3D KOCH-TYPE CRYSTALS

GIOVANNI FERRER AND ALEJANDRO VÉLEZ - SANTIAGO

Abstract. We consider the construction of a family $\{K_N\}$ of 3-dimensional Koch-type surfaces, with a corresponding family of 3-dimensional Koch-type "snowflake analogues" $\{C_N\}$, where $N > 1$ are integers with $N \not\equiv 0 \,(\mathrm{mod}\ 3)$. We first establish that the Koch surfaces K_N are s_N-sets with respect to the s_N-dimensional Hausdorff measure, for $s_N = \log(N^2+2)/\log(N)$ the Hausdorff dimension of each Koch-type surface K_N. Using self-similarity, one deduces that the same result holds for each Koch-type crystal C_N. We then develop lower and upper approximation monotonic sequences converging to the s_N-dimensional Hausdorff measure on each Koch-type surface K_N, and consequently, one obtains upper and lower bounds for the Hausdorff measure for each set C_N. As an application, we consider the realization of Robin boundary value problems over the Koch-type crystals C_N, for $N > 2$.

1. Introduction

The aim of this paper is to give rise to 3-dimensional Koch-type fractal sets which exhibit some analogies in some sense to both the Koch curve and the Koch snowflake. These 3-dimensional fractal sets will be called Koch N-surfaces and Koch N-crystals, respectively (see Section 3 for illustrations and precise definitions of these sets). Although the geometry of these sets and the corresponding pre-fractal sets may have been considered and visualized, in our knowledge, there is no concrete mathematical construction and analysis of Koch-type surfaces and Koch-type crystals, up to the present time. Using geometric and self-similarity tools, we deduce the generation of a family of compact invariant self-similar sets, which correspond precisely to Koch N-surfaces K_N (for $N \in \mathbb{N}$ with $N \not\equiv 0 \,(\mathrm{mod}\ 3)$). From here, using standard methods as in [2, 3, 10], we compute the Hausdorff dimension s_N of each Koch N-surface K_N, and obtain that $\{K_N\}$ form a family of s_N-set with respect to the s_N-dimensional Hausdorff measure. The self-similar properties of each K_N lead to the construction of a family of Koch N-crystals $\{C_N\}$, whose boundaries (in the case $N > 2$) are also s_N-sets with respect to the same values s_N and same measures. In particular, when $N > 2$ with $N \not\equiv 0 \,(\mathrm{mod}\ 3)$, the crystals $\{C_N\}$ can be regarded as a family of open connected domains with Koch-type fractal boundaries. This plays an important role in certain applications, which we will consider at the end of the paper.

We then generalize tools developed by Jia [4, 5] (for 2-dimensional fractals) to establish the main results of the paper, which consist on approximating the s_N-dimensional Hausdorff measure of each Koch N-surface K_N by means of increasingly precise upper and lower bounds. To be more precise, we will establish the existence of a decreasing sequence $\{a_n(N)\}$ of positive numbers, and an increasing sequence $\{a'_n(N)\}$ of positive numbers, such that

$$a'_n(N) \le \mathscr{H}^{s_N}(K_N) \le a_n(N), \quad \text{for each } n \in \mathbb{N}, \quad \text{and} \quad \limsup_{n\to\infty} a'_n(N) = \mathscr{H}^{s_N}(K_N) = \liminf_{n\to\infty} a_n(N). \tag{1.1}$$

Some applications to boundary value problems over the family $\{C_N\}$ of Koch N-crystals will be addressed.

Fractals play a role in many areas in Mathematics, with multiple applications to other fields. Concerning Koch-type fractal sets, there is a vast amount of research done over the classical Koch snowflake domain (see image below).

2010 *Mathematics Subject Classification.* 28A80, 28A78, 37F35.
Key words and phrases. Koch surface, Koch crystal, Hausdorff measure, Hausdorff dimension, Self-similarity.

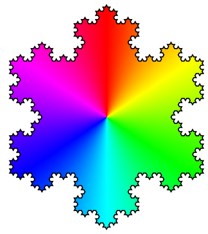


Figure 0: The Koch snowflake domain

In particular, the fact that the interior of the Koch snowflake domain is an open connected set, and the boundary is a self-similar d-set (for $d = \log(4)/\log(3)$), has allowed the well posedness and regularity results for boundary value problems over such region. One can refer to the works in [7, 8, 9, 11] (among many others). The interior of the Koch snowflake is an example of a finitely connected (ε, δ)-domain (e.g. Definition 6), which in views of [6] is equivalent to say that the interior of the domain satisfies the p-extension property in the sense of [6, pag. 1] (also called a Jones domain). It is important to point out that the exact value of the of the d-Hausdorff measure for the classical Koch snowflake (refer to Figure 0) is unknown, up to the present time. Approximation sequences fulfilling a statement as in (1.1) were developed by Jia [5], and this work motivates the generalization to the 3D case, which is the heart of the present paper.

In the case of 3-dimensional domains, the equivalence provided by [6] for finitely connected Jordan curves in $\mathbb{R}^2$ is no longer valid. Furthermore, there are little literature concerning domains in $\mathbb{R}^3$ with fractal boundaries that may exhibit sufficient geometric properties, allowing the interior to be an (ε, δ)-domain, and the boundary to be a d-set. Thus, motivated from the structure and construction of the Koch snowflake domain, we have assembled a family of 3-dimensional connected domains whose fractal boundaries can be viewed as the limit of a sequence of pre-fractal sets (which are Lipschitz) having similar structure as the Koch curve. It follows that many of the properties of the snowflake domain are inherited by the Koch-type surfaces and crystals, which opens the door for multiple extensions and applications. In particular, one can define partial differential equations over the interior of the Koch cube, and obtain solvability and regularity results. These latter applications will be discussed in more detail in Section 7.

The paper is organized in the following way. Section 2 provides an overview of the basic concepts, definitions and results concerning self-similar sets and the geometry of domains. In Section 3, we give a precise definitions and constructions for the Koch N-surfaces K_N, and the existence of a family $\{C_N\}$ of Koch crystals. Geometrical motivations and justifications are also provided. At the end, we show that each Koch N-surface is a s_N-set with respect to the s_N-dimensional Hausdorff measure, for $s_N = \log(N^2+2)/\log(N)$. In Section 4, we provide all the machinery needed to provide concrete definitions for the sequences $\{a_n\}$ and $\{a'_n\}$ mentioned in the previous paragraphs, and we state the main results of the paper, which consists in the fulfillment of (1.1). Some more general useful results are also established in this section, whose validity extend to more general classes of fractal self-similar sets. Section 5 is purely devoted to the proof of the main result of the paper for the particular case $N = 2$, while Section 6 takes care of the proof of the main result (1.1) when $N > 2$. Finally, Section 7 presents an example of a quasi-linear partial differential equation with Robin boundary conditions over the Koch N-crystals, for $N > 2$. We show that the structure of these crystals, which can be viewed as domains with fractal boundaries, allows the Robin problem to be well posed, solvable, and with fine regularity results.

2. Preliminaries

In this section, we collect some basic definitions and results who will play a role in the subsequent sections.

Definition 1. *We denote the **Hausdorff distance** of $A, B \subset \mathbb{R}^n$ by*

$$d_H(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

Definition 2. *A mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a **similitude** if there exists $0 < r < 1$, such that*

$$|S(x) - S(y)| = r|x - y|,\ \textit{for}\ x, y \in \mathbb{R}^n.$$

Similitudes are exactly those maps S which can be written as

$$S(x) = rg(x) + z,\ \textit{for}\ x \in \mathbb{R}^n,$$

for some $g \in O(n)$, $z \in \mathbb{R}^n$ and $0 < r < 1$. We say that r is the contraction ratio of S.

Definition 3. *Let $S = \{S_1, \ldots, S_N\}$ $(N \geq 2)$ be a finite sequence of similitudes with contraction ratios $\{r_1, \ldots, r_N\}$ $(0 < r_i < 1)$.*

*(a) We that a non-empty compact set K is **invariant** under S, if*

$$K = \bigcap_{i=1}^{N} S_i K.$$

(b) If in addition,

$$\mathscr{H}^s(S_i(K) \cap S_j(k)) = 0, \text{ for } i \neq j, \qquad \text{for } s = dim_{\mathscr{H}}(K),$$

*then we call the invariant set K **self-similar**.*

*(c) The **similarity dimension** of K is defined as the unique $s \geq 0$, such that*

$$\sum_{i=1}^{N} r_i^s = 1.$$

In views of [2], it is known that for any such S, there exists a unique invariant compact set.

Definition 4. *We say that a family of similitudes $S = \{S_1, \ldots, S_N\}$ $(N \geq 2)$ satisfies the **open set condition** if there exists a non-empty open set V such that*

$$\bigcup_{i=1}^{N} S_i(V) \subset V, \quad \text{and} \quad S_i(V) \cap S_j(V) = \emptyset \text{ whenever } i \neq j.$$

Definition 5. *Let $K \subset \mathbb{R}^n$ be a compact set, $s \in [0, N]$ and μ a positive measure supported K. We say that K is a s**-set** with respect to the measure μ, if there exist constants a, b, $R > 0$, such that*

$$ar^s \leq \mu(K \cap B(x, r)) \leq br^s, \quad \text{for all } x \in K \quad, 0 < r \leq R.$$

*In this case, we call μ an s**-Ahlfors measure** on K.*

The following result is important.

Theorem 1. *(see [3, 10]) If the family $S = \{S_1, \ldots, S_N\}$ with contraction ratios $r_1, \ldots, r_N$ satisfies the open set condition, then the invariant compact set K under S is self-similar, with $0 < \mathscr{H}^s(K) < \infty$, for $s = dim_{\mathscr{H}} K$. Furthermore, s equals the similarity dimension of K, and K is a s-set with respect to $\mathscr{H}^s$.*

We conclude this section with the following geometric definition of a domain, introduced by Jones [6].

Definition 6. *An open set $\Omega \subseteq \mathbb{R}^N$ is called an (ε, δ)**- domain**, if there exists $\delta \in (0, +\infty]$ and there exists $\varepsilon \in (0, 1]$, such that for each x, $y \in \Omega$ with $|x - y| \leq \delta$, there exists a continuous rectifiable curve $\gamma : [0, t] \to \Omega$, such that $\gamma(0) = x$ and $\gamma(t) = y$, with the following properties:*

(i) $l(\{\gamma\}) \leq \frac{1}{\varepsilon}|x - y|$.

