity, suppose $c \in Z(C)$. Then we may find a function $g \in C(\mathrm{Prim}\,(C))$ so that $\mathcal{F}(g) = c$, where $\mathcal{F}\colon C(\mathrm{Prim}\,(C)) \to Z(C)$ is the isomorphism given by the Dauns–Hofmann Theorem (Theorem 1.1.7.7). As ${}_{C}X_{C(T)}$ is an imprimitivity bimodule, it induces a Rieffel homeomorphism $h_X\colon T \to \mathrm{Prim}\,(C)$. We then have, for all $x \in X$,

$$c \triangleright x = \mathcal{F}(g) \triangleright x = x \triangleleft (g \circ h_X) = \eta(g \circ h_X) \triangleright x$$

by Proposition 1.1.9.3, where we treated the Dauns–Hofmann isomorphism $\mathcal{F}$ as the identity on $C(T)$. Since the left C action is faithful, we conclude that $c = \eta(g \circ h_X)$, and so η is an isomorphism $C(T) \to Z(C)$. A similar argument shows that $\kappa\colon C(T) \to Z(C)$ is an isomorphism as well.

For the general case, suppose ${}_{\mu}B_{\nu}$ is an inverse of ${}_{\phi}A_{\psi}$. Applying the above argument to the 1-morphism ${}_{\eta}C_{\kappa} = {}_{\phi}(A \otimes_S B)_{\nu}$ proves that $\phi \otimes 1_B$ is a $*$-isomorphism onto $Z(A \otimes_S B)$. However,

$$\phi(C(T)) \otimes_S 1_B \subseteq Z(A) \otimes_S 1_B \subseteq Z(A \otimes_S B)$$

from which we conclude that ϕ is a $*$-isomorphism $C(T) \to Z(A)$ as

$$Z(A) \cong Z(A) \otimes_S C(S) \cong Z(A) \otimes_S 1_B \subseteq Z(A \otimes_S B)$$

Similarly, applying the argument to ${}_{\mu}(B \otimes_T A)_{\psi}$ proves that $\psi\colon C(S) \to Z(A)$ is a $*$-isomorphism. We then see that $\psi^{-1} \circ \phi\colon C(T) \to C(S)$ is a $*$-isomorphism, from which we conclude that $T \cong S$ by Gelfand duality. □

The characterization of π_0 follows immediately from the preceding lemma.

Theorem 4.2.1.2. *The collection of equivalence classes of objects in* $\mathsf{AbC^*Alg}$ *is given by*

$$\pi_0(\mathsf{AbC^*Alg}) = \pi_0(\mathsf{CHaus}) = \{\textit{compact Hausdorff spaces}\}/\textit{homeomorphism}.$$

We continue our analysis of the central $*$-isomorphisms for elements $\pi_1(\mathsf{AbC^*Alg}, C(T))$. We will often use the following lemma about the center of 1-morphisms. It also gives a suggestion of the semidirect product decomposition of $\pi_1(\mathsf{AbC^*Alg}, C(T))$.

Proposition 4.2.1.3. *Let T be a compact Hausdorff space and ${}_{\phi}A_{\psi}$ a 1-automorphism of $C(T)$. Then ${}_{\phi}A_{\psi} \cong {}_{\phi}Z(A)_{\psi} \otimes_T {}_{\psi}A_{\psi} \cong {}_{\psi^{-1}\circ\phi}C(T)_{\mathrm{id}} \otimes_T {}_{\psi}A_{\psi}$. When $A = C(T)$, we have ${}_{\phi}C(T)_{\psi} \cong {}_{\psi^{-1}\circ\phi}C(T)_{\mathrm{id}}$.*

Proof. It is routine to see that ${}_{\phi}A_{\psi}$ and ${}_{\phi}Z(A)_{\psi} \otimes_T {}_{\psi}A_{\psi}$ are isomorphic as C*-algebras via the map $a \mapsto 1 \otimes_T a$. It is clear that this $*$-isomorphism preserves the central maps ϕ and ψ. Thus, they are equivalent 1-morphisms in $\mathsf{AbC^*Alg}$ by Lemma 4.2.0.1. Now, considering the $*$-isomorphism $\psi\colon {}_{\psi^{-1}\circ\phi}C(T)_{\mathrm{id}} \to {}_{\phi}Z(A)_{\psi}$, we see that this is another isomorphism of C*-algebras that respects the central maps ψ (resp. ϕ) and id (resp. $\psi^{-1} \circ \phi$). Once again, Lemma 4.2.0.1 proves they are equivalent 1-morphisms in $\mathsf{AbC^*Alg}$. The special case when $A = C(T)$ immediately follows. □

When $A = C(T)$, we see that the central $*$-homomorphisms can be moved to one side of the 1-morphism. This allows us to construct a map from Homeo(T) to $\pi_1(\mathsf{AbC^*Alg}, C(T))$. The following proposition describes how these morphisms compose and proves they form a subgroup.

Proposition 4.2.1.4 (Arithmetic with Homeomorphisms)**.** *Let T be a compact Hausdorff space, and let ϕ and ψ be automorphisms of $C(T)$. Then the 1-automorphisms ${}_{\phi}C(T)_{\mathrm{id}} \otimes_T {}_{\psi}C(T)_{\mathrm{id}}$ and ${}_{\psi\phi}C(T)_{\mathrm{id}}$ are equivalent. Furthermore, ${}_{\phi}C(T)_{\mathrm{id}}$ and ${}_{\psi}C(T)_{\mathrm{id}}$ are equivalent if and only if $\phi = \psi$. Therefore, the map $\Pi\colon \mathrm{Homeo}(T) \to \pi_1(\mathsf{AbC^*Alg}, C(T)))$ given by $\Phi \mapsto {}_{\Phi^*}C(T)_{\mathrm{id}}$ is an injective group homomorphism.*

Proof. Define a $*$-isomorphism $\rho\colon {}_{\phi}C(T)_{\mathrm{id}} \otimes_T {}_{\psi}C(T)_{\mathrm{id}} \to {}_{\psi\circ\phi}C(T)_{\mathrm{id}}$ given by

$$f \otimes_T g \mapsto \psi(f)g.$$

Note that this respects the central maps, in that $\rho \circ (\phi \otimes_T 1) = \psi \circ \phi$ and $\rho \circ (1 \otimes_T \mathrm{id}) = \mathrm{id}$. Thus these are equivalent 1-morphisms in $\mathsf{AbC^*Alg}$ by Lemma 4.2.0.1.

Now, suppose ${}_{\phi}C(T)_{\mathrm{id}}$ and ${}_{\psi}C(T)_{\mathrm{id}}$ are equivalent; that is, there is a $C(T)$-$C(T)$ imprimitivity bimodule X satisfying the properties

$$\textbf{(a):}\ \phi(f) \triangleright x = x \triangleleft \psi(f) \text{ and } \textbf{(b):}\ \mathrm{id}(f) \triangleright x = x \triangleleft \mathrm{id}(f)$$

for all $x \in X$ and $f \in C(T)$. Then, in particular,

$$\phi(f) \triangleright x \overset{\textbf{(a)}}{=} x \triangleleft \psi(f) \overset{\textbf{(b)}}{=} \psi(f) \triangleright x.$$

As this holds for all $x \in X$, and X is full, it follows that $\phi(f) = \psi(f)$, from which we conclude that $\phi = \psi$.

To show that Π is a group homomorphism from Homeo(T) to $\pi_1(\mathsf{AbC^*Alg}, C(T))$, let Φ and Ψ belong to Homeo(T). Then

$$\begin{aligned}\Pi(\Phi) \otimes_T \Pi(\Psi) &= {}_{\Phi^*}C(T)_{\mathrm{id}} \otimes_T {}_{\Psi^*}C(T)_{\mathrm{id}} \\ &\cong {}_{\Psi^* \circ \Phi^*}C(T)_{\mathrm{id}} \\ &= {}_{(\Phi \circ \Psi)^*}C(T)_{\mathrm{id}} \\ &= \Pi(\Phi \circ \Psi).\end{aligned}$$

We conclude that Π is an injective group homomorphism. $\square$

Having obtained a good handle on the central $*$-homomorphisms, we now construct a correspondence between spectra. Because we have maps from $C(T)$ to $Z(A)$ rather than from T to $\hat{A}$, we need to formally understand the connection between $\hat{A}$ and $Z(A)$. The following result is well-known, but we will need an explicit description of the restriction map in our setting.

