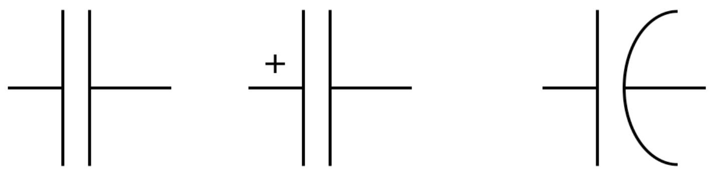
Lecture 2: Resistors and Capacitors

Capacitance:

- Capacitance (C) is defined as the ratio of charge (Q) to voltage (V) on an object:
  - $C = Q/V$ = Coulombs/Volt = Farad
    - ☞ Capacitance of an object depends on geometry and its dielectric constant.
  - Symbol(s) for capacitors:



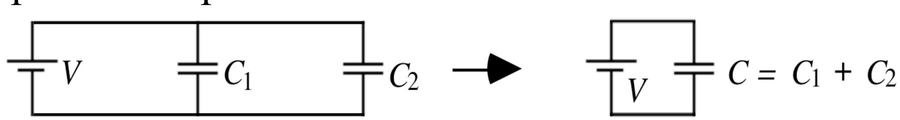
polarized

  - A capacitor is a device that stores electric charge (memory devices).
  - A capacitor is a device that stores energy

$$E = \frac{Q^2}{2C} = \frac{CV^2}{2}$$

  - Capacitors are easy to fabricate in small sizes (µm), use in chips

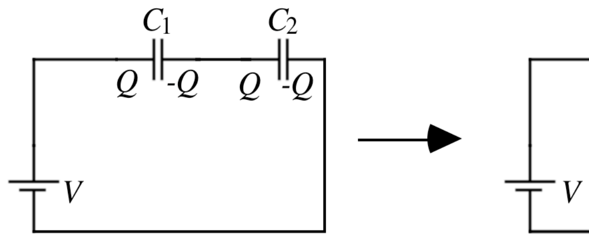
- How to combine capacitance:
  - capacitors in parallel adds like resistors in series:



$$C_{tot} = C_1 + C_2 = \sum C_i$$

Total capacitance is *more* than individual capacitance!

- capacitors in series add like resistors in parallel:



$$C = \frac{C_1 C_2}{C_1 + C_2}$$

$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} = \sum \frac{1}{C_i}$$

Total capacitance is *less* than individual capacitance!

- Energy and Power in Capacitors
  - How much energy can a "typical" capacitor store?
    - Pick a 4 μF Cap (it would read 4 mF) rated at 3 kV

      $$E = \tfrac{1}{2}CV^2 = \tfrac{1}{2}4\times 10^{-6} \bullet 3000^2 = 18 \text{ J}$$
    - This is the same as dropping a 2 kg weight (about 4 pounds) 1 meter
  - How much power is dissipated in a capacitor?

$$Power = \frac{dE}{dt}$$

$$= \frac{d}{dt}\left(\frac{CV^2}{2}\right)$$

$$P = CV\frac{dV}{dt}$$

    - dV/dt must be finite otherwise we source (or sink) an infinite amount of power!

THIS WOULD BE UNPHYSICAL.

- the voltage across a capacitor cannot change instantaneously
  - a useful fact when trying to guess the transient (short term) behavior of a circuit
- the voltage across a resistor can change instantaneously
  - the power dissipated in a resistor does not depend on dV/dt:

    $P = I^2R$ or V^2/R

- Why do capacitors come in such small values?
  - Example: Calculate the size of a 1 Farad parallel capacitor with air between the plates.

$$C = \frac{k\varepsilon_0 A}{d}$$

k = dielectric constant (= 1 for air)

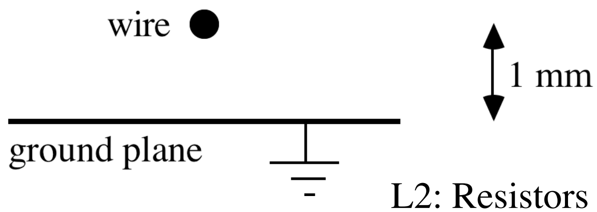
$\varepsilon_0 = 8.85 \times 10^{-12} N^{-1} m^{-2}$

d = distance between plates (assumed 1 mm)

A = area of plates $= 1.1 \times 10^8 m^2$!!!