(ii) $dist(z, \partial\Omega) \geq \varepsilon \min\{|x - z|, |y - z|\}, \ \forall z \in \{\gamma\}$.

*Also, an (ε, ∞)-domain is called an **uniform domain**.*

3. Koch Surfaces and the Koch Crystals

In this section we construct the family of fractal domains central to this paper and provide several main properties.

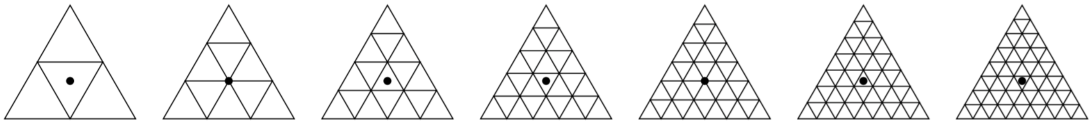


FIGURE 1. The point $(0,0,0)$ together with $T_2,\dots,T_8$ respectively

3.1. **Construction.** Let T be the compact region in the xy plane of $\mathbb{R}^3$ enclosed by the equilateral triangle of side length 1 which is centered at the origin with vertices

$$p_1=\frac{\sqrt{3}}{6}(\cos(0),\sin(0),0)\qquad p_2=\frac{\sqrt{3}}{6}\left(\cos\left(2\pi/3\right),\sin\left(2\pi/3\right),0\right)\qquad p_3=\frac{\sqrt{3}}{6}\left(\cos\left(4\pi/3\right),\sin\left(4\pi/3\right),0\right).$$

Then for $N>1$, we consider the following triangulations T_N of T consisting of N^2 equilateral triangles of scale $1/N$ (see Figure 1).
Note that there does not exists a middle triangle in $T_{3(N-1)}$ (for $N>1$), that is, a unique triangle containing the origin. With this in mind, we may use these T_N to define the following family of fractals analogous to the construction of the Koch curve.

Definition 7. *Let $N>1$ such that $N\not\equiv 0\,(\bmod\ 3)$. We define the **Koch** $\mathbf{N}$**-Surface** $\mathbf{K_N}$ to be the compact self-similar invariant set under the mappings $\mathfrak{F}_N=\{F_{i,N}\}_{i=1}^{N^2+2}$ of ratio $1/N$ that send T to each equilateral triangle except for the middle one in T_N, together with three additional mappings which send T to the three equilateral triangles that form a regular tetrahedron with the removed middle triangle.*

One may think of $\mathfrak{F}_N$ as acting on T by extruding the middle triangle of scale $1/N$ into the other 3 sides of the regular tetrahedron with such base. This is seen explicitly in the following example.

Example 1. *The **Koch 2-Surface** $\mathbf{K_2}$ is the compact self-similar invariant set under the family of mappings $\mathfrak{F}_2=\{F_{i,2}\}_{i=1}^{6}$ given by*

$$F_{1,2}(x,y,z)=\left(\frac{x+\frac{\sqrt{3}}{3}}{2},\ \frac{y}{2},\ \frac{z}{2}\right)$$

$$F_{2,2}(x,y,z)=\left(\frac{x+\frac{\sqrt{3}}{3}\cos\left(\frac{2\pi}{3}\right)}{2},\ \frac{y+\frac{\sqrt{3}}{3}\sin\left(\frac{2\pi}{3}\right)}{2},\ \frac{z}{2}\right)$$

$$F_{3,2}(x,y,z)=\left(\frac{x+\frac{\sqrt{3}}{3}\cos\left(\frac{4\pi}{3}\right)}{2},\ \frac{y+\frac{\sqrt{3}}{3}\sin\left(\frac{4\pi}{3}\right)}{2},\ \frac{z}{2}\right)$$

$$F_{4,2}(x,y,z)=\left(-\frac{x}{6}+\frac{\sqrt{2}}{3}z+\frac{\sqrt{3}}{18},\ -\frac{y}{2},\ \frac{\sqrt{2}}{3}x+\frac{z}{6}+\frac{\sqrt{6}}{18}\right)$$

$$F_{5,2}(x,y,z)=\left(\frac{x}{12}+\frac{\sqrt{3}}{4}y-\frac{\sqrt{2}}{6}z-\frac{\sqrt{3}}{36},\ -\frac{\sqrt{3}}{12}x+\frac{y}{4}+\frac{\sqrt{6}}{6}z+\frac{1}{12},\ \frac{\sqrt{2}}{3}x+\frac{z}{6}+\frac{\sqrt{6}}{18}\right)$$

$$F_{6,2}(x,y,z)=\left(\frac{x}{12}-\frac{\sqrt{3}}{4}y-\frac{\sqrt{2}}{6}z-\frac{\sqrt{3}}{36},\ \frac{\sqrt{3}}{12}x+\frac{y}{4}-\frac{\sqrt{6}}{6}z-\frac{1}{12},\ \frac{\sqrt{2}}{3}x+\frac{z}{6}+\frac{\sqrt{6}}{18}\right)$$

We will adopt the custom of writing $F_j(\cdot,\cdot,\cdot):=F_{j,2}(\cdot,\cdot,\cdot)$ ($j\in\{1,\dots,6\}$) since it is a particularly difficult case, requiring closer examination.

Note that for $1\le j\le 3$, F_j contracts T by a factor of $\frac{1}{2}$ and leaves p_j fixed. Thus the maps $\{F_j\}_{i=1}^{3}$ generate Sierpiński gaskets as seen in Figure 3.
We are now ready to define the fractals of main interest for this paper.

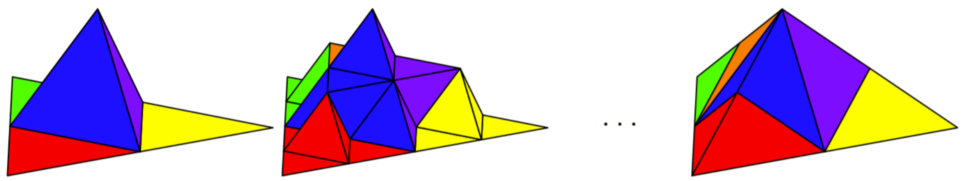


FIGURE 2. Koch 2-Surface Iterations

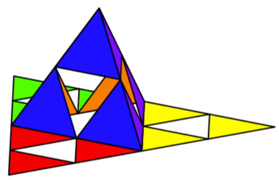


FIGURE 3. Applying F_1, F_2, F_3 and then $\mathfrak{F}_2$ to the triangle T.

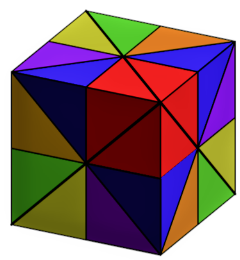


FIGURE 4. Koch 2-Surface Iterations

Definition 8. *Let $N > 1$ such that $N \not\equiv 0 \,(\mathrm{mod}\ 3)$. We define the **Koch N-Crystal** $\mathcal{C}_\mathbf{N}$ as the closed set enclosed by four congruent Koch N-Surfaces, each pair of which intersect at precisely one edge. We then define $\partial\mathcal{C}_N$ as the boundary of $\mathcal{C}_N$.*

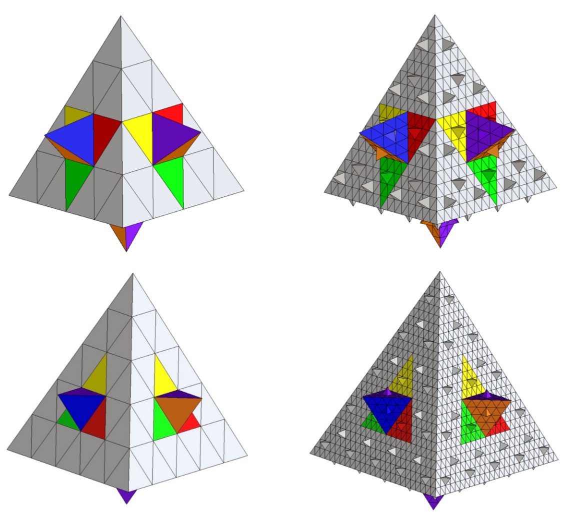


FIGURE 5. Koch 4-Crystal and 5-Crystal, First and Second Iterations

3.2. **Properties.** For $N \not\equiv 0 \pmod 3$, let K_N be the Koch N-surface generated by the iterated function system $\mathfrak{F}_N$. One can see that each $\mathfrak{F}_n$ satisfies the open set condition by considering the bounded open set enclosed by the tetrahedron with vertices p_1, p_2, p_3, and the highest point $p_4 \in K_N$, i.e. $\pi_z(p_4) = \max\{z \mid (x, y, z) \in K_N\}$. Thus, by Theorem 1, it follows that $s_N = \dim_{\mathscr{H}}(K_N) = \log(N^2+2)/\log N$, which is the solution of the equation

$$\sum_{k=1}^{N^2+2} \left(\frac{1}{N}\right)^{s_N} = (N^2+2)\left(\frac{1}{N}\right)^{s_N} = 1.$$

If $N > 2$, then $\partial\mathcal{C}_N$ is the union of four copies of K_N, and thus $\dim_{\mathscr{H}}(\partial\mathcal{C}_N) = \dim_{\mathscr{H}}(K_N) = \log(N^2+2)/\log N$ due to the stability of the Hausdorff dimension. Furthermore, $\mathscr{H}^{s_N}$ is an s_N-Ahlfors measure on $\partial\mathcal{C}_N$ for each $N \in \mathbb{N} \setminus \{1\}$ with $N \not\equiv 0 \pmod 3$, where $s_N = dim_{\mathscr{H}}(\partial\mathcal{C}_N)$. Moreover, it is clearly seen that the interior of the set $\mathcal{C}_N \subseteq \mathbb{R}^3$ is an uniform domain.

In the case when $N = 2$, we see from Figure 4 that, while K_2 is a fractal of Hausdorff dimension $s_2 = \log 6/\log 2$, the figure K_2 is the cube of side-length 1 and ∂K_2 is just its boundary of dimension 2.

4. Bounds for Hausdorff Measure

In this section, we present the machinery needed in order to develop a process to compute sharp bounds for the Hausdorff measure of the Koch N-surfaces K_N and N-crystals C_N. The process will lead to an approximation tool to compute the Hausdorff measure of these fractal sets. Some key general results will be stated and proved. In the end, we will state the main results of the paper.

We start with the following definition.