Lemma 4.2.1.5. *Let $\lambda \in \hat{A}$. Then the $*$-homomorphism $\lambda|_{Z(A)} : Z(A) \to \mathbb{C}$ belongs to $\widehat{Z(A)}$, and the map* res: $\hat{A} \to \widehat{Z(A)}$ *given by $\lambda \mapsto \lambda|_{Z(A)}$ is a continuous surjection. This map is injective if and only if $\hat{A}$ is Hausdorff.*

Proof. By Lemma 1.1 in [Nil95], $\lambda|_{Z(A)}$ is in $\widehat{Z(A)}$. Furthermore, the map $\ker(\lambda) \mapsto \ker(\lambda|_{Z(A)})$ is a continuous map from $\operatorname{Prim} A$ to $\operatorname{Prim} Z(A)$ with dense range. Since A

is unital, $\operatorname{Prim} A$ is compact by Proposition 1.1.7.6, so this map is actually onto. Thus, $\lambda \mapsto \ker(\lambda|_{Z(A)})$ is a continuous surjection q of $\hat{A}$ onto $\operatorname{Prim} Z(A)$. As $Z(A)$ is abelian, $\ker\colon \widehat{Z(A)} \to \operatorname{Prim} Z(A)$ is a homeomorphism, so we have $\ker^{-1} \circ q\colon \hat{A} \to \widehat{Z(A)}$ is a continuous surjection and, in particular, $(\ker^{-1} \circ q)(\lambda) = \lambda|_{Z(A)}$. The argument is summarized in the below diagrams.

$$\begin{tikzcd}
\widehat{A} \arrow[r, "\text{res}"] \arrow[d, "\ker"'] \arrow[dr, "q"] & \widehat{Z(A)} \arrow[d, "\ker"] \\
\operatorname{Prim} A \arrow[r, "- \cap Z(A)"'] & \operatorname{Prim} Z(A)
\end{tikzcd}$$

$$\begin{tikzcd}
\lambda \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & \lambda|_{Z(A)} \arrow[d, mapsto] \\
\ker(\lambda) \arrow[r, mapsto] & \ker(\lambda|_{Z(A)})
\end{tikzcd}$$

If the restriction map is injective, then it is a continuous bijection from a compact space to a Hausdorff space and is therefore a homeomorphism, from which it follows that $\hat{A}$ is Hausdorff. Conversely, if $\hat{A}$ is Hausdorff, then $\ker\colon \hat{A} \to \operatorname{Prim} A$ must be injective because the topology on $\hat{A}$ is pulled back from the topology on $\operatorname{Prim} A$. But, then $\ker\colon \hat{A} \to \operatorname{Prim} A$ is a homeomorphism, and so $\operatorname{Prim} A$ is Hausdorff. Thus, as $\operatorname{Prim} A$ is also compact, the map $- \cap Z(A)$ is a homeomorphism by Theorem 2.2 of [Nil95]. It follows that res is a homeomorphism and, in particular, is injective. $\square$

A 1-morphism composed with its inverse will be Morita equivalent to $C(T)$, so there is much that can be said about the structure of this composition. In particular, the spectrum of the relative tensor product $A \otimes_T A^{-1}$ will be homeomorphic to T. We want to use this fact to construct a homeomorphism of $\hat{A}$ and T. As a first step, the following remark relates the spectrum of A to the spectrum of $A \otimes_T A^{-1}$.

Remark 4.2.1.6. Let ${}_{\phi}A_{\psi}$ and ${}_{\mu}B_{\nu}$ be two 1-automorphisms of $C(T)$. Observe that the pairs $(\lambda, \rho) \subset \hat{A} \times \hat{B}$ satisfying $\lambda|_{Z(A)} \circ \psi = \rho|_{Z(B)} \circ \mu$ lie in the spectrum of $A \otimes_T B$. For, we have an inclusion of $\hat{A} \times \hat{B}$ into the spectrum of $A \otimes_{\max} B$ by sending $(\lambda, \rho) \mapsto \lambda \otimes \rho$ (This map is a homeomorphism onto its range in the spectrum of $A \otimes_{\min} B$ by [RW98, Theorem B.45], and $A \otimes_{\max} B$ surjects onto $A \otimes_{\min} B$). We then see that the representations $\lambda \otimes \rho$ satisfying $\lambda|_{Z(A)} \circ \psi = \rho|_{Z(B)} \circ \mu$ are precisely the representations of this form that restrict to 0 on the balancing ideal I_T, and so are in the spectrum of $A \otimes_T B$.

Having established the preceding relationship, we can now work towards constructing the desired homeomorphism $\hat{A} \to T$. As an intermediate step, we will prove that the spectrum is Hausdorff, which will subsequently allow us to apply Lemma 4.2.1.5.

Lemma 4.2.1.7. *Suppose that ${}_{\phi}A_{\psi}$ is a* 1*-automorphism. Then the spectrum $\hat{A}$ is Hausdorff.*

Proof. By Lemma 4.2.1.5, it suffices to show that the restriction map $[\lambda] \mapsto [\lambda|_{Z(A)}]$ is injective. Let ${}_{\mu}B_{\nu}$ be $({}_{\phi}A_{\psi})^{-1}$, so that ${}_{\phi}(A \otimes_T B)_{\nu}$ is Morita equivalent to ${}_{\mathrm{id}}C(T)_{\mathrm{id}}$. To show that it is injective, we consider two representations λ_1 and λ_2 in $\hat{A}$ and assume $\mathrm{res}(\lambda_1) = \mathrm{res}(\lambda_2)$; that is, $\lambda_1|_{Z(A)}$ and $\lambda_2|_{Z(A)}$ are unitarily equivalent. As $Z(A)$ is an abelian C*-algebra, this happens if and only if $\lambda_1|_{Z(A)} = \lambda_2|_{Z(A)}$ as maps into $\mathbb{C}$. As $Z(B) \cong C(T)$ by Lemma 4.2.1.1, we may choose $\rho \in \hat{B}$ so that $\rho|_{Z(B)} = \lambda_1|_{Z(A)} \circ (\psi \circ \mu^{-1}) \in \widehat{Z(B)}$ by Lemma 4.2.1.5. We then see that $\lambda_i \otimes \rho$ is an element of $\widehat{A \otimes_T B}$ for $i = 1, 2$ by Remark 4.2.1.6. Furthermore, we claim $\lambda_1 \otimes_T \rho$ and $\lambda_2 \otimes_T \rho$ are unitarily equivalent irreducible representations. For, since ${}_{\phi}(A \otimes_T B)_{\nu}$ is Morita equivalent to $C(T)$, the spectrum of ${}_{\phi}(A \otimes_T B)_{\nu}$ is homeomorphic to T, and, in particular, is Hausdorff by Theorem 1.1.9.1. Then, by Lemma 4.2.1.5, the restriction map is a homeomorphism. Since the restrictions of $\lambda_i \otimes_T \rho$ are equal, they are unitarily equivalent. However, we then have $\lambda_1 \otimes_T \rho(1_B)$ and $\lambda_2 \otimes_T \rho(1_B)$ are unitarily equivalent, and it follows that λ_1 and λ_2 are unitarily equivalent by [RW98, Lemma B.47]. We conclude that $[\lambda_1] = [\lambda_2]$ in $\hat{A}$, so that the restriction map is injective. □

We now construct homeomorphisms $\hat{A} \to T$ from the central $*$-isomorphisms.

Proposition 4.2.1.8. *If ${}_\phi A_\psi$ is a 1-automorphism of $C(T)$, then $\hat{A} \cong T$. In particular, $\hat{\phi}, \hat{\psi}\colon \hat{A} \to T$ given by $\hat{\phi}(\lambda) = \lambda|_{Z(A)} \circ \phi$ and $\hat{\psi}(\lambda) = \lambda|_{Z(A)} \circ \psi$ are homeomorphisms.*

Proof. By Lemma 4.2.1.7, $\hat{A}$ is Hausdorff, and hence $\hat{A} \cong \widehat{Z(A)}$ by Lemma 4.2.1.5. On the other hand, $Z(A) \cong C(T)$ by Lemma 4.2.1.1, so we conclude $\hat{A} \cong \widehat{Z(A)} \cong T$. To describe these isomorphisms more explicitly, recall that the map from $\hat{A} \to \widehat{Z(A)}$ is given by sending $\lambda \mapsto \lambda|_{Z(A)}$. Furthermore, our identification of $Z(A)$ with $C(T)$ is given by ϕ, and so the identification $\widehat{Z(A)} \to \widehat{C(T)} = T$ is given by $\rho \mapsto \rho \circ \phi$. Composing these maps yields a map $\lambda \mapsto \lambda|_{Z(A)} \circ \phi$, which is the map $\hat{\phi}$. By a similar argument, $\hat{\psi}$ is a homeomorphism as well. $\square$

Having proved each morphism ${}_\phi A_\psi$ in $\pi_1(\mathsf{AbC^*Alg}, C(T))$ has spectrum $\hat{A} \cong T$, we can now show that the C*-algebra A has continuous-trace.