  - square plate of 6.5 miles per side
  - breakthroughs in capacitor technologies (driven by the computer industries)
    - fabrication of 0.5-5 F capacitors of small size (1-2 cm high) and low cost (< $5)

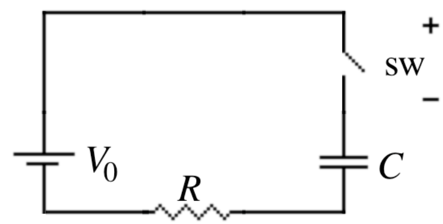
- How small can we make capacitors?
  - A wire near a ground plane has $C \sim 0.1$ pf $= 10^{-13}$ F.



- Some words to the wise on capacitors and their labeling.
  - Typical capacitors are multiples of micro Farads (10^{-6} F) or picoFarads (10^{-12} F).
    - Whenever you see mF it almost always is micro, *not* milli F and *never* mega F.
    - picoFarad (10^{-12} F) is sometimes written as pf and pronounced *puff*.
  - no *single* convention for labeling capacitors
    - Many manufacturers have their own labeling scheme (See Horowitz and Hill lab manual).

- Resistors and Capacitors
  - Examine voltage and current vs. time for a circuit with one R and one C.
    - Assume that at $t < 0$ all voltages are zero, $V_R = V_C = 0$.
    - At $t \geq 0$ the switch is closed and the battery (V_0) is connected.
    - Apply Kirchhoff's voltage rule:

$$V_0 = V_R + V_C$$
$$= IR + \frac{Q}{C}$$
$$= R\frac{dQ}{dt} + \frac{Q}{C}$$



    - Solve the differential equation by differentiating both sides of above equation:

$$\frac{dV_0}{dt} = \frac{1}{C}\frac{dQ}{dt} + R\frac{d^2Q}{dt^2}$$
$$0 = \frac{I}{C} + R\frac{dI}{dt}$$
$$\frac{dI}{dt} = -\frac{I}{RC}$$

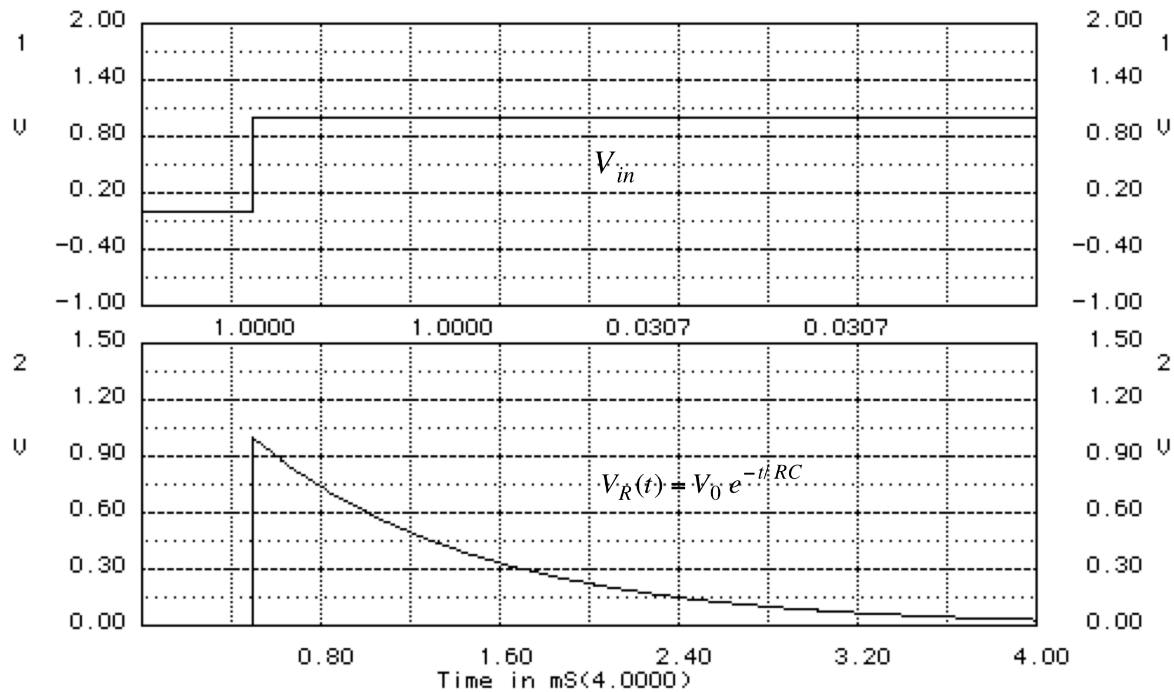
- This is just an exponential decay equation with time constant RC (sec).
- The current as a function of time through the resistor and capacitor is:

$$I(t) = I_0 e^{-t/RC}$$

- What's $V_R(t)$?
  - By Ohm's law:

$$V_R(t) = I_R \cdot R$$
$$= I_0 R\, e^{-t/RC}$$
$$= V_0\, e^{-t/RC}$$

  - At $t = 0$ all the voltage appears across the resistor, $V_R(0) = V_0$.
  - At $t = \infty$, $V_R(\infty) = 0$.



- What's $V_C(t)$?
  - Easiest way to answer is to use the fact that $V_0 = V_R + V_C$ is valid for all t.

$$V_C = V_0 - V_R$$

$$V_C = V_0\left(1 - e^{-t/RC}\right)$$

    - ☞ At $t = 0$ all the voltage appears across the resistor so $V_C(0) = 0$.
    - ☞ At $t = \infty$, $V_C(\infty) = V_0$.

- Suppose we wait until $I = 0$ and then short out the battery.

$$0 = V_R + V_C$$

$$V_R = -V_C$$

$$R\frac{dQ}{dt} = -\frac{Q}{C}$$

$$\frac{dQ}{dt} = -\frac{Q}{RC}$$

  - Solving the exponential equation yields,

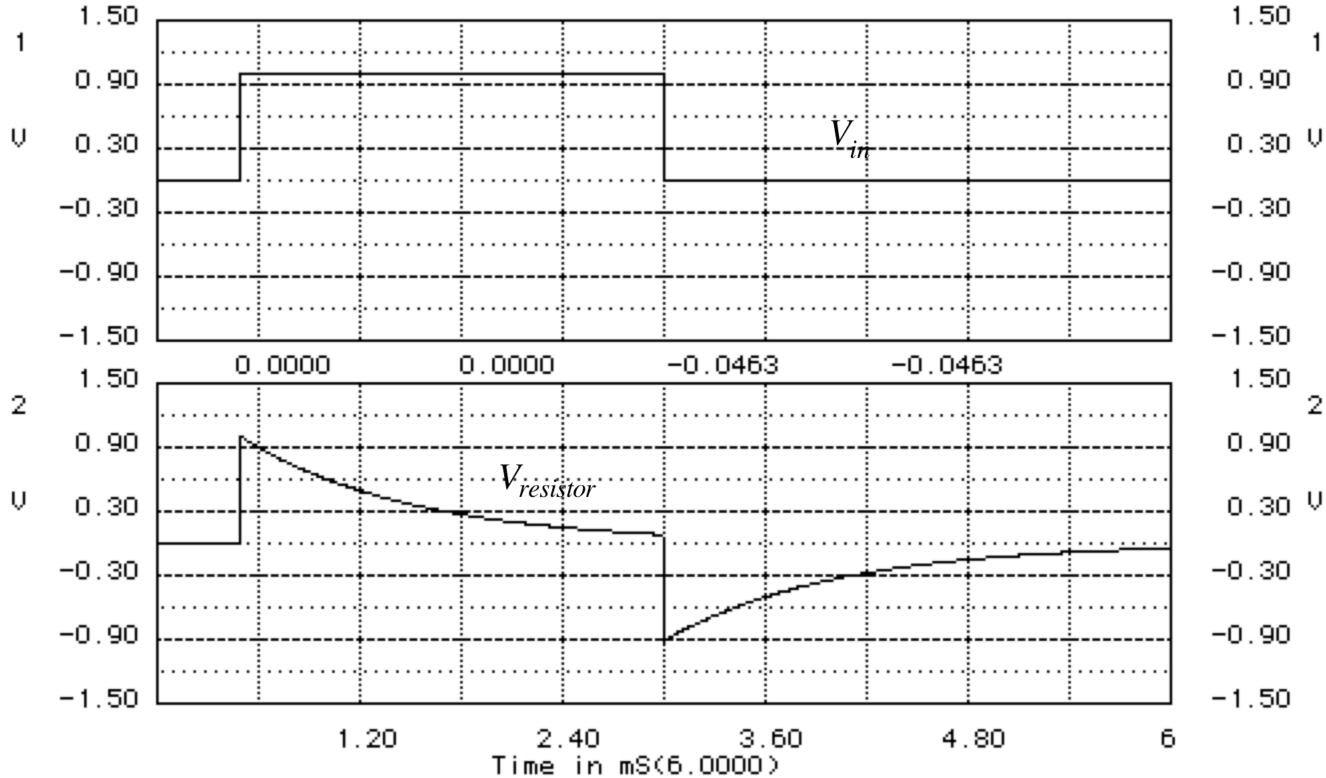
$$Q(t) = Q_0 e^{-t/RC}$$

  - We can find V_C using $V = Q/C$,

$$V_C(t) = V_0 e^{-t/RC}$$

  - Finally we can find the voltage across the resistor using $V_R = -V_C$,

$$V_R(t) = -V_0 e^{-t/RC}$$



- Suppose $V_C(t) = V_0 \sin\omega t$ instead of DC
  - ☞ What happens to V_C and I_C?

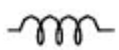
$$Q(t) = CV(t)$$
$$= CV_0 \sin \omega t$$
$$I_C = dQ/dt$$
$$= \omega C V_0 \cos \omega t$$
$$= \omega C V_0 \sin(\omega t + \pi/2)$$

☞ **current in capacitor varies like a sine wave too, but 90^0 out of phase with voltage.**

- We can write an equation that looks like Ohm's law by defining V^*:
  $V^* = V_0 \sin(\omega t + \pi/2)$
  - ☞ the relationship between the voltage and current in C looks like:
    $$V^* = I_C / \omega C$$