Definition 9. *Let K be the unique non-empty compact self-similar invariant set under an iterated function system (IFS) $\mathfrak{F} = \{F_j\}_{j=1}^M$ satisfying the open set condition (OSC) where F_j has ratio $0 < r_j$. Let $\boldsymbol{M} := \{1, 2, \dots, M\}$ and $n \geq 1$. We define the word space associated to K as $\Omega := \boldsymbol{M}^{\mathbb{N}}$ and $\Omega_n := \boldsymbol{M}^n$ with the n-truncation map $[\,\cdot\,]_n : \Omega \to \Omega_n$ defined for a word $\omega = \omega_1\omega_2\cdots \in \Omega$ by $[\omega]_n := \omega_1\cdots\omega_n$.*

We will not concern ourselves with the trivial case when $M = 1$. Notice that there is a relation between the word space Ω and the attractor K of an IFS with M maps, where we identify points in K with infinite words, and regions with finite words. Namely for $\omega \in \Omega_n$, we define $K_{(\omega)} := F_\omega(K)$ where $F_{\omega_1\omega_2\cdots\omega_n}$ is given inductively as $F_{\omega_2\cdots\omega_n} \circ F_{\omega_1}$. Moreover for $\omega \in \Omega$, we define the point $K_{(\omega)}$ as the unique point in $\bigcap_{n\in\mathbb{N}} K_{[\omega]_n}$. We will denote the natural probability measure on K as μ where for $\omega = \omega_1\cdots\omega_n \in \Omega_n$, we have that $\mu(K_{(\omega)}) = r_{\omega_1}^s \cdots r_{\omega_n}^s$. Since $\mathfrak{F}$ satisfies the OSC, we also have that $\mu(K_{(\omega)}) = \sum_{j=0}^M \mu(K_{(\omega j)}) = \sum_{j=0}^M r_j^s\, \mu(K_{(\omega)})$.

Definition 10. *Let $K_n := \{K_{(\omega)} \mid \omega \in \Omega_n\}$ be the set of n-cells of K, where we reserve the notation $\Delta_i^{(n)}$ for elements of K_n, which we call n-cells. We also define $K_0 := K$.*

We now present Proposition 1.1 in [4].

Proposition 1. *For $n \geq 1$, $1 \leq k \leq M^n$, and $s = \dim_{\mathscr{H}} K$, let*

$$b_k := \min_{\left\{\Delta_i^{(n)}\right\}_{i=1}^k \subseteq K_n} \left\{ \frac{\left|\bigcup_{i=1}^k \Delta_i^{(n)}\right|^s}{\mu\left(\bigcup_{i=1}^k \Delta_i^{(n)}\right)} \right\}$$

and let $a_n = \min_{1\leq k\leq M^n}\{b_k\}$. If there exists a constant $\mathbf{a} > 0$ such that $a_n \geq \mathbf{a}$ for all n, then $\mathscr{H}^s(K) \geq \mathbf{a}$. We think of $|\bigcup_{i=1}^k \Delta_i^{(n)}|^s$ as a loose analog of s-dimensional volume, and since μ is the natural probability measure on K, we can then think of b_k as finding the collection of k n-cells with the most "density". By means of this analogy, a_n can be thought of as finding the collection of n-cells with the most "density" for every n.

The sequence defined in Proposition 1 has a special consequence, as the following proposition taken from [4] describes.

Proposition 2. *For $n \geq 1$, the sequence $\{a_n\}$ defined in Proposition 1 is decreasing, with $\lim_{n\to\infty} a_n = \mathscr{H}^s(K)$.*

One of the goals of this paper consists in finding a sequence of constants **a** (as in Proposition 1) which increase towards $\mathscr{H}^s(K)$. This will be achieved using case-by-case analysis. To proceed, we add some additional definitions and notations.

Definition 11. *We say a proposition P on the subsets of K is a (valid)* ***case****, if*

- *For every n, there is a family $\{\Delta_i^{(n)}\}_{i=1}^k \subseteq K_n$ such that $\bigcup_{i=1}^k \Delta_i^{(n)}$ satisfies P.*
- *For $n \geq 1$ and $\{\Delta_i^{(n)}\}_{i=1}^k \subseteq K_n$ such that $\bigcup_{i=1}^k \Delta_i^{(n)}$ satisfies case P, there is a family $\{\Delta_j^{(n-1)}\}_j \subseteq K_{n-1}$ such that $\bigcup_{i=1}^k \Delta_i^{(n)} \subseteq \bigcup_j \Delta_j^{(n-1)}$ and $\bigcup_j \Delta_j^{(n-1)}$ satisfies case P.*

Throughout the main proof of the paper, we shall consider the cases when $\{\Delta_i^{(n)}\}_{i=1}^k$ intersect three, two, one, or none of the bottom 1-cells of K. These are examples of cases in the sense of Definition 11.

Definition 12. *In view of the notations in Definition 10, we say that $\{\Delta_i^{(n)}\}_{i=1}^k \subset K_n$ is P-**scaleable** for a proposition P if $\bigcup_{i=1}^k \Delta_i^{(n)}$ satisfies P and there exists a similarity S of ratio r such that each $S^{-1}(\Delta_i^{(n)})$ is unique and in K_{n-1}, with $\bigcup_{i=1}^k S^{-1}(\Delta_i^{(n)})$ satisfying P.*

One should think of P as defining a case in our proof. We observe that if $\{\Delta_i^{(n)}\} \subset K_n$ is P-scaleable, then there exists an unique family in K_n whose union coincides with that of $\{S^{-1}(\Delta_i^{(n)})\} \subset K_{n-1}$, thus satisfying P as well. One then obtains the following result, which will be applied at times in the proof of the central result of the paper.

Lemma 1. *Let*

$$a_n^{(P)} = \min_{1\leq k\leq M^n} \min_{\{\Delta_i^{(n)}\}_{i=1}^k \subseteq K_n} \left\{ \frac{\left|\bigcup_{i=1}^k \Delta_i^{(n)}\right|^s}{\mu\left(\bigcup_{i=1}^k \Delta_i^{(n)}\right)} \;\middle|\; \bigcup_{i=1}^k \Delta_i^{(n)} \text{ satisfies case } P \right\}$$

Furthermore, let

$$a_n' = \min_{1\leq k\leq M^n} \min_{\{\Delta_i^{(n)}\}_{i=1}^k \subseteq K_n} \left\{ \frac{\left|\bigcup_{i=1}^k \Delta_i^{(n)}\right|^s}{\mu\left(\bigcup_{i=1}^k \Delta_i^{(n)}\right)} \;\middle|\; \bigcup_{i=1}^k \Delta_i^{(n)} \text{ satisfies case } P \text{ and is not } P\text{-scaleable} \right\}.$$

Then $a_n^{(P)} = a_n'$. That is, we may exclude P-scaleable families from consideration when calculating lower bounds for $\alpha^{(P)} := \lim_{n\to\infty} a_n^{(P)}$.

Proof. Clearly $a_n^{(P)} \leq a_n'$. Suppose $\{\Delta_i^{(n)}\}_{i=1}^k \subset K_n$ is P-scaleable. Then $|S^{-1}(\bigcup_{i=1}^k \Delta_i^{(n)})| = r^{-1}|\bigcup_{i=1}^k \Delta_i^{(n)}|$. Now note that every $S^{-1}(\Delta_i^{(n)}) \in K_{n-1}$ is the union of $\{\Delta_{i_j}^{(n)}\}_{j=1}^M \subset K_n$. By considering the family $\bigcup_{i=1}^k \{\Delta_{i_j}^{(n)}\}_{j=1}^M \subset K_n$, this satisfies case P since $\bigcup_{i=1}^k \bigcup_{j=1}^M \Delta_{i_j} = \bigcup_{i=1}^k S^{-1}(\Delta_i^{(n)})$, and $Mk \leq M^n$ with

$$\frac{\left|\bigcup_{i=1}^k \Delta_i^{(n)}\right|^s}{\mu\left(\bigcup_{i=1}^k \Delta_i^{(n)}\right)} = \frac{r^s\left|S^{-1}\left(\bigcup_{i=1}^k \Delta_i^{(n)}\right)\right|^s}{\mu\left(\bigcup_{i=1}^k \Delta_i^{(n)}\right)} = \frac{\left|\bigcup_{i,j} \Delta_{i_j}^{(n)}\right|^s}{\sum_{i=1}^k r^{-s}\mu\left(\Delta_i^{(n)}\right)} = \frac{\left|\bigcup_{i,j} \Delta_{i_j}^{(n)}\right|^s}{\mu\left(\bigcup_{i,j} \Delta_{i_j}^{(n)}\right)},$$

since $\sum_{i=1}^k r^{-s}\mu(\Delta_i^{(n)}) = \sum_{i=1}^k \mu(S^{-1}(\Delta_i^{(n)})) = \sum_{i=1}^k \mu(\bigcup_{j=1}^M \Delta_{i_j}^{(n)}) = \mu(\bigcup_{i,j} \Delta_{i_j}^{(n)})$. We may repeat this process if the family $\bigcup_{i=1}^K \{\Delta_{i_j}^{(n)}\}_{j=1}^M$ is P-scaleable and so forth. We must eventually obtain a family that is not P-scaleable. Indeed, if one were able to apply this process n times, then $M^n k \leq M^n$ and $k = 1$. Thus, the family obtained after n steps must be exactly K_n, which is not P-scaleable due to the uniqueness condition $\{S^{-1}(\Delta_i^{(n)})\}$ would need to satisfy. Thus the value $|\bigcup_{i=1}^k \Delta_i^{(n)}| \,/\, \mu(\bigcup_{i=1}^k \Delta_i^{(n)})$ must be larger than the value achieved by some family which is not P-scaleable. Therefore $a_n^{(P)} \geq a_n'$. $\square$

The following key result will be constantly applied in the proof of the central result of the paper, and it has value of its own, and can be applied to general fractals. We present the general version below.

Theorem 2. *Let $\{a_n\}$, $\{a_n^{(P)}\}$ be as in Proposition 1 and Lemma 1, respectively, and let $\beta \leq |\bigcup_{i=1}^k \Delta_i^{(n)}|$, for every $\{\Delta_i^{(n)}\}_{i=1}^k \in K_n$ such that $\bigcup_{i=1}^k \Delta_i^{(n)}$ satisfies P. Then*

$$\alpha_n^{(P)} \geq a_n |K|^s \exp\left(\frac{-s\gamma_n}{\beta(1-r_{\max})}\right),$$

where $r_{\max} := \max_{1\leq i\leq M} r_i$, $r_{\min} := \min_{1\leq i\leq M} r_i$, and $\gamma_n := 2r_{\max}^n \left\{\max_{1\leq \ell,k\leq M} d_H(F_\ell(K), F_k(K))\right\}$.