Theorem 4.2.1.9. *If ${}_\phi A_\psi$ is a 1-automorphism of $C(T)$, then A is a continuous-trace algebra over T.*

Proof. By Proposition 4.2.1.8, we know that $\hat{A} \cong T$. Then, if ${}_\mu B_\nu$ is an inverse for ${}_\phi A_\psi$, we know that, in particular, $A \otimes_T B$ is Morita equivalent to $C(T)$. Thus $A \otimes_T B$ is a continuous-trace C*-algebra by Proposition 5.15 of [RW98]. Now, define the set

$$\Delta := \{\lambda \otimes \rho : (\lambda, \rho) \in \hat{A} \times \hat{B}, \lambda \circ \psi = \rho \circ \mu\}.$$

Then the map $\mathrm{Tr}_{1_A \otimes 1_B}$ on $\widehat{A \otimes_T B}$ restricts to a continuous function on Δ. Define $\gamma\colon \hat{A} \to \hat{B}$ by $\gamma = \hat{\mu}^{-1} \circ \hat{\psi}$. Observe that $\lambda \mapsto (\lambda \otimes \gamma(\lambda))$ is a homeomorphism $\hat{A} \cong \Delta$. As $A \otimes_T B$ is unital, we know this means that $1_A \otimes_T 1_B$ is a continuous-trace element by Remark 1.1.9.10. It follows that $\dim_{A \otimes_T B}$ is continuous on $\widehat{A \otimes_T B}$ and, in particular, takes finite values in $\mathbf{N}$. Importantly, since $\dim_{A \otimes_T B} = \dim_A \cdot \dim_B$, we conclude that both $\dim_A$ and $\dim_B$ take on finite values. Thus,

$$\dim_A(\lambda) = \frac{\dim_A(\lambda)\dim_B(\gamma(\lambda))}{\dim_B(\gamma(\lambda))} = \underbrace{\dim_{A\otimes_T B}(\lambda \otimes_T \gamma(\lambda))}_{\text{continuous}}\ \underbrace{\dim_B(\gamma(\lambda))^{-1}}_{\text{upper semicontinuous}}$$

where upper semicontinuity of $[\lambda] \mapsto \dim_B(\gamma(\lambda))^{-1}$ comes from the fact that $\dim_B$ is lower semicontinuous and nonzero by Theorem 1.1.9.7. Therefore, $\dim_A$ is the product of two positive upper semicontinuous functions and is therefore upper semicontinuous. However, it is always lower semicontinuous (again by Theorem 1.1.9.7), so we conclude that $\dim_A$ is continuous. This means that 1_A is a continuous-trace element of A, and so A has continuous-trace by Remark 1.1.9.10. $\square$

We now wish to build a group homomorphism from $H^3_{\mathrm{tor}}(T;\mathbb{Z})$ to $\pi_1(\mathsf{AbC^*Alg}, T)$. Some care must be taken in constructing the map because our 1-morphisms are constructed using unital C*-algebras.

Lemma 4.2.1.10. *If ${}_\phi A_\phi$ is an invertible 1-morphism of $C(T)$, then its Dixmier–Douady class $\delta(A)$ lies in $H^3_{\mathrm{tor}}(T;\mathbb{Z})$. Conversely, given any $\delta \in H^3_{\mathrm{tor}}(T;\mathbb{Z})$, there is a 1-morphism ${}_\psi B_\psi$ of $C(T)$ with $\delta(B) = \delta$.*

Proof. We begin by showing that the Dixmier–Douady invariant is a torsion element of $H^3(T;\mathbb{Z})$. Given a compact Hausdorff space T, we may write T as a disjoint union $T = \sqcup_{i=1}^n T_i$ for some connected components T_i (finitely many because T is compact). Thus, since the C*-algebra A in ${}_\phi A_\phi$ is a unital continuous-trace algebra over T by Theorem 4.2.1.9, we may write A as the direct sum

$$A = \bigoplus_{i=1}^n \phi(\chi_{T_i})A.$$

Observe that $A_i := \phi(\chi_{T_i})A$ is a continuous-trace C*-algebra over T_i. Because T_i is connected, the function $\dim_{A_i}$ must be constant, and so A_i is a homogeneous C*-algebra. Thus, by

Theorem IV.1.7.23 of [Bla06], the Dixmier–Douady class $\delta(A_i)$ must be a torsion element of $H^3(T_i;\mathbb{Z})$. Since cohomology respects direct sums, $\delta(A) = \sum_{i=1}^n \delta(A_i)$ must be a torsion element in $H^3(T;\mathbb{Z})$ as well.

For the reverse direction, given a torsion element $\delta \in H^3(T;\mathbb{Z})$, by [Gro95, Corollaries 1.5 & 1.7] (c.f. [Bla06, Theorem IV.1.7.24]), there is a homogeneous C*-algebra B whose spectrum is identified with T and whose Dixmer–Douady class is $\delta(B) = \delta$. This identification of T with $\hat{B}$ corresponds to a $*$-isomorphism $\psi\colon C(T) \to Z(B)$ by the Dauns–Hofmann Theorem (Theorem 1.1.7.7). Furthermore, B is a unital C*-algebra by Theorem 3.2 of [Fel61]. Therefore, ${}_\psi B_\psi$ defines an invertible 1-morphism $C(T) \to C(T)$ with $\delta(B) = \delta$. □

Because the Dixmier–Douady classification assumes a single identification $\hat{A} \cong T$ and our 1-morphisms have two such identifications, we need to ensure that the morphisms with identical central maps genuinely form a subgroup of $\pi_1(\mathsf{AbC^*Alg}, C(T))$. The first part of the proof will also be important for proving the exactness of the split exact sequence we construct in Theorem 4.2.1.12.

Lemma 4.2.1.11. *The set of equivalence classes of* 1*-morphisms of the form* ${}_\phi A_\phi$ *is a subgroup of* $\pi_1(\mathsf{AbC^*Alg}, T)$*.*

Proof. We will first show that the equivalence classes of this form only consist of elements where both $*$-homomorphisms $C(T) \to Z(B)$ are equal. Suppose ${}_\phi A_\phi$ and ${}_\mu B_\nu$ are equivalent via an imprimitivity bimodule ${}_A X_B$. Because this defines a 2-equivalence in $\mathsf{AbC^*Alg}$, X satisfies the properties

$$\textbf{(a):}\ \phi(f) \triangleright x = x \triangleleft \mu(f) \text{ and } \textbf{(b):}\ \phi(f) \triangleright x = x \triangleleft \nu(f)$$

for all $x \in X$ and $f \in C(T)$. Then, in particular,

$$x \triangleleft \mu(f) \overset{\textbf{(a)}}{=} \phi(f) \triangleright x \overset{\textbf{(b)}}{=} x \triangleleft \nu(f).$$

As X is full as a Hilbert B-module, and the above holds for all $x \in X$ and $f \in C(T)$, we conclude that $\nu = \mu$ by Remark 1.1.8.2. So any representative of these equivalence classes will have a pair of identical central $*$-homomorphisms.

Now, to show that this is a subgroup, first note that the set contains the identity element ${}_{\mathrm{id}}C(T)_{\mathrm{id}}$. Consider two 1-automorphisms ${}_{\phi}A_{\phi}$ and ${}_{\psi}B_{\psi}$ of $C(T)$. Then their composite is ${}_{\phi}(A \otimes_T B)_{\psi}$, which will be in the purported subgroup if and only if the left and right maps from $C(T) \to Z(A \otimes_T B)$ are equal. However, for $f \in C(T)$, we have

$$\phi(f) \otimes_T 1_B = 1_A \otimes_T \psi(f).$$

Therefore, the composite may be more formally written

$${}_{\phi \otimes 1_B}(A \otimes_T B)_{1_A \otimes \psi} = {}_{\phi \otimes 1_B}(A \otimes_T B)_{\phi \otimes 1_B}.$$

We conclude that the set forms a subgroup of $\pi_1(\mathsf{AbC^*Alg}, T)$. ☐

We now have all of the necessary background to decompose $\pi_1(\mathsf{AbC^*Alg}, C(T))$ as a semidirect product.