Proof. Since $\mathscr{H}^s(K) = \mathscr{H}^s(K/|K|)/|K|^s$, we may suppose that $|K| = 1$. We now construct a proof motivated by the procedure found in [5]. Indeed, since P is a case for $n > 1$ and the family of n-cells $\{\Delta_i^{(n)}\}_i$, there exists a collection of $(n-1)$-cells $\Delta_j^{(n-1)} \in K_{n-1}$ such that $\Delta_i^{(n)} \subset \Delta_j^{(n-1)}$, $\bigcup_j \Delta_j^{(n-1)}$ satisfies case P, and $\Delta_1^{(n-1)}, \ldots, \Delta_{k_{n-1}}^{(n-1)}$ are all taken to be distinct. Next, we claim that

$$\left|\bigcup_{j=1}^{k_{n-1}} \Delta_j^{(n-1)}\right| \leq d + \gamma_{n-1}, \qquad \text{for } d := \left|\bigcup_{i=1}^{k} \Delta_i^{(n)}\right|. \tag{4.1}$$

To establish the claim, we proceed as follows. Note that for $x, y \in \bigcup_j \Delta_j^{(n-1)}$,

$$\|x-y\| \leq \min_{a\,\in \bigcup_i \Delta_i^{(n)} \cap \Delta_x^{(n-1)}} \{\|x-a\|\} + d + \min_{b\,\in \bigcup_i \Delta_i^{(n)} \cap \Delta_y^{(n-1)}} \{\|b-y\|\}.$$

Taking the supremum over $x, y \in \bigcup_j \Delta_j^{(n-1)}$ and re-scaling $\Delta_x^{(n-1)}$ onto K, this yields

$$\begin{aligned}
\left|\bigcup_j \Delta_j^{(n-1)}\right| &\leq d + 2 \sup_{x\in\bigcup_j \Delta_j^{(n-1)}} \min_{a\in\bigcup_i \Delta_i^{(n)}\cap\Delta_x^{(n-1)}} \{\|x-a\|\} \\
&\leq d + 2r_{\max}^{n-1} \max_{x\in K} \max_{\{\Delta_k^{(1)}\}_k \subseteq K_1} \min_{a\in\bigcup_k \Delta_k^{(1)}} \{\|x-a\|\} \\
&= d + 2r_{\max}^{n-1} \max_{x\in K} \max_{F_k(K)\in K_1} \min_{a\in F_k(K)} \{\|x-a\|\} \\
&= d + 2r_{\max}^{n-1} \max_{1\leq k\leq M} d_H(K, F_k(K))
\end{aligned}$$

where the latter value is precisely $d + \gamma_{n-1}$. Thus (4.1) is established, as desired. From here, taking into account the monotonicity of the function $f(x) = (x+\gamma_{n-1})/x$ for $x > 0$ and a fixed n, we deduce that $|\bigcup_i \Delta_i^{(n-1)}|/d \leq (d+\gamma_{n-1})/d \leq (\beta+\gamma_{n-1})/\beta$, from where one obtains

$$\frac{d^s}{\mu\left(\bigcup_i \Delta_i^{(n)}\right)} \geq \left(\frac{\beta}{\beta+\gamma_{n-1}}\right)^s \frac{\left|\bigcup_i \Delta_i^{(n-1)}\right|^s}{\mu\left(\bigcup_j \Delta_j^{(n-1)}\right)}. \tag{4.2}$$

Taking the infimum over both sides in (4.2), the previous chain of inequalities can be expressed as follows:

$$a_n^{(P)} \geq \left(\frac{\beta}{\beta+\gamma_{n-1}}\right)^s \frac{|\bigcup_i \Delta_i^{n-1}|^s}{\mu\left(\bigcup_j \Delta_j^{(n-1)}\right)} \geq \left(\frac{\beta}{\beta+\gamma_{n-1}}\right)^s a_{n-1}^{(P)}$$

Then, for any $m \geq 1$, proceeding inductively, we arrive at

$$a_{n+m}^{(P)} \geq a_n^{(P)} \prod_{i=n}^{m+1} \left(\frac{\beta}{\beta+\gamma_i}\right)^s = a_n^{(P)} \prod_{i=n}^{m+1} \left(1+\frac{\gamma_i}{\beta}\right)^{-s}. \tag{4.3}$$

Taking logarithms on both sides in (4.3), and using the inequality $\ln(1+x) < x$, valid for $x > 0$, we find that

$$\ln(a_{n+m}^{(P)}) \geq \ln(a_n^{(P)}) - s\sum_{i=n}^{m+1} \ln\left(1+\frac{\gamma_i}{\beta}\right) \geq \ln(a_n^{(P)}) - s\sum_{i=n}^{m+1} \frac{\gamma_i}{\beta}. \tag{4.4}$$

Proceeding as in Propositions 1 and Proposition 2, one sees that the sequence $\{a_n^{(P)}\}$ is decreasing and bounded. Setting $\alpha^{(P)} := \lim_{n\to\infty} a_n^{(P)}$, and letting $m \to \infty$ in (4.4), we have

$$\ln(\alpha^{(P)}) \geq \ln(a_n^{(P)}) - s\left[\frac{\gamma_n/\beta}{1-r_{\max}}\right] = \ln\left(a_n^{(P)} \exp\left(\frac{-s\gamma_n}{\beta(1-r_{\max})}\right)\right).$$

Therefore, $\alpha^{(P)} \geq a_n^{(P)} \exp\left(\frac{-s\gamma_n}{\beta(1-r_{\max})}\right) \geq a_n \exp\left(\frac{-s\gamma_n}{\beta(1-r_{\max})}\right)$, completing the proof. $\square$

An useful form of the preceding theorem for particular types of sets K reads as follows.

Corollary 1. *Under the assumptions and notations of Theorem 2, assume that $|K| = 1$, $r_{\max} = r_{\min} = r$, and $\max_{\ell,k} d_H(F_\ell(K), F_k(K)) = 1 - r$. Then*

$$\alpha_n^{(P)} \geq a_n \exp\left(-\frac{s \cdot 2r^n}{\beta}\right).$$

It is easily verified that fractals such as the Cantor Set, Sierpinski Gasket, Koch Curve, and Koch N-surfaces all satisfy the conditions in Corollary 1.

We now present the central results of the paper.

Theorem 3. *Let a_n be a sequence given as in Proposition 1 for the Koch 2-surface K_2 as seen in 3.1. Then for every $n \in \mathbb{N}$, the Hausdorff measure of K_2 satisfies the following estimation:*

$$a_n \;\geq\; \mathscr{H}^{s_2}(K_2) \;\geq\; a_n \exp\left(-\frac{s_2(\sqrt{2}+\sqrt{6})}{2^{n-6}}\right), \tag{4.5}$$

where we recall that $s_2 := \frac{\log(6)}{\log(2)}$.

One can calculate and find out that $a_1 = b_3 = 2\left|K_{(4)} \cup K_{(5)} \cup K_{(6)}\right|^{s_2} = 2\left(\frac{\sqrt{6}}{4}\right)^{s_2}$.

Theorem 4. *Let $N > 2$ such that $N \not\equiv 0\,(mod\ 3)$, and let a_n be a sequence as in Proposition 1 for the Koch N-surface K_N defined by Definition 7. Then for every $n \in \mathbb{N}$, the Hausdorff measure of the surface K_N satisfies the following estimation:*

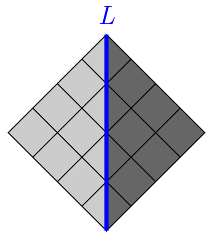
$$a_n \;\geq\; \mathscr{H}^{s_N}(K_N) \;\geq\; a_n \exp\left(-\frac{s_N\sqrt{6}}{N^{n-3}}\right)$$

where we recall that $s_N := \frac{\log(N^2+2)}{\log(N)}$

5. Proof of Theorem 3

The leftmost inequality follows immediately from Proposition 2. We now focus on the remaining inequality. Let $\{\Delta_i^{(n)}\}_i$ be a collection of n-cells and let $d = |\bigcup_{i=1}^k \Delta_i^{(n)}|$ be the diameter of such collection. We will organize the proof by cases, based on how many of the bottom 1-cells $K_{(1)}, K_{(2)}, K_{(3)}$ the family $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects. These cases will subdivided by how many of the top 1-cells $K_{(4)}, K_{(5)}, K_{(6)}$ the family $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects and strategically excluding scenarios when $\bigcup_{i=1}^k \Delta_i^{(n)}$ is scaleable. This will allow us find a constant $\beta > 0$ as in Theorem 2 and Corollary 1; that is, $\beta \leq d$ whenever $\{\Delta_i^{(n)}\}$ satisfies the case under consideration. In order to find such a value, we will consider families $\{\Delta_i^{(n)}\}_{i=1}^k$ such that k is as small as possible (so they have minimal diameter for a fixed n) while satisfying the case in question. As $n \to \infty$, we may imagine $\{\Delta_i^{(n)}\}_{i=1}^k$ as approximating a family of points in K. The diameter of this set of points is meant to yield an optimal value for $\beta > 0$. We first provide a definition, and a technical lemma, which will be applied throughout the proof of the first main result. In this lemma, we provide diagrams depicting these sets of points whose diameter is β; and color each point bases on the region they must intersect. This is meant to help the reader get their bearing on this process of finding these values for β. Later in the proof of Theorem 3, we simply color all dots with green. We will also graph the segments between these points (in green) in order to aid the reader in computing the diameters of these collections of points.