Theorem 4.2.1.12. *There is a split exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & H^3_{\mathrm{tor}}(T;\mathbb{Z}) \arrow[r, "1"] & \pi_1(\mathsf{AbC^*Alg}, T) \arrow[r, "2"] & \mathrm{Homeo}(T) \arrow[r] \arrow[l, dashed, bend right, "3"'] & 0.
\end{tikzcd}$$

In particular, $\pi_1(\mathsf{AbC^*Alg}, T) = H^3_{\mathrm{tor}}(T;\mathbb{Z}) \rtimes \mathrm{Homeo}(T)$.

Proof. We describe the aforementioned split short exact sequence as follows:

1. Given a Dixmier-Douady class $\delta \in H^3_{\mathrm{tor}}(T;\mathbb{Z})$, there is a corresponding unital continuous-trace C*-algebra A^δ with a $*$-isomorphism $\psi\colon C(T) \to Z(A^\delta)$ by Lemma 4.2.1.10. Notice ${}_{\mathrm{id}}A^\delta_{\mathrm{id}} \in \pi_1(\mathsf{AbC^*Alg}, T)$ as the Dixmier-Douady class of $A^\delta \otimes_T (A^\delta)^{\mathrm{op}}$ is $0 \in H^3_{\mathrm{tor}}(T; Z)$;

hence, $A^\delta \otimes_T (A^\delta)^{\mathrm{op}}$ is equivalent to ${}_{\mathrm{id}}C(T)_{\mathrm{id}}$ in $\mathsf{AbC^*Alg}$, where we consider $(A^\delta)^{\mathrm{op}}$ as a 1-morphism with the same central $*$-homomorphisms as A^δ. Note that as any two continuous-trace C*-algebras with the same Dixmier–Douady class are Morita equivalent over $C(T)$, the above map does not depend upon the choice of A^δ. Furthermore, as the C*-algebras A^δ have an identical pair of central maps $C(T) \to Z(A)$, the 1-composition of these morphisms agrees with the composition in the Brauer group. Thus, this defines a genuine group homomorphism $H^3_{\mathrm{tor}}(T;\mathbb{Z}) \to \pi_1(\mathsf{AbC^*Alg}, T)$, which is injective by Theorem 1.1.9.15.

2. Define $\Lambda\colon \pi_1(\mathsf{AbC^*Alg}, T) \to \mathrm{Homeo}(T)$ by $\Lambda({}_\phi A_\psi) = \widehat{\psi^{-1} \circ \phi}$ (recalling that $\phi\colon C(T) \to Z(A)$ is an isomorphism). We want to verify that this is a well-defined function on π_1; thus, suppose ${}_\phi A_\psi$ and ${}_{\phi'} A'_{\psi'}$ are 2-isomorphic via $X\colon A \Rightarrow A'$. Recall that this imprimitivity bimodule X satisfies, for all $f \in C(T)$ and $x \in X$,

$$\textbf{(a):}\ \phi(f) \triangleright x = x \triangleleft \phi'(f) \text{ and } \textbf{(b):}\ \psi(f) \triangleright x = x \triangleleft \psi'(f).$$

This allows us to show, for all $f \in C(T)$ and $x \in X$, that

$$\psi((\psi'^{-1} \circ \phi')(f)) \triangleright x \overset{\textbf{(b)}}{=} x \triangleleft \psi'((\psi'^{-1} \circ \phi')(f)) = x \triangleleft \phi'(f) \overset{\textbf{(a)}}{=} \phi(f) \triangleright x.$$

Thus, since X is a full bimodule, we conclude that $(\psi \circ \psi'^{-1} \circ \phi')(f) = \phi(f)$ for all $f \in C(T)$. In particular, this means that $\phi^{-1} \circ \psi = \phi'^{-1} \circ \psi'$, so the above map is well-defined on π_1.

To show that this is a group homomorphism, given two 1-morphisms ${}_\phi A_\psi$ and ${}_\mu B_\nu$ in $\pi_1(\mathsf{AbC^*Alg}, T)$, the 1-morphism ${}_\phi A \otimes_T B_\nu$ has associated central homomorphisms

$\phi \otimes_T 1_B$ and $1_A \otimes_T \nu$. If we consider an element $f \in C(T)$, we have

$$\begin{aligned}
(1_A \otimes_T \nu)^{-1}(\phi \otimes_T 1_B)(f) &= (1_A \otimes_T \nu)^{-1}(\phi(f) \otimes_T 1_B) \\
&= (1_A \otimes_T \nu)^{-1}(\psi(\psi^{-1}\phi(f)) \otimes_T 1_B) \\
&= (1_A \otimes_T \nu)^{-1}(1_A \otimes_T \mu(\psi^{-1}\phi(f)) \\
&= \nu^{-1}\mu\psi^{-1}\phi(f).
\end{aligned}$$

We therefore see that $\Lambda(A \otimes_T B) = \widehat{\nu^{-1}\mu\psi^{-1}\phi} = (\widehat{\psi^{-1}\phi}) \circ (\widehat{\nu^{-1}\mu}) = \Lambda(A) \circ \Lambda(b)$, and so Λ is a group homomorphism. Furthermore, we see that this is surjective, as for any $\Phi \in \mathrm{Homeo}(T)$, we have $\Lambda({}_{\Phi^*}C(T)_{\mathrm{id}}) = \Phi$. Finally, we show that this sequence is exact at π_1. $H^3_{\mathrm{tor}}(T;\mathbb{Z})$ is identified with morphisms of the form ${}_{\phi}A_{\phi}$ by Lemma 4.2.1.11, which is contained in the kernel of Λ. Because any morphism in $\ker(\Lambda)$ has the form ${}_{\phi}A_{\phi}$, which has a single central $*$-homomorphism, its equivalence class in $\pi_1(\mathsf{AbC^*Alg}, C(T))$ is determined solely by its Dixmier–Douady class. Thus, $H^3_{\mathrm{tor}}(T;\mathbb{Z})$ surjects onto $\ker(\Lambda)$.

3. Define (iii) to be the function $\Phi \mapsto {}_{\Phi^*}C(T)_{\mathrm{id}}$, which is a group homomorphism by Proposition 4.2.1.4. As noted in defining the map (ii), we have $\Lambda({}_{\Phi^*}C(T)_{\mathrm{id}}) = \Phi$ for any $\Phi \in \mathrm{Homeo}(T)$, so we conclude (iii) is a splitting. □

Remark 4.2.1.13. The previous short exact sequence is not trivial in general, i.e., $\pi_1(\mathsf{AbC^*Alg}, T)$ is not a direct sum $H^3_{\mathrm{tor}}(T;\mathbb{Z}) \oplus \mathrm{Homeo}(T)$. Indeed, let $S \coloneqq \Sigma\mathbb{RP}^2$, the suspension of the real projective plane. Note that $H^3(S;\mathbb{Z}) = \mathbb{Z}_2$, and so with $T \coloneqq S \sqcup S$, we have $H^3(T;\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, which we may also write as $\{a, b | ab = ba, a^2 = b^2 = 1\}$. Let A be a unital continuous-trace C*-algebra over S with $\delta(A) = a$, where the identification of spectra corresponds to $\phi\colon C(S) \to Z(A)$. Construct a 1-automorphism of $C(T)$ as

$${}_{\phi\oplus\mathrm{id}}A \oplus C(S)_{\phi\oplus\mathrm{id}}$$

First, note that we have $\delta(A \oplus C(S)) = a$. Now, let ψ be the swap-automorphism of $C(T) = C(S) \oplus C(S)$ given by $\psi(f,g) = (g,f)$. Then

$$\begin{aligned}
&({}_{\psi}C(T)_{\mathrm{id}}) \otimes_T ({}_{\phi\oplus\mathrm{id}}A \oplus C(S)_{\phi\oplus\mathrm{id}}) \otimes_T ({}_{\psi}C(T)_{\mathrm{id}})^{-1} \\
\cong&({}_{\psi}C(T)_{\mathrm{id}}) \otimes_T ({}_{\phi\oplus\mathrm{id}}A \oplus C(S)_{\phi\oplus\mathrm{id}}) \otimes_T ({}_{\psi^{-1}}C(T)_{\mathrm{id}}) \\
\cong&{}_{(\phi\oplus\mathrm{id})\circ\psi}A \oplus C(S)_{(\phi\oplus\mathrm{id})\circ\psi}
\end{aligned}$$

via the isomorphism $\sigma((f_1,f_2) \otimes_T (a,h) \otimes_T (g_1,g_2)) = (\phi(f_1)a\phi(g_2), f_2hg_1)$. This respects the central $*$-homomorphisms; for, given $(f,g) \in C(T) = C(S) \oplus C(S)$, we have

$$\begin{aligned}
(\sigma \circ (\psi \otimes_T 1_{A\oplus C(S)} \otimes_T 1_{C(T)}))(f,g) &= \sigma((g,f) \otimes_T 1_{A\oplus C(S)} \otimes_T 1_{C(T)}) \\
&= (\phi(g), f) \\
&= ((\phi \oplus \mathrm{id}) \circ \psi)(f,g)
\end{aligned}$$

and

$$\begin{aligned}
(\sigma \circ (1_{C(T)} \otimes_T 1_{A\oplus C(S)} \otimes_T \mathrm{id}))(f,g) &= \sigma(1_{C(T)} \otimes_T 1_{A\oplus C(S)} \otimes_T (f,g)) \\
&= (\phi(g), f) \\
&= ((\phi \oplus \mathrm{id}) \circ \psi)(f,g).
\end{aligned}$$

Thus, σ defines a 2-equivalence in $\mathsf{AbC^*Alg}$ by Lemma 4.2.0.1. Now, note that for $f, g \in C(S)$, we have

$$(\phi \oplus \mathrm{id})(f,g) = \phi(f) \oplus g$$

and

$$(\phi \oplus \mathrm{id}) \circ \psi(f,g) = \phi(g) \oplus f.$$

Therefore, the part of the C*-algebra with trivial Dixmier–Douady class – $C(S)$ – is now living over the first copy of S in $S \sqcup S$ rather than the second copy. The part with non-trivial

Dixmier–Douady class – A – is now living over the second copy of S. So we have

$$\delta({}_{(\phi\oplus\mathrm{id})\circ\psi}A \oplus C(S)_{(\phi\oplus\mathrm{id})\circ\psi}) = b$$

which is not $a = \delta({}_{\phi\oplus\mathrm{id}}A \oplus C(S)_{\phi\oplus\mathrm{id}})$. So the two 1-automorphisms cannot be equivalent; that is, our semidirect product is not, in general, a direct sum.

4.2.2 Results for π_2 and π_3

Having completed our analysis of π_1, we move on to describe π_2 and π_3. We will begin with some important facts for line bundles. The following results are well-known in the non-unitary setting. We include these for completeness while adapting them for the unitary setting, where the same object may be equipped with potentially different unitary structures. We recall the following definition of the Picard group.

Definition 4.2.2.1. For a compact Hausdorff space T, the Picard group $\mathrm{Pic}(T)$ is given by (isomorphism classes of) complex line bundles over T with multiplication given by the fiberwise tensor product $\otimes$.

Because we consider invertible 2-morphisms ${}_{\mathrm{id}}C(T)_{\mathrm{id}} \Rightarrow {}_{\mathrm{id}}C(T)_{\mathrm{id}}$, our bimodules will be equipped with but a single action of $C(T)$. However, they will be imprimitivity bimodules over $C(T)$ and therefore have two $C(T)$-valued inner products $\langle \cdot | \cdot \rangle$ and $\langle \cdot , \cdot \rangle$. A priori, these inner products may be quite different, but the following lemma proves that they mutually determine each other.

Lemma 4.2.2.2. *Let $E \to T$ be a line bundle over T with two Hermitian metrics $\langle \cdot , \cdot \rangle$ and $\langle \cdot | \cdot \rangle$ which equip the space of sections $\Gamma(E)$ with the structure of a $C(T)$-$C(T)$ imprimitivity bimodule, i.e.,*

$${}_{C(T)}\langle f, g\rangle h = f\langle g|h\rangle_{C(T)} \qquad \forall f, g, h \in \Gamma(E).$$

Then $\langle \cdot, \cdot \rangle = \overline{\langle \cdot | \cdot \rangle}$.

Proof. First, if $E = T \times \mathbb{C}$ is the trivial line bundle, we may use the constant section $1 \in \Gamma(E)$ to compute, for any $f, g \in \Gamma(E)$,

$$_{C(T)}\langle f, g \rangle = {}_{C(T)}\langle f, g \rangle 1 = f \langle g | 1 \rangle_{C(T)} = \langle g | f \rangle_{C(T)} = \overline{\langle f | g \rangle}_{C(T)}.$$

The case of a general line bundle $E \overset{p}{\to} T$ follows by a simple partition of unity argument. Indeed, choose a partition of unity $\{\sigma_i \colon T \to \mathbb{C}\}_i$ on T such that each $K_i := \overline{\operatorname{supp}}\, \sigma_i \subseteq T$ is compact and locally trivializable, i.e. $p^{-1}(K_i) \cong K_i \times \mathbb{C}$ as bundles. Applying our previous argument over each K_i, we deduce

$$_{C(T)}\langle f, g \rangle = \sum_i {}_{C(T)}\langle \sigma_i f, g \rangle = \sum_i \overline{\langle \sigma_i f | g \rangle}_{C(T)} = \overline{\langle f | g \rangle}_{C(T)}.$$

for all $f, g \in \Gamma(E)$. As the inner products on $\Gamma(E)$ agree up to complex conjugation, it follows that the Hermitian metrics $_L\langle \cdot, \cdot \rangle$ and $\langle \cdot | \cdot \rangle_R$ agree up to complex conjugation as well. □

This means that our line bundles that produce imprimitivity bimodules will really only have a single choice of Hermitian metric. However, the Picard group $\operatorname{Pic}(T)$ does not involve Hermitian metrics. The next lemma shows that the choice of Hermitian metric is essentially superfluous.

Lemma 4.2.2.3. *Let* $E \to T$ *be a line bundle with two hermitian structures* $\langle \cdot | \cdot \rangle_1$ *and* $\langle \cdot | \cdot \rangle_2$. *Then there is a unitary isomorphism between the (right) Hilbert C*-modules* $(\Gamma(E), \langle \cdot | \cdot \rangle_1)$ *and* $(\Gamma(E), \langle \cdot | \cdot \rangle_2)$.

Proof. By Theorem 2.5 of [Ket], there is an isometry $f \colon (E, \langle \cdot | \cdot \rangle_1) \to (E, \langle \cdot | \cdot \rangle_2)$. That is, f is a bundle map such that $\langle f(e_1) | f(e_2) \rangle_2 = \langle e_1, e_2 \rangle_1$. Furthermore, as f is also a bundle isomorphism (from [Ket]), we have that $f^\dagger = f^{-1}$, in the sense that

$$\langle f(e_1) | e_2 \rangle_2 = \langle e_1 | f^{-1}(e_2) \rangle_1.$$

By Lemma 1.3.3.2, Γ is a $\dagger$-functor, so $\Gamma(f)\colon (\Gamma(E), \langle\cdot|\cdot\rangle_1) \to (\Gamma(E), \langle\cdot|\cdot\rangle_2)$ is a unitary isomorphism. $\square$

Having justified that the pair of Hermitian metrics on E that yield an imprimitivity bimodule $\Gamma(E)$ do not affect the isomorphism class of $\Gamma(E)$, the characterization of π_2 as $\mathrm{Pic}(T)$ follows from the monoidal version of the Serre–Swan Theorem. We produce the precise statement of this equivalence in the below corollary, but we refer the reader to §1.3.3 for the proof.

Corollary 4.2.2.4 (Serre–Swan)**.** *There is a monoidal $\dagger$-equivalence $E \mapsto \Gamma(E)$ between the monoidal $\dagger$-categories of Hermitian line bundles over T and Hilbert $C(T)$-bimodules.*

We can now prove that $\pi_2(\mathsf{AbC^*Alg}, C(T))$ is isomorphic to $\mathrm{Pic}(T)$.

Theorem 4.2.2.5. *For $T \in \mathsf{AbC^*Alg}$, $\pi_2(\mathsf{AbC^*Alg}, T) \cong \mathrm{Pic}(T)$. Furthermore, when T has the homotopy type of a CW-complex, $\pi_2(\mathsf{AbC^*Alg}, T) \cong H^2(T;\mathbb{Z})$.*

Proof. By the proposition on page 291 of [Rie82], a $C(T)$-$C(T)$ imprimitivity bimodule is finitely generated and projective. Thus, using Lemma 4.2.2.2 and Corollary 4.2.2.4, we see that unitary isomorphism classes of imprimitivity bimodules over $C(T)$ are in correspondence with unitary isomorphism classes of Hermitian line bundles over T. As every complex line bundle over the compact space T admits a Hermitian metric ([Hat03, Proposition 1.2]), Lemma 4.2.2.3 proves that the latter are in correspondence with isomorphism classes of line bundles over T, i.e., $\pi_2(\mathsf{AbC^*Alg}, T) \cong \mathrm{Pic}(T)$. Since the correspondence from Corollary 4.2.2.4 is monoidal, these are indeed isomorphic as groups.