Definition 13. *Let $A_\ell \subset K$ be a square of side length $\frac{\ell\sqrt{2}}{2}$ with a distinguished diagonal L of length ℓ. Furthermore, let $(A_\ell)_n := \{\Delta \cap A_\ell \mid \Delta \in K_n\}$ be the set of n-cells of A_l. We say that $\{s_i\}_{i=1}^k$ is P-***scaleable in** *A_ℓ, if there exists a P-scaleable $\{\Delta_i\}_{i=1}^k \subset K_n$ such that $\Delta_i \cap A_\ell = s_i$ for all $1 \leq i \leq k$.*

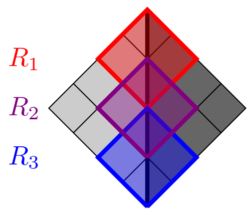


Lemma 2. *Let $\{\Delta_i^{(n)}\}_{i=1}^k \subset K_n$ and $\Theta_i := \Delta_i \cap A_\ell \neq \varnothing$ for all $1 \leq i \leq k$. Suppose that the following conditions hold*

- $|\bigcup_{i=1}^k \Delta_i^{(n)}| \leq |\bigcup_{i=1}^k \Theta_i|$,
- *$\bigcup_{i=1}^k \Theta_i$ intersects both regions left and right of the distinguished diagonal L,*
- *$\{\Delta_i^{(n)}\}_{i=1}^k$ is P-scaleable if and only if $\{\Theta_i\}_{i=1}^k$ is P-scaleable in A_ℓ.*

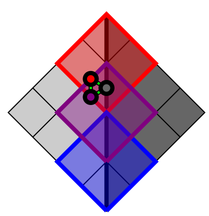
Then $d := |\bigcup_{i=1}^k \Delta_i^{(n)}| \geq \ell(\sqrt{3}-1)\left(\frac{1}{2}\right)^4$.

Proof. We define R_1, R_2, R_3 to be the following regions.

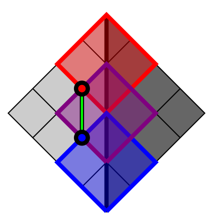


We provide bounds by cases depending on how many of the regions R_i the figure $\bigcup_i \Theta_i$ is contained in.

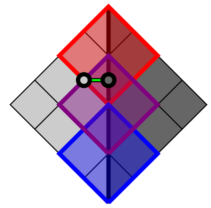
- When $\bigcup_i \Theta_i$ lies in R_1, R_2, or R_3, it follows that $\bigcup_i \Theta_i$ is P-scaleable in A_ℓ and can be excluded from consideration by Lemma 1.
- When $\bigcup_i \Theta_i$ belongs to either $R_1 \bigcup R_2$, or $R_2 \bigcup R_3$, and none of the previous cases, we have that $\bigcup_i \Theta_i \not\subset R_1 \cap R_2$ and $\bigcup_i \Theta_i \not\subset R_2 \cap R_3$. By symmetry, we can assume $\bigcup_i \Theta_i \subset R_1 \bigcup R_2$ and $\bigcup_i \Theta_i \not\subset R_1 \cap R_2$. Thus, $\bigcup_i \Theta_i$ must intersect $R_1 \setminus R_2$, $R_2 \setminus R_1$, H_L, and H_R. A quick calculation shows that $d \geq \ell(\sqrt{3}-1)\left(\frac{1}{2}\right)^4$



- When $\bigcup_i \Theta_i \subset R_1 \bigcup R_2 \bigcup R_3$ and none of the previous cases, one sees that $\bigcup_i \Theta_i$ must intersect $R_1 \setminus R_2$ and $R_3 \setminus R_2$. Then $d \geq \ell\left(\frac{1}{2}\right)^2$.



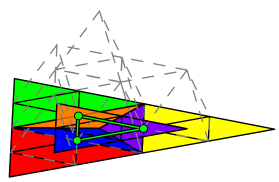
- When none of the previous cases are satisfied, we have that $\bigcup_i \Theta_i \not\subset R_1 \bigcup R_2 \bigcup R_3$. By symmetry we may assume $\bigcup_i \Theta_i$ intersects $H_L \setminus (\bigcup_i R_i)$ and H_R. Then $d \geq \ell\left(\frac{1}{2}\right)^3$.



Combining all cases, we conclude that $d \geq \ell(\sqrt{3}-1)\left(\frac{1}{2}\right)^4$, as claimed. $\square$

We now proceed to continue with the proof of Theorem 3, which we divide in four main cases. These are, when $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects three, two, one, or none of the 1-cells $K_{(1)}$, $K_{(2)}$, $K_{(3)}$. However, before examining each case, we will note that when $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained in a 1-cell, there must exist a similarity of ratio $\frac{1}{2}$ onto K, making these families scaleable. By Lemma 1, we will exclude these from consideration.

Case 1. When $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects $K_{(1)}$, $K_{(2)}$, and $K_{(3)}$, we have that $d \geq \frac{1}{4}$. This follows from the fact that there exists a projection π onto the triangle T.



Note that $d \geq |\pi(\bigcup_{i=1}^k \Delta_i^{(n)})|$ and by symmetry, the minimum $\frac{1}{4}$ is obtained by the triangle with vertices $F_{12}(p_3)$, $F_{23}(p_1)$, $F_{31}(p_2)$. By Corollary 1, $\alpha^{(1)} \geq a_n \exp\left(-s_2/2^{n-3}\right)$.

Case 2. Suppose $\bigcup_{i=1}^k \Delta_i^{(n)}$ only intersects two of the base 1-cells of K. We will assume these are $K_{(1)}$ and $K_{(2)}$, since the other arguments follow by rotations of $\mathbb{R}^3$. We will provide a similar argument to Case 1, by introducing a sequence $a_n^{(2)}$. However, this case will rely on Lemma 1, since $\bigcup_{i=1}^k \Delta_i^{(n)}$ can be arbitrarily close to the point $F_1(p_2) = F_2(p_1)$ as $n \to \infty$, implying that we cannot find lower bound for d unless we exclude Case 2-scaleable families from consideration. We will show that $\alpha^{(2)} \geq a_n \exp\left(-s_2\sqrt{2}/2^{n-3}\right)$, dividing this part into two sub-cases.

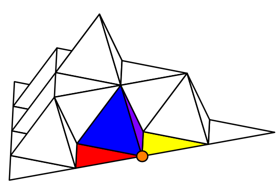


FIGURE 6. Critical corner for Case 2

Case 2a. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained in the critical region $K_{(12)} \cup K_{(43)} \cup K_{(52)} \cup K_{(21)}$, we can scale this corner by 2 into $K_{(1)} \cup K_{(2)} \cup K_{(4)} \cup K_{(5)}$. By Lemma 1, we may exclude this case from consideration.

Case 2b. Assume that $\bigcup_{i=1}^k \Delta_i^{(n)}$ is not contained in the critical region $K_{(12)} \cup K_{(43)} \cup K_{(52)} \cup K_{(21)}$. By symmetry, we may suppose $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects $K_1 \setminus (K_{(12)} \cup K_{(43)} \cup K_{(52)} \cup K_{(21)})$. We then see from Figure 7 that $d \geq \frac{\sqrt{2}}{8}$.

By Corollary 1, $\alpha^{(2b)} \geq a_n^{(2b)} \exp\left(-s_2/2^{n-3}\right)$. Putting $\alpha^{(2)} := \alpha^{(2b)}$ and $a_n^{(2)} := a_n^{(2b)}$, we obtain our desired sequence. Furthermore,

$$\alpha^{(2)} \geq a_n^{(2)} \exp\left(-s_2\sqrt{2}/2^{n-3}\right) \geq a_n \exp\left(-s_2\sqrt{2}/2^{n-3}\right).$$

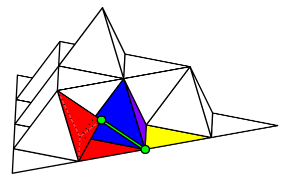


FIGURE 7. Case 2b calculation

Case 3. Suppose $\bigcup_{i=1}^k \Delta_i^{(n)}$ only intersects one of the base 1-cells of K. We may assume that this cell is $K_{(1)}$ by symmetry. We will subdivide this case by considering when $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects $K_{(5)}$ and $K_{(6)}$, either of these exclusively, or neither. Defining $\alpha^{(3)} := \min\{\alpha^{(3a)}, \alpha^{(3b)}, \alpha^{(3c)}\}$ and $a^{(3)} := \min\{a^{(3a)}, a^{(3b)}, a^{(3c)}\}$, we will obtain our desired sequence. Furthermore, we will show $\alpha^{(3)} \geq a_n^{(3)} \exp\left(-s_2(\sqrt{2}+\sqrt{6})/2^{n-6}\right) \geq a_n \exp\left(-s_2(\sqrt{2}+\sqrt{6})/2^{n-6}\right)$.

Case 3a. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects $K_{(5)}$ and $K_{(6)}$ as well, we see from the symmetries of K and Figure 8 that $d \geq \frac{\sqrt{6}-2}{2}$.

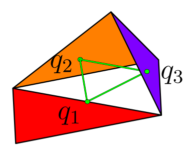


FIGURE 8. $K_{(1)}, K_{(5)}, K_{(6)}$ together with points q_1, q_2, q_3 in green

where $p_{i,j} := (p_i + p_j)/2$, and

$$q_1 = \frac{p_{1,2} + p_{1,3}}{2}$$
$$q_2 = (\sqrt{6}-2)p_{1,3} + \frac{3-\sqrt{6}}{2}p_{2,3} + \frac{3-\sqrt{6}}{2}p_4$$
$$q_3 = (\sqrt{6}-2)p_{1,2} + \frac{3-\sqrt{6}}{2}p_{2,3} + \frac{3-\sqrt{6}}{2}p_4$$

Case 3b. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ is intersects $K_{(5)}$ or $K_{(6)}$ exclusively, an argument similar to that of Case 2 holds, and we see that $d \geq \frac{1}{8}$.