Finally, when T has the homotopy type of a CW-complex, by Proposition 3.10 of [Hat03], the map sending a line bundle E to its first Chern class $c_1(E)$ is an isomorphism from $\mathrm{Pic}(T)$ to $H^2(T;\mathbb{Z})$. $\square$

Proving that $\pi_3(\mathsf{AbC^*Alg}, C(T)) = C(T)^\times$ is straightforward, especially because we have "topped-out" the structure of $\mathsf{AbC^*Alg}$ and the only equivalence relation on our 3-morphisms is equality.

Theorem 4.2.2.6. *For $T \in \mathsf{AbC^*Alg}$, $\pi_3(\mathsf{AbC^*Alg}, T) \cong C(T)^\times$.*

Proof. Recall that $\mathrm{id}_{\mathrm{id}_T} = C(T)$ as a Hilbert C*-bimodule over itself. Since $C(T)$ is abelian and unital,

$$\mathrm{End}({}_{C(T)}C(T)_{C(T)}) \cong Z(C(T)) = C(T).$$

Hence, the invertible maps are given by $\pi_3(\mathsf{AbC^*Alg}, C(T)) \cong C(T)^\times$. $\square$

4.3 Actions on π_2 and π_3

Having computed the three homotopy groups π_1, π_2 and π_3, we are now ready to compute some of the additional data that classifies this homotopy 3-type. We give the following definition of the actions of $\pi_1(\mathsf{AbC^*Alg}, C(T))$ on $\pi_2(\mathsf{AbC^*Alg}, C(T))$ and $\pi_3(\mathsf{AbC^*Alg}, C(T))$.

Definition 4.3.0.1. We define the actions of π_1 on π_2 and π_3 as follows: Let ${}_\phi A_\psi$ be in $\pi_1(\mathsf{AbC^*Alg}, C(T))$, and choose a 2-isomorphism $Y\colon A \otimes_T {}_{\mathrm{id}}C(T)_{\mathrm{id}} \otimes_T A^{-1} \Rightarrow {}_{\mathrm{id}}C(T)_{\mathrm{id}}$. For $X \in \pi_2(\mathsf{AbC^*Alg}, C(T))$, we define $A \curvearrowright X$ as the bimodule

$$Y^{-1} \otimes (\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{-1}}) \otimes Y.$$

This action is well-defined as both π_1 and π_2 are defined up to equivalence. For $f \in \pi_3(\mathsf{AbC^*Alg}, C(T))$, we define $A \curvearrowright f$ using the 3-morphism

$$\tilde{f} := \mathrm{id}_{Y^{-1}} \otimes (1_A \otimes_T f \otimes_T 1_{A^{-1}}) \otimes \mathrm{id}_Y$$

as a 3-automorphism of $A \curvearrowright \mathrm{id}_{C(T)}$. However, in general, $A \curvearrowright \mathrm{id}_{C(T)}$ is not equal to $\mathrm{id}_{C(T)}$, only isomorphic. Thus, we choose a 3-isomorphism $g\colon A \curvearrowright \mathrm{id}_{C(T)} \Rrightarrow \mathrm{id}_{C(T)}$ and define $A \curvearrowright f := g \circ \tilde{f} \circ g^{-1}$. Again, this is well-defined, as everything is up to equivalence.

Remark 4.3.0.2. Recall that $A \otimes_T {}_{\mathrm{id}}C(T)_{\mathrm{id}} \otimes_T A^{-1}$ is $*$-isomorphic to ${}_{\mathrm{id}}C(T)_{\mathrm{id}}$ in a way that respects the action, so we can construct an invertible bimodule Y from this isomorphism by Lemma 4.2.0.1. In this case, 2-composing with Y and Y^{-1} simply yields $\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{-1}}$ as a $C(T)$-$C(T)$ bimodule with the actions given by the $*$-isomorphism.

4.3.1 Actions by Cohomology

Because $H^3_{\mathrm{tor}}(T; \mathbb{Z})$ is an abelian group, intuition suggests that we should be able to braid the 1-morphisms in the definition of the actions. This is made precise in the following lemma.

Lemma 4.3.1.1. *Let* $A := {}_\phi A_\phi \in \pi_1(\mathsf{AbC^*Alg}, C(T))$*. Then the functor defined by* $X \mapsto Y^{-1} \otimes (\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}}) \otimes Y$ *and* $f \mapsto \mathrm{id}_{Y^{-1}} \otimes (1_A \otimes_T f \otimes_T 1_{A^{\mathrm{op}}}) \otimes \mathrm{id}_Y$ *is naturally unitarily isomorphic to the identity functor (for* $X \in \pi_2(\mathsf{AbC^*Alg}, C(T))$ *and* $f \in \pi_3(\mathsf{AbC^*Alg}, C(T))$*).*

Proof. We have that $A \otimes_T {}_{\mathrm{id}}C(T)_{\mathrm{id}} \otimes_T A^{\mathrm{op}}$ is $*$-isomorphic to $A \otimes_T A^{\mathrm{op}}$ via the map

$$a \otimes_T f \otimes_T b \mapsto a\phi(f) \otimes_T b.$$

This $*$-isomorphism preserves the central $*$-homomorphisms, and thus induces a 2-equivalence

$$Z\colon A \otimes_T {}_{\mathrm{id}}C(T)_{\mathrm{id}} \otimes_T A^{\mathrm{op}} \Rightarrow A \otimes_T A^{\mathrm{op}}$$

by Lemma 4.2.0.1. By Remark 4.3.0.2, we then have that $Z^{-1} \otimes (\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}}) \otimes Z$ is isomorphic to $\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}}$ as an $A \otimes_T A^{\mathrm{op}}$ bimodule via the actions

$$(a \otimes_T b) \triangleright (c \otimes_T x \otimes_T d) = ac \otimes_T x \otimes_T db$$

and

$$(c \otimes_T x \otimes_T d) \triangleleft (a \otimes_T b) = ca \otimes_T x \otimes_T bd,$$

and is furthermore equipped with the inner products

$$\langle a \otimes_T x \otimes_T b | c \otimes_T y \otimes_T d \rangle = a^\dagger c \phi(\langle x | y \rangle) \otimes_T db^\dagger$$

and

$$\langle a \otimes_T x \otimes_T b, c \otimes_T y \otimes_T d\rangle = ac^\dagger \phi(\langle x, y\rangle) \otimes_T d^\dagger b.$$

Note that multiplication in the third tensor factor is reversed because the elements lie in A^{op}. We first claim that $Z^{-1} \otimes (\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}}) \otimes Z$ is naturally 3-isomorphic to the $A \otimes_T A^{\mathrm{op}}$ bimodule $X \otimes_T \mathrm{id}_A \otimes_T \mathrm{id}_{A^{\mathrm{op}}}$, where the actions and inner products are given similarly. The only subtlety here is that the tensor permutation map τ given by $\tau(a \otimes_T x \otimes_T b) = x \otimes_T a \otimes_T b$ is well-defined. Well-definedness of this map heavily relies upon the fact that ${}_\phi A_\phi$ has identical central homomorphisms. For, given $f \in C(T)$, we have

$$\begin{tikzcd}
a \otimes_T x \otimes_T \phi(f)b \arrow[r, mapsto, "\tau"] \arrow[d, equal] & x \otimes_T a \otimes_T \phi(f)b \arrow[d, equal] \\
a\phi(f) \otimes_T x \otimes_T b \arrow[r, mapsto, "\tau"] \arrow[d, equal] & x \otimes_T a\phi(f) \otimes_T b \arrow[d, equal] \\
a \otimes_T (f \triangleright x) \otimes_T b \arrow[r, mapsto, "\tau"] & (f \triangleright x) \otimes_T a \otimes_T b
\end{tikzcd}$$

Furthermore, because the inner products of simple tensors in $\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}}$ and their images under τ produce the same elements of $A \otimes_T A^{\mathrm{op}}$, we see that τ is isometric on the span of simple tensors and therefore extends to all of $\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}}$ by continuity. Furthermore, we see that τ is adjointable with $\tau^\dagger = \tau^{-1}$, for

$$\begin{aligned}
\langle \tau(a \otimes_T x \otimes_T b) | y \otimes_T c \otimes_T d\rangle &= \langle x \otimes_T a \otimes_T b | y \otimes_T c \otimes_T d\rangle \\
&= a^\dagger c \langle x|y\rangle \otimes_T db^\dagger \\
&= \langle a \otimes_T x \otimes_T b | c \otimes_T y \otimes_T d\rangle \\
&= \langle a \otimes_T x \otimes_T b | \tau^{-1}(y \otimes_T c \otimes_T d)\rangle
\end{aligned}$$