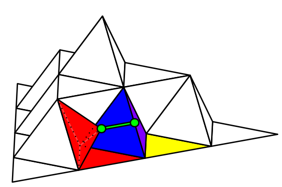


FIGURE 9. Case 3b calculation

Case 3c. The difficult case arises when $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained in $K_{(1)} \cup K_{(4)}$, since there is now a critical face where the diameter $|\bigcup_{i=1}^k \Delta_i^{(n)}|$ can approach 0 as $n \to \infty$. This case must be subdivided into further sub-situations, depending on the region where $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained.
We first define the following regions

$$R_1 := K_{(21)} \cup K_{(34)} \qquad R_2 := K_{(51)} \cup K_{(64)} \qquad R_3 := K_{(61)} \cup K_{(54)} \qquad R_4 := K_{(31)} \cup K_{(24)}$$
$$R_1' := K_{(321)} \cup K_{(234)} \qquad R_2' := K_{(351)} \cup K_{(234)} \qquad R_3' := K_{(261)} \cup K_{(354)} \qquad R_4' := K_{(231)} \cup K_{(324)}$$

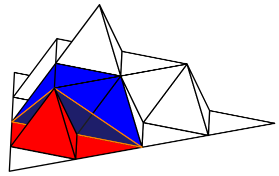
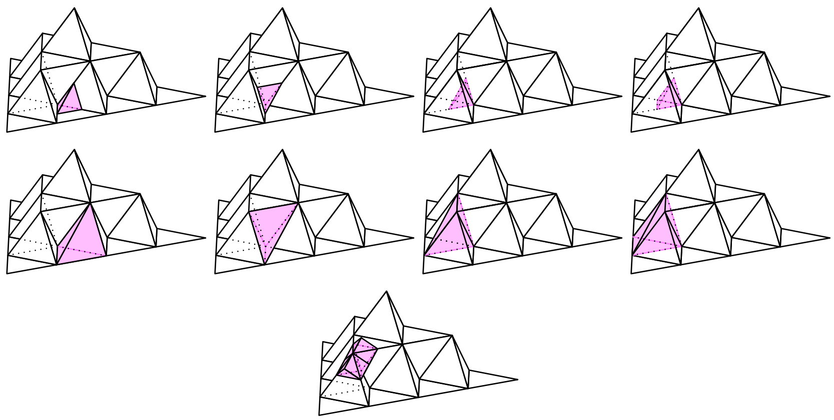


FIGURE 10. Critical region for Case 3

$$R_5 = K_{(141)} \cup K_{(151)} \cup K_{(161)} \cup K_{(414)} \cup K_{(514)} \cup K_{(614)}$$

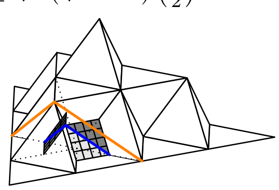


We then divide this case into the following subcases

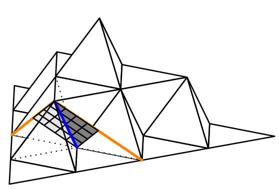
(i) $\bigcup_i \Delta_i^{(n)} \subset R_1,\ R_2,\ R_3,\ R_4$, or R_5; or $\bigcup_i \Delta_i^{(n)} \subset R_1' \cup R_2' \cup R_3' \cup R_4'$.
(ii) $\bigcup_i \Delta_i^{(n)} \subset R_1 \cup R_2$ or $R_3 \cup R_4$ and none of the previous cases
(iii) $\bigcup_i \Delta_i^{(n)} \subset R_2 \cup R_3$ and none of the previous cases
(iv) $\bigcup_i \Delta_i^{(n)} \subset R_1 \cup R_2 \cup R_3 \cup R_4$ and none of the previous cases
(v) None of the previous cases

Case 3c.i. There exist the following similarities into $R_1 \cup R_4$ of ratio $\frac{1}{2}$. By Lemma 1, we may exclude this case from consideration.

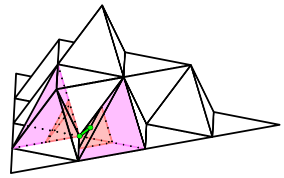
Case 3c.ii. Since $\bigcup \Delta_i^{(n)} \not\subset R_1' \cup R_2' \cup R_3' \cup R_4'$, there is a projection π onto the following squares with $\ell = \sqrt{2}\left(\frac{1}{2}\right)^3$. By Lemma 2, we obtain $d \geq \sqrt{2}(\sqrt{3}-1)\left(\frac{1}{2}\right)^7$.



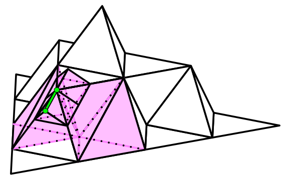
Case 3c.iii. There exists a projection π onto the following square with $l = \left(\frac{1}{2}\right)^2$. By Lemma 2, we obtain $d \geq (\sqrt{3}-1)\left(\frac{1}{2}\right)^6$.



Case 3c.iv We may assume that $\bigcup_{i=1}^{k} \Delta_i^{(n)}$ intersects R_1 and R_3, or R_2 and R_4 by exclusion of previous cases. Then $d \geq \sqrt{2}\left(\frac{1}{2}\right)^4$.



Case 3c.v. When $\bigcup_i s_i \not\subset \bigcup_{i=1}^5 R_i$, a simple calculation yields $d \geq \sqrt{2}\left(\frac{1}{2}\right)^4$.

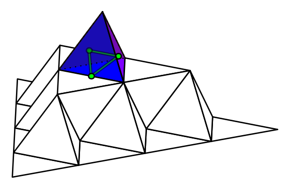


As in Case 2, we apply Corollary 1 and obtain

$$\alpha^{(3)} = \alpha^{(3c)} = \alpha^{(3c.ii)} \geq a_n^{(3)} \exp\left(-s_2(\sqrt{2}+\sqrt{6})/2^{n-6}\right) \geq a_n \exp\left(-s_2(\sqrt{2}+\sqrt{6})/2^{n-6}\right).$$

Case 4. Suppose that $\bigcup_{i=1}^k \Delta_i^{(n)}$ does not intersect any of the base 1-cells. We will subdivide this case by considering when $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects three or two of the 1-cells $K_{(4)}$, $K_{(5)}$, and $K_{(6)}$. Defining $\alpha^{(4)} := \min\{\alpha^{(4a)}, \alpha^{(4b)}\}$ and $a^{(4)} := \min\{a^{(4a)}, a^{(4b)}\}$, we obtain our desired sequence. Furthermore, $\alpha^{(4)} \geq a_n^{(4)} \exp\left(-s_2(1+\sqrt{3})/2^{n-5}\right) \geq a_n \exp\left(-s_2(1+\sqrt{3})/2^{n-5}\right)$.

Case 4a. Consider when $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects $K_{(4)}, K_{(5)}, K_{(6)}$. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ is not contained in the upper corner $K_{(41)} \cup K_{(51)} \cup K_{(61)}$, we see from the symmetries of K and the following diagram that $d \geq \frac{\sqrt{6}-2}{4}$. This is identical to the argument in Case 3a.



In the case where $\bigcup_{i=1}^k \Delta_i^{(n)} \subseteq K_{(41)} \cup K_{(51)} \cup K_{(61)}$, there exists a similarity into $R_4 \cup R_5 \cup R_6$ of ratio $\frac{1}{2}$, making the family case (4a)-scaleable.

Case 4b. Consider when $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects only two of $K_{(4)}, K_{(5)}, K_{(6)}$. By symmetry, one can assume that $K_{(4)}$ and K_5 are such sets. Note that there is now a critical edge of intersection between the points p_4 and $F_1(p_2) = F_2(p_1)$. However, there exists a suitable projection π onto the square with $l = \frac{1}{2}$ seen in Figure 11. An application of Lemma 2 gives $d \geq (\sqrt{3}-1)\left(\frac{1}{2}\right)^5$.

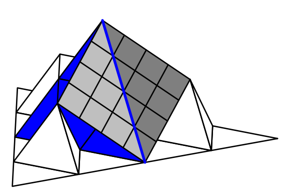


FIGURE 11. Critical region for case

We then see that

$$\alpha^{(4)} = \alpha^{(4b)} \geq a_n^{(4b)} \exp\left(-s_2(1+\sqrt{3})/2^{n-5}\right) \geq a_n \exp\left(-s_2(1+\sqrt{3})/2^{n-5}\right).$$

Henceforth, combining all the above cases, we obtain
$\mathscr{H}^s(K) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \min\{a_n^{(1)}, a_n^{(2)}, a_n^{(3)}, a_n^{(4)}\} = \min\{\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}\}$

$$\geq a_n \min\left\{\exp\left(-s_2/2^{n-3}\right), \exp\left(-s_2\sqrt{2}/2^{n-3}\right), \exp\left(-s_2(\sqrt{2}+\sqrt{6})/2^{n-6}\right), \exp\left(-s_2(1+\sqrt{3})/2^{n-5}\right)\right\}$$
$$= a_n \exp\left(-s_2(\sqrt{2}+\sqrt{6})/2^{n-6}\right).$$

Combining the above inequality with Proposition 2, we are led to the inequality (4.5), completing the proof.

6. Proof of Theorem 4

This demonstration will be akin to that of the proof of Theorem 3. Let $N > 2$ be such that $N \not\equiv 0 \pmod 3$ and let $\{\Delta_i^{(n)}\}_i$ be a collection of n-cells of K_N with diameter $d := |\bigcup_{i=1}^k \Delta_i^{(n)}|$. We will also organize this proof by cases, based on how many of the bottom 1-cells tangent at an edge to the peak of K_N the family $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects. In order to make this procedure as similar to what was done for K_2, we will choose to denote the 1-cells adjacent to the peak by $K_{(1)}, K_{(2)}, K_{(3)}$ and those 1-cells making up the peak by $K_{(4)}, K_{(5)}, K_{(6)}$. However, due to the difficulty in naming most of the 1-cells in K_N, we will depend on

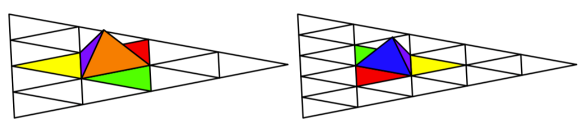
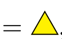
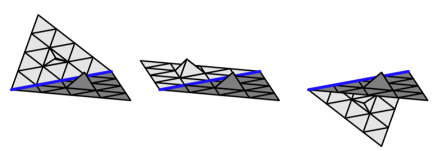


Figure 12. Koch 4-surface and 5-Surface, First Iterations

diagrams to refer to the regions they depict throughout the proof. For example, $K_{(1)} =$ , $K_{(2)} =$ , and $K_{(3)} =$ . We will organize the proof by how many of these distinguished base 1-cells , , and  the set $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects. As before, we will note that when $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained in a 1-cell, there exists a similarity of ratio $\frac{1}{N}$ onto K, making these families scaleable. By Lemma 1, we will exclude these from consideration. We will also provide another technical lemma which will play an analogous role to that of Lemma 2.