It is routine to see that τ is natural; the following diagram clearly commutes for any 3-morphism $g\colon X \to X'$.

$$\begin{tikzcd}
\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}} \arrow[r, "1_A \otimes_T g \otimes_T 1_{A^{\mathrm{op}}}"] \arrow[d, "\tau"] & \mathrm{id}_A \otimes_T X' \otimes_T \mathrm{id}_{A^{\mathrm{op}}} \arrow[d, "\tau'"] \\
X \otimes_T \mathrm{id}_A \otimes_T \mathrm{id}_{A^{\mathrm{op}}} \arrow[r, "g \otimes_T 1_A \otimes_T 1_{A^{\mathrm{op}}}"'] & X' \otimes_T \mathrm{id}_A \otimes_T \mathrm{id}_{A^{\mathrm{op}}}
\end{tikzcd}$$

Now, choose a 2-equivalence $Y' \colon A \otimes_T A^{\mathrm{op}} \Rightarrow {}_{\mathrm{id}}C(T)_{\mathrm{id}}$, and set $Y \coloneqq Z \otimes Y'$. We then have natural isomorphisms between the following:

$$\begin{aligned}
Y^{-1} \otimes (\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}}) \otimes Y &\cong (Y')^{-1} \otimes Z^{-1} \otimes (\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{\mathrm{op}}}) \otimes Z \otimes Y' \\
&\cong (Y')^{-1} \otimes (X \otimes_T \mathrm{id}_A \otimes_T \mathrm{id}_{A^{\mathrm{op}}}) \otimes Y' \\
&\cong X \otimes_T ((Y')^{-1} \otimes Y') \\
&\cong X \otimes_T C(T) \\
&\cong X
\end{aligned}$$

These isomorphisms are all natural because 1-composition is natural and the only maps used to construct the isomorphisms (other than τ) are unitors, associators, and interchangers, which are all natural as well. Furthermore, τ is a unitary isomorphism, and all coherence data in $\mathsf{AbC^*Alg}$ is unitary, so this natural isomorphism is, in fact, unitary. □

Because these 1-morphisms may be braided, the actions given by morphisms in $H^3_{\mathrm{tor}}(T;\mathbb{Z})$ are automatically trivial.

Corollary 4.3.1.2. *For* ${}_\phi A_\phi \in \pi_1(\mathsf{AbC^*Alg}, C(T))$*, the actions of* ${}_\phi A_\phi$ *on* $\pi_2(\mathsf{AbC^*Alg}, C(T))$ *and* $\pi_3(\mathsf{AbC^*Alg}, C(T))$ *are trivial.*

4.3.2 Actions by Homeomorphisms

In this subsection we will describe the actions by 1-morphisms in Homeo(T). Because the central $*$-isomorphisms differ, these morphisms cannot be braided as we did the morphisms in $H^3_{\mathrm{tor}}(T;\mathbb{Z})$. The following lemma describes the image of an invertible bimodule X under this action.

Lemma 4.3.2.1. *For a* 1*-morphism* $A := {}_{\phi}C(T)_{\mathrm{id}} \in \pi_1(\mathsf{AbC^*Alg}, C(T))$ *and* $X \in \pi_2(\mathsf{AbC^*Alg}, C(T))$*, we claim that* $\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{-1}}$ *is isomorphic to the bimodule* $\phi(X)$*, which has the same underlying vector space* X *with action* $\blacktriangleright$ *and inner product* $(\cdot|\cdot)$ *defined as follows:*

$$f \blacktriangleright x = \phi(f) \triangleright x$$

$$(x|y) = \phi^{-1}(\langle x|y\rangle)$$

where $\triangleright$ *and* $\langle\cdot|\cdot\rangle$ *are the original actions and inner product on* X*.*

Proof. First, recall that $A^{-1} = {}_{\phi^{-1}}C(T)_{\mathrm{id}}$. Furthermore, remember that id_A is $C(T)$ as a $C(T)$-$C(T)$ with the usual inner products and actions. Now, observe that

$$A \otimes_T C(T) \otimes_T A^{-1} \cong C(T)$$

via the isomorphism

$$\sigma\colon f \otimes g \otimes h \mapsto \phi^{-1}(fg)h.$$

Furthermore, this isomorphism respects the central $*$-homomorphisms from $C(T)$. For, given $f \in C(T)$,

$$\sigma(\phi \otimes_T 1_{C(T)} \otimes_T 1_{A^{-1}})(f) = \sigma(\phi(f) \otimes_T 1_{C(T)} \otimes_T 1_{A^{-1}}) = f$$

and

$$\sigma(1_A \otimes_T 1_{C(T)} \otimes_T \mathrm{id})(f) = \sigma(1_A \otimes_T 1_{C(T)} \otimes_T f) = f,$$

and so σ is a $*$-isomorphism from $A \otimes_T C(T) \otimes_T A^{-1}$ to ${}_{\mathrm{id}}C(T)_{\mathrm{id}}$ that respects the central maps. Thus, the corresponding imprimitivity bimodule Y will be a 2-isomorphism in $\mathsf{AbC^*Alg}$

by Lemma 4.2.0.1. By Remark 4.3.0.2, we have

$$\begin{aligned} A \curvearrowright X &= Y^{-1} \otimes (\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{-1}}) \otimes Y \\ &\cong \mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{-1}} \end{aligned}$$

where the resulting bimodule $\mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{-1}}$ as a $C(T)$-$C(T)$ bimodule with actions defined using σ. Furthermore, it is clear that there is a vector space isomorphism $\Sigma\colon \mathrm{id}_A \otimes_T X \otimes_T \mathrm{id}_{A^{-1}} \to X$ given by $\Sigma(f \otimes_T x \otimes_T g) = f\phi(g) \triangleright x$. Thus, we define $\phi(X)$ to have underlying vector space X given by this isomorphism. We use the maps σ and Σ to determine the $C(T)$-action on $\phi(X)$ as follows:

$$\begin{aligned} f \blacktriangleright x &= \Sigma(\sigma^{-1}(f) \triangleright \Sigma^{-1}(x)) \\ &= \Sigma((1_A \otimes_T \phi(f) \otimes_T 1_{A^{-1}}) \triangleright (1_A \otimes_T x \otimes_T 1_{A^{-1}})) \\ &= \Sigma(1_A \otimes_T (\phi(f) \triangleright x) \otimes_T 1_{A^{-1}}) \\ &= \phi(f) \triangleright x \end{aligned}$$

We define the inner product in a similar manner:

$$\begin{aligned} (x|y) &= \sigma(\langle \Sigma^{-1}(x) | \Sigma^{-1}(y) \rangle) \\ &= \sigma(\langle 1_A \otimes_T x \otimes_T 1_{A^{-1}} | 1_A \otimes_T y \otimes_T 1_{A^{-1}} \rangle) \\ &= \sigma(\langle 1_A | 1_A \rangle \otimes_T \langle x|y \rangle \otimes_T \langle 1_{A^{-1}} | 1_{A^{-1}} \rangle) \\ &= \phi^{-1}(\langle x|y \rangle) \end{aligned}$$

With these definitions, $\Sigma\colon (A \curvearrowright X) \to \phi(X)$ is manifestly a 3-isomorphism in $\mathsf{AbC^*Alg}$. $\square$

Having computed how 1-morphisms in $H^3_{\mathrm{tor}}(T;\mathbb{Z})$ and $\mathrm{Homeo}(T)$ individually act on $\pi_2(\mathsf{AbC^*Alg}, C(T))$, we can now describe how a general 1-morphism acts.

Corollary 4.3.2.2. *Let* ${}_\phi A_\psi$ *belong to* $\pi_1(\mathsf{AbC^*Alg}, C(T))$ *and* X *to* $\pi_2(\mathsf{AbC^*Alg}, C(T))$. *Then* $A \curvearrowright X$ *is the bimodule* $\psi^{-1}\phi(X)$ *given by the same underlying vector space* X *with actions and inner product as follows:*

$$f \blacktriangleright x = (\psi^{-1} \circ \phi) \triangleright x$$

$$(x|y) = (\phi^{-1} \circ \psi)(\langle x|y \rangle).$$

Proof. By Proposition 4.2.1.3, ${}_\phi A_\psi$ is equivalent to ${}_{\psi^{-1}\phi}C(T)_{\mathrm{id}} \otimes_T {}_\psi A_\psi$ in the first homotopy group $\pi_1(\mathsf{AbC^*Alg}, C(T))$. By our previous work, we have

$${}_\phi A_\psi \curvearrowright X \cong {}_{\psi^{-1}\phi}C(T)_{\mathrm{id}} \curvearrowright ({}_\psi A_\psi \curvearrowright X) \cong {}_{\psi^{-1}\phi}C(T)_{\mathrm{id}} \curvearrowright X \cong \psi^{-1}\phi(X)$$

as desired. $\square$

We would also like to describe the actions on π_2 in terms of line bundles and, in particular, in terms of the first Chern class in $H^2(T;\mathbb{Z})$. The following theorem describes this relationship.