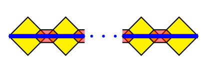
Lemma 3. *Consider two Koch N-surfaces of scale $1/N$ intersecting at a base edge of length $1/N$ forming a dihedral angle of $\theta = \arccos(1/3), 0, \arccos(1/3) - \pi$.*

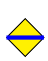
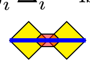


Then if $\bigcup_{i=1}^k \Delta_i^{(n)}$ is a family of n-cells that intersects both Koch N-surfaces, it follows that

$$d := \left|\bigcup_{i=1}^k \Delta_i^{(n)}\right| \geq \frac{\sqrt{6}}{3N^3}.$$

Proof. Note that we can cover the critical edge L where the two Koch N-surfaces by N self-similar copies  of scale $1/N$ of the whole figure, see yellow-shaded regions in Figure 13. We can further cover the points where adjacent yellow-shaded regions meet by $N-1$ hexagonal regions  of scale $1/N^2$, see red-shaded regions in Figure 13.

We first consider the case when $\bigcup_i \Delta_i^{(n)} \subseteq$  L

1. If $\bigcup_i \Delta_i^{(n)} \subseteq$  , then $\bigcup_i \Delta_i^{(n)}$ is scaleable.
2. If not, suppose $\bigcup_i \Delta_i^{(n)} \subseteq$  .
   (a) If $\bigcup_i \Delta_i^{(n)} \subset$  , then $\bigcup_i \Delta_i^{(n)}$ is scaleable.

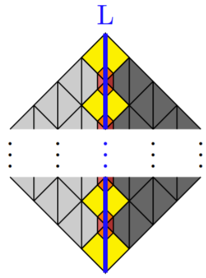
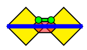
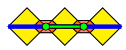
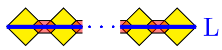


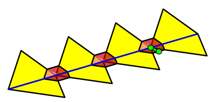
FIGURE 13. Planar representation of two Koch N-surfaces meeting at an edge

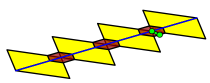
   (b) Otherwise, we see from  that $d \geq 1/N^3$ regardless of θ.

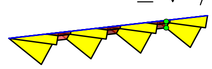
(3) If neither (1) or (2) hold, we see from  that $d \geq 1/N^2$ regardless of θ.

Now we consider $\bigcup_i \Delta_i^{(n)} \not\subseteq$ 

.

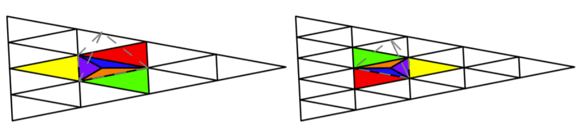
(1) When $\theta = \arccos(1/3)$, we see from  that $d \geq \sqrt{3}/2N^3$

(2) When $\theta = 0$, we see from  that $d \geq \sqrt{3}/2N^3$.

(3) When $\theta = \arccos(1/3) - \pi$, we see from  that $d \geq \sqrt{6}/3N^3$. □

We now proceed to complete the body of the proof of Theorem 4, which again is divided into four main cases.

Case 1. When $\bigcup_{i=1}^{k} \Delta_i^{(n)}$ intersects all of the distinguished base 1-cells , , and , we have that $d \geq \frac{1}{2N}$. This follows from the fact that there exists a projection π onto the triangle T.



Note that $d \geq |\pi(\bigcup_{i=1}^{k} \Delta_i^{(n)})|$ and by symmetry, the infimum $\frac{1}{2N}$ is obtained by the triangle with vertices $F_{12}(p_3)$, $F_{23}(p_1)$, $F_{31}(p_2)$. By Corollary 1, $\alpha^{(1)} \geq a_n \exp\left(-\frac{4 s_N}{N^{n-1}}\right)$.

Case 2. Suppose $\bigcup_{i=1}^{k} \Delta_i^{(n)}$ only intersects two of the distinguished base 1-cells of K. We will assume these are $K_{(1)}$ and $K_{(2)}$, since the other arguments follow by rotations of $\mathbb{R}^3$. We will provide a similar argument to Case 1, by introducing a sequence $a_n^{(2)}$. However, this case will rely on Lemma 1, since $\bigcup_{i=1}^{k} \Delta_i^{(n)}$ can be arbitrarily close to the point $F_1(p_2) = F_2(p_1)$ as $n \to \infty$, implying that we cannot find lower bound for d unless we exclude Case 2-scaleable families from consideration. We will show that $\alpha^{(2)} \geq a_n \exp\left(-\frac{4\sqrt{3}\, s_N}{3N^{n-2}}\right)$ divide this part into two sub-cases.

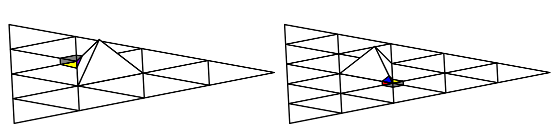
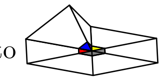
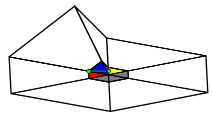


FIGURE 14. Critical corner for Case 2

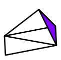
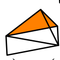
Case 2a. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained in the critical region , we can scale this corner by 2 into  By Lemma 1, we may exclude this case from consideration.

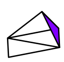
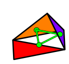
Case 2b. Assume that $\bigcup_{i=1}^k \Delta_i^{(n)}$ is not contained in the critical region  . We see from the following diagram that $d \geq \sqrt{3}/2N^2$.

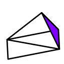
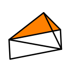
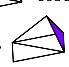
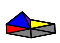


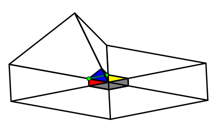
By Corollary 1, $\alpha^{(2b)} \geq a_n^{(2b)} \exp\left(-\frac{4\sqrt{3}\, s_N}{3N^{n-2}}\right)$. Putting $\alpha^{(2)} := \alpha^{(2b)}$ and $a_n^{(2)} := a_n^{(2b)}$, we obtain our desired sequence. Furthermore,

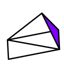
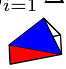
$$\alpha^{(2)} \geq a_n^{(2)} \exp\left(-\frac{4\sqrt{3}\, s_N}{3N^{n-2}}\right) \geq a_n \exp\left(-\frac{4\sqrt{3}\, s_N}{3N^{n-2}}\right).$$

Case 3. Suppose $\bigcup_{i=1}^k \Delta_i^{(n)}$ only intersects one of the three distinguished base 1-cells of K. We may assume that this cell is  by symmetry. We will subdivide this case by considering when $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects  and  , either of these exclusively, or neither. Defining $\alpha^{(3)} := \min\{\alpha^{(3a)}, \alpha^{(3b)}, \alpha^{(3c)}\}$ and $a^{(3)} := \min\{a^{(3a)}, a^{(3b)}, a^{(3c)}\}$, we will obtain our desired sequence. Furthermore, we will show $\alpha^{(3)} \geq a_n \exp\left(-\frac{\sqrt{6}\, s_N}{N^{n-3}}\right)$.

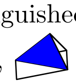
Case 3a. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects both  and  , we see from symmetry and  that $d \geq \frac{\sqrt{6}-2}{N}$. Finding this bound is identical to the calculation when $N = 2$.

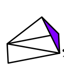
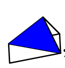
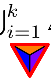
Case 3b. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects  or  exclusively, an argument similar to that of Case 2 holds. Indeed, we may suppose $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects  by symmetry and exclude when $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained in the critical region  by Lemma 1. We then see from the following diagram that $d \geq \frac{\sqrt{6}}{3N^2}$.



Case 3c. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ does not intersect  or  then $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects the two Koch n-surfaces (in red and blue)  of scale $1/N$ meeting at an angle of $\theta = \arccos(1/3)$. So by lemma 3, $d \geq \frac{\sqrt{6}}{3N^3}$.

By Corollary 1, $\alpha^{(3)} = \alpha^{(3c)} \geq a_n^{(3c)} \exp\left(-\frac{\sqrt{6}\, s_N}{N^{n-3}}\right) \geq a_n \exp\left(-\frac{\sqrt{6}\, s_N}{N^{n-3}}\right)$.

Case 4. Suppose that $\bigcup_{i=1}^k \Delta_i^{(n)}$ does not intersect any of the three distinguished base 1-cells of K. We will subdivide this case by considering how many of the 1-cells , ,  the set $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects. Defining $\alpha^{(4)} := \min\{\alpha^{(4a)}, \alpha^{(4b)}\}$ and $a^{(4)} := \min\{a^{(4a)}, a^{(4b)}\}$, we obtain our desired sequence. Furthermore, we show that $\alpha^{(4)} \geq a_n^{(4)} \exp\left(-\frac{\sqrt{6}\, s_N}{N^{n-3}}\right) \geq a_n \exp\left(-\frac{\sqrt{6}\, s_N}{N^{n-3}}\right)$.

Case 4a. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects , , and , an argument similar to that of Case 2b and 3b holds. Indeed, we may exclude when $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained in the uppermost corner  (see Figure 15) by Lemma 1. Now supposing $\bigcup_{i=1}^k \Delta_i^{(n)} \not\subset$ , it follows from the symmetries of K_N and from the following diagram that $d \geq \frac{\sqrt{6}-2}{N^2}$. Finding this bound is identical to the calculation when $N = 2$.

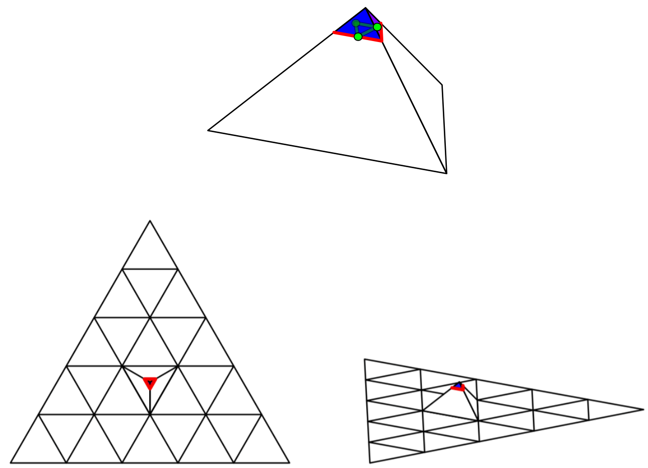
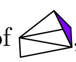
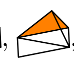
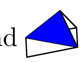
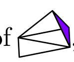
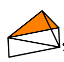
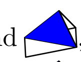
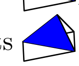
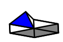
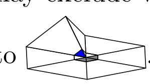
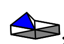
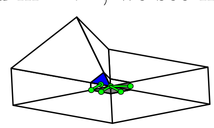


FIGURE 15. Uppermost corner of K_5

In the case where $\bigcup_{i=1}^k \Delta_i^{(n)} \subseteq$ , there exists a similarity into $R_4 \cup R_5 \cup R_6$ of ratio $\frac{1}{N}$, making the family Case (4a)-scaleable.

Case 4b. Consider when $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects only two of , , and  , so that $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects two Koch n-surfaces of scale $1/N$ meeting at an angle of $\theta = \arccos(1/3) - \pi$. By lemma 3, $d \geq \frac{\sqrt{6}}{3N^3}$.

Case 4c. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects only one of , , and , then an identical argument to Cases 2b and 3b holds. First, we may suppose $\bigcup_{i=1}^k \Delta_i^{(n)}$ only intersects  by symmetry. Then, we may exclude when $\bigcup_{i=1}^k \Delta_i^{(n)}$ is contained in the critical region  since we may scale this corner by N into . On the other hand, if $\bigcup_{i=1}^k \Delta_i^{(n)}$ is not contained in , we see from the following diagram that $d \geq \sqrt{3}/2N^2$.



Case 4d. If $\bigcup_{i=1}^k \Delta_i^{(n)}$ does not intersect any of the 1-cells, we may assume without loss of generality that $\bigcup_{i=1}^k \Delta_i^{(n)}$ intersects adjacent 1-cells $\Delta_k^{(1)}, \Delta_\ell^{(1)}$. If $\Delta_k^{(1)}$ and $\Delta_\ell^{(1)}$ intersect at an edge (forming a dihedral angle of $\theta = 0$), it follows from Lemma 3 that $d \geq \sqrt{6}/3N^3$. On the other hand, if $\Delta_k^{(1)}$ and $\Delta_\ell^{(1)}$ only intersect at a point p then an argument similar to Case 2b and 3b holds. Indeed, consider the region R consisting of the six 1-cells which intersect at p. Note that there exists a similar region R' of six 2-cells which intersect at p.

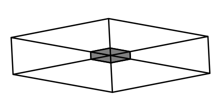


FIGURE 16. The region R together with R' in gray

Now if $\bigcup_{i=1}^k \Delta_i^{(n)} \subset R'$, we can scale R' by a factor of N into R. So by Lemma 1, we may exclude this case from consideration. When $\bigcup_{i=1}^k \Delta_i^{(n)} \not\subset R'$, we see from the following diagram that $d \geq \sqrt{3}/2N^2$.

We then see that

$$\alpha^{(4)} = \alpha^{(4b)} \geq a_n^{(4b)} \exp\left(-\frac{\sqrt{6}\, s_N}{N^{n-3}}\right) \geq a_n \exp\left(-\frac{\sqrt{6}\, s_N}{N^{n-3}}\right).$$

Henceforth, combining all the above cases, we obtain

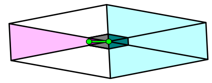


FIGURE 17. $\Delta_k^{(1)}$ in pink (left) with the possible $\Delta_\ell^{(1)}$ in blue (right)

$$\mathscr{H}^s(K) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \min\{a_n^{(1)}, a_n^{(2)}, a_n^{(3)}, a_n^{(4)}\} = \min\{\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}\}$$
$$\geq a_n \min\left\{\exp\left(-\frac{4s_N}{N^{n-1}}\right), \exp\left(-\frac{4\sqrt{3}\,s_N}{3N^{n-2}}\right), \exp\left(-\frac{\sqrt{6}\,s_N}{N^{n-3}}\right)\right\} = a_n \exp\left(-\frac{\sqrt{6}\,s_N}{N^{n-3}}\right).$$

Combining the above inequality with Proposition 2, we are lead to the inequality (4.5), completing the proof.

7. Application

We now present an application to the theory of Partial Differential Equations over the Koch N-crystals.

7.1. **Preliminaries.** To begin, denote by $\Omega_N := \text{int}(C_N) \subseteq \mathbb{R}^3$ the interior of the Koch N-Crystal defined by Definition 8 with boundary $\Gamma_N := K_N$ (as in Definition 7), for $N > 2$ with $N \not\equiv 0 \,(\text{mod } 3)$. As discussed at the end of section 3, one has that Ω_N is an uniform domain whose boundary Γ_N is a s_N-set with respect to the s_N-dimensional Hausdorff measure, where $s_N := \log(N^2+2)/\log(N)$. Given a set E, we denote by $L^p(E,\mu)$ the L^p-based space of μ-measurable functions, and we write λ_3 as the 3-dimensional Lebesgue measure. Also, for a domain D, we denote by $W^{1,p}(D)$ the well-known p-Sobolev spaces, for $1 \leq p \leq \infty$. When $p = 2$, we write $H^1(D) := W^{1,2}(D)$. Finally, by $\overline{\Omega}_N$ we denote the closure of the set Ω_N.

For non-Lipschitz domains, the normal derivative may not be well-defined. Thus, in order to pose our boundary value problem, an appropriate interpretation of the normal outward vector needs to be addressed. The following definition is presented in advance (e.g. [1, 11]).

Definition 14. *Let η be a s-Ahlfors measure supported on ∂D and let $u \in W^{1,1}_{loc}(D)$ fulfills $\nabla u \cdot \nabla v \in L^1(D)$ for all $v \in C^1(\overline{D})$. If there exists $f \in L^1_{loc}(\mathbb{R}^N)$ with*

$$\int_D \nabla u \nabla v \, dx = \int_D f v \, dx + \int_{\partial D} v \, d\eta,$$

*for all $v \in C^1(\overline{D})$, then we say that η is the **generalized normal derivative** of u, and we denote*

$$\frac{\partial u}{\partial \nu_s} := \eta.$$

We now present the following example.

7.2. **The Robin boundary value problem.** Consider now the realization of the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega_N, \\ \dfrac{\partial u}{\partial \nu_{s_N}} + \beta u = g & \text{on } \Gamma_N, \end{cases} \tag{7.1}$$

for $f \in L^p(\Omega_N)$, $g \in L^q(\Gamma_N)$, and $\beta \in L^\infty(\Gamma_N)$ with $\operatorname{ess}\inf_{x\in\Gamma_N} \beta(x) \geq b_0$ for some constant $b_0 > 0$. Then equation (7.1) turns out to be a Robin boundary value problem over the Koch N-crystal. In fact, [11] shows that one can define the Robin problem over more classes of irregular domains, in which the Koch N-crystals form family of domains fulfilling the required properties.

A function $u \in H^1(\Omega_N)$ is said to be a **weak solution** of the Robin problem (7.1), if

$$\mathcal{E}_N(u,\varphi) = \int_{\Omega_N} f\varphi \, dx + \int_{\Gamma_N} g\varphi \, d\mathscr{H}^{s_N}$$

for all $\varphi \in H^1(\Omega_N)$, where

$$\mathcal{E}_N(u,\varphi) = \int_{\Omega_N} \nabla u \nabla \varphi \, dx + \int_{\Gamma_N} \beta u \varphi \, d\mathscr{H}^{s_N}.$$

Since Ω_N is an uniform domain, Γ_N is a s_N-set with respect to $\mathscr{H}$, and $s_N \in (1,3)$, the conclusions in [11] imply the following important result.

Theorem 5. *If $f \in L^p(\Omega_N)$ and $g \in L^q(\Gamma_N)$ for $p > \frac{3}{2}$ and $q > \frac{s_N}{s_N - 1}$, then the Robin problem (7.1) admits a unique weak solution $u \in H^1(\Omega_N)$, and there exists a constant $\delta \in (0,1)$ such that $u \in C^{0,\delta}(\overline{\Omega}_N)$, that is, u is globally Hölder continuous. Furthermore, there is a constant $C > 0$ (independent of u), such that*

$$\|u\|_{C^{0,\delta}(\overline{\Omega}_N)} \leq C \left(\|f\|_{p,\Omega_N} + \|g\|_{q,\Gamma_N} \right).$$

The above result is also valid in the quasi-linear case involving the p-Laplace operator, for $6(s_N+2)^{-1} < p < \infty$ (under some modifications on q and r; see [11]). Recently a generalization of this result has been obtained to the Robin problem involving variable exponents and anisotropic structures. For more details, refer to [1].

Acknowledgements

We would like to thank Ernesto Ferrer for his help in modeling the second and third iterations of the Koch 4-Crystal and 5-Crystal, seen in Figure 5.

The second author is supported by The Puerto Rico Science, Technology and Research Trust under agreement number 2022-00014.

References

[1] M.-M. Boureanu and A. Vélez - Santiago. Fine regularity for elliptic and parabolic anisotropic Robin problems with variable exponents. *J. Differential Equations* **266** (2019), 8164–8232.
[2] K. J. Falconer, K. J. *Fractal geometry: Mathematical foundations and applications.* Chichester: Wiley, 1990.
[3] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math.* **30** (1981), 713–747.
[4] B. Jia. Bounds of Hausdorff measure of the Sierpinski gasket. *J. Math. Anal. Appl.* **330** (2007), 1016–1024.
[5] B. Jia. Bounds of the Hausdorff measure of the Koch curve. *Applied Mathematics and Computation* **190** (2007), 559–565.
[6] P. W. Jones. Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.* **147** (1981), 71–88.
[7] M. R. Lancia, A. Vélez - Santiago, P. Vernole. A quasi-linear nonlocal Venttsel' problem of Ambrosetti–Prodi type on fractal domains. *Discrete & Continuous Dynamical Systems - Series A* **39** (2019), 4487–4518.
[8] M. R. Lancia, A. Vélez - Santiago, P. Vernole. Quasi-linear Venttsel' problems with nonlocal boundary conditions on fractal domains. *Nonlinear Analysis: Real World Applications* **35** (2017), 265–291.
[9] M. L. Lapidus and M. Pang. Eigenfunctions of the Koch snowflake domain. *Commun. Math. Phys.* **172** (1995), 359–376.
[10] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge Univ. Press, 1995.
[11] A. Vélez - Santiago. Global regularity for a class of quasi-linear local and nonlocal elliptic equations on extension domains. *J. Functional Analysis* **269** (2015), 1–46.

Giovanni Ferrer
Department of Mathematics, The Ohio State University, 100 Math Tower, 231 West 18th Avenue, Columbus, OH, 43210, U.S.A.
Email address: ferrer.40@osu.edu

Alejandro Vélez - Santiago
Department of Mathematical Sciences, University of Puerto Rico at Mayagüez, Mayagüez, PR 00681
Email address: alejandro.velez2@upr.edu, dr.velez.santiago@gmail.com