Theorem 4.3.2.3. *Let* $(\omega, \Phi) \in H^3_{\mathrm{tor}}(T;\mathbb{Z}) \rtimes \mathrm{Homeo}(T)$, *and let* E *be a line bundle over* T. *Then* $(\omega, \Phi) \curvearrowright E$ *is the pullback bundle* $(\Phi^{-1})^*(E)$. *Furthermore, when* T *has the homotopy type of a CW-complex, the action of* (ω, Φ) *on* $H^2(T;\mathbb{Z})$ *corresponds to the pullback along* Φ^{-1}.

Proof. By the previous theorem, we know that $(\omega, \Phi) \curvearrowright \Gamma(E) \cong \Phi^*(\Gamma(E))$, where the $C(T)$-action has been twisted by Φ^* (that is, by precomposition with Φ). Define the pullback bundle $(\Phi^{-1})^*(E)$ to have total space

$$(\Phi^{-1})^*(E) \coloneqq \{(t, e) \in T \times E : \Phi^{-1}(t) = p(e)\} \subseteq T \times E$$

with projection map $p_\Phi(t, e) = t = (\Phi \circ p)(e)$. We see that precomposition with Φ^{-1} defines a group isomorphism from $\Phi^*(\Gamma(E))$ to $\Gamma((\Phi^{-1})^*(E))$ by sending a continuous section $g \in \Gamma(E)$

to $(g \circ \Phi^{-1})(t) = (t, (g \circ \Phi^{-1})(t))$. To verify this, note this does produce a section in $\Gamma((\Phi^{-1})^* E)$; for, if $g \in \Gamma(E)$, we have

$$(\Phi \circ p) \circ (g \circ \Phi^{-1}) = \Phi \circ \mathrm{id}_T \circ \Phi^{-1} = \mathrm{id}_T.$$

Also, this map has an inverse sending $h \in \Gamma((\Phi^{-1})^*(E))$ to $h \circ \Phi$ (with a mild abuse of notation by identifying E with $\mathrm{id}^*(E)$). Note that $(\Phi^{-1})^* \colon \Phi^*(\Gamma(E)) \to \Gamma(\Phi^{-1})^*(E))$ also intertwines the $C(T)$-module actions; for any $f \in C(T)$, we have

$$(\Phi^{-1})^*(f \blacktriangleright g) = (\Phi^{-1})^*((f \circ \Phi)g) = f(g \circ \Phi^{-1}) = f \triangleright (\Phi^{-1})^*(g).$$

We conclude that $(\Phi^{-1})^*$ is an isomorphism of $C(T)$-modules, and it follows that $(\omega, \Phi) \curvearrowright E \cong (\Phi^{-1})^*(E)$.

Furthermore, in the case where T has the homotopy type of a CW-complex, we know that line bundles are classified by their first Chern class $c_1(E) \in H^2(T; \mathbb{Z})$. We then have that $(\omega, \Phi) \curvearrowright c_1(E) = (\Phi^{-1})^* c_1(E)$, the pullback of the first Chern class (as the pullback of line bundles corresponds to the pullback of the Chern classes). □

We are now ready to analyze the action of 1-morphisms in Homeo(T) on the third homotopy group $\pi_3(\mathsf{AbC^*Alg}, C(T))$. As we can concretely describe the image of the identity bimodule $\mathrm{id}_{C(T)}$ under this action, we essentially trace how a morphism in π_3 acts through these isomorphisms.

Theorem 4.3.2.4. *Let $f \in \pi_3(\mathsf{AbC^*Alg}, T)$ and $A := {}_{\phi}C(T)_{\mathrm{id}} \in \pi_1(\mathsf{AbC^*Alg}, T)$. Then $A \curvearrowright f = \phi^{-1}(f)$.*

Proof. By definition, we have $A \curvearrowright f = \mathrm{id}_{Y^{-1}} \otimes (1_A \otimes_T f \otimes_T 1_{A^{-1}}) \otimes \mathrm{id}_Y$. However, because we can choose Y to be implemented by an isomorphism σ as in the proof of Lemma 4.3.2.1, we can determine how $A \curvearrowright f$ acts on our representative $\phi(\mathrm{id}_{C(T)})$ by composing $\Sigma \circ (1_A \otimes_T f \otimes_T 1_{A^{-1}}) \circ \Sigma^{-1}$ (where Σ also comes from the proof of Lemma 4.3.2.1). We then see that

$$\begin{aligned}(\Sigma \circ (1_A \otimes_T f \otimes_T 1_{A^{-1}}) \circ \Sigma^{-1})(g) &= \Sigma(1_A \otimes_T f \otimes_T 1_{A^{-1}})(1_A \otimes_T g \otimes_T 1_{A^{-1}}) \\ &= \Sigma(1_A \otimes_T fg \otimes_T 1_{A^{-1}}) \\ &= fg.\end{aligned}$$

Thus, $A \curvearrowright f$ acts on $\phi(\mathrm{id}_{C(T)})$ by multiplying by f. However, $\phi(\mathrm{id}_{C(T)})$ is not explicitly equal to $\mathrm{id}_{C(T)}$, which is the 2-morphism on which the elements of π_3 act. Therefore, we need to use an appropriate isomorphism $\phi(\mathrm{id}_{C(T)}) \Rrightarrow \mathrm{id}_{C(T)}$. Now, consider the map $\eta\colon \phi(\mathrm{id}_{C(T)}) \to \mathrm{id}_{C(T)}$ given by

$$\eta(g) = \phi^{-1}(g).$$

This map is clearly invertible with $\eta^\dagger = \eta^{-1}$, for

$$\langle \eta(g) | h \rangle = \phi^{-1}(\overline{g})h = \phi^{-1}(\overline{g}\phi(h)) = (g|\phi(h)) = (g|\eta^{-1}(h)).$$

So this is a 3-equivalence from $\phi(\mathrm{id}_{C(T)})$ to $\mathrm{id}_{C(T)}$. To see how $A \curvearrowright f$ acts on $\mathrm{id}_{C(T)}$, we compute (for $g \in \mathrm{id}_{C(T)}$):

$$\begin{aligned}(\eta \circ (A \curvearrowright f) \circ \eta^{-1})(g) &= (\eta \circ (A \curvearrowright f))(\phi(g)) \\ &= \eta(f\phi(g)) \\ &= \phi^{-1}(f)g.\end{aligned}$$

Thus, we see that $A \curvearrowright f$ is equal to $\phi^{-1}(f)$ in π_3. □

As was the case with π_2, we can also describe the action of a general invertible 1-morphism on π_3.

Corollary 4.3.2.5. *Given* ${}_{\phi}A_{\psi} \in \pi_1(\mathsf{AbC^*Alg}, C(T))$ *and* $f \in \pi_3(\mathsf{AbC^*Alg}, C(T))$*, we have that* ${}_{\phi}A_{\psi} \curvearrowright f = (\psi^{-1} \circ \phi)(f)$*.*

Proof. As in the proof of Corollary 4.3.2.2, we use Proposition 4.2.1.3 to write ${}_{\phi}A_{\psi} \cong {}_{\psi^{-1}\circ\phi}C(T)_{\mathrm{id}} \otimes_T {}_{\psi}A_{\psi}$. Therefore,

$$\begin{aligned} {}_{\phi}A_{\psi} \curvearrowright f &= ({}_{\psi^{-1}\circ\phi}C(T)_{\mathrm{id}} \otimes_T {}_{\psi}A_{\psi}) \curvearrowright f \\ &= {}_{\psi^{-1}\circ\phi}C(T)_{\mathrm{id}} \curvearrowright f \\ &= (\phi^{-1} \circ \psi)(f) \end{aligned}$$

□

To give a characterization in terms of topological data, the next corollary immediately follows from Corollary 4.3.2.5.

Corollary 4.3.2.6. *Given* $(\omega, \Phi) \in H^3_{\mathrm{tor}}(T; \mathbb{Z}) \rtimes \mathrm{Homeo}(T)$ *and* $f \in \pi_3(\mathsf{AbC^*Alg}, T)$*, we have* $(\omega, \Phi) \curvearrowright f = f \circ \Phi^{-1}$*.*

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