    $$= I_C R^*$$
    - ☞ $1/\omega C$ can be identified as a kind of resistance, <u>capacitive reactance</u>:
      $X_C \equiv 1/\omega C$ (Ohms)
      - ❑ $X_C = 0$ for $\omega = \infty$
        - ☞ high frequencies: a capacitor looks like a short circuit
      - ❑ $X_C = \infty$ for $\omega = 0$
        - ☞ low frequencies: a capacitor looks like an open circuit (high resistance).

Inductance:

- Define inductance by: $V = LdI/dt$
  - Unit: Henry
  - Symbol:



  - Inductors are usually made from a coil of wire
    - tend to be bulky and are hard to fabricate in small sizes (μm), not used in chips.
  - Two inductors next to each other (transformer) can step up or down a voltage
    - no change in the frequency of the voltage
    - provide isolation from the rest of the circuit

- How much energy is stored in an inductor?

$$dE = VdQ$$

$$I = \frac{dQ}{dt}$$

$$dE = VIdt$$

$$V = L\frac{dI}{dt}$$

$$dE = LIdI$$

$$E = L\int_0^I IdI$$

☞ $$E = \tfrac{1}{2}LI^2$$

- How much power is dissipated in an inductor?

$$Power = \frac{dE}{dt}$$

$$= \frac{d}{dt}\left(\frac{LI^2}{2}\right)$$

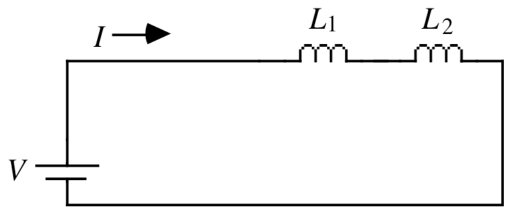
☞ $$P = LI\frac{dI}{dt}$$

❑ *dI*/*dt* must be finite as we can't source (or sink) an infinite amount of power in an inductor!

THIS WOULD BE UNPHYSICAL.

☞ the *current* across an inductor *<u>cannot</u>* change *instantaneously.*

- Two inductors in series:



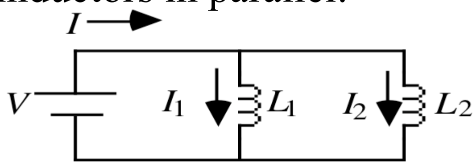
- Apply Kirchhoff's Laws,

$$V = V_1 + V_2$$
$$= L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt}$$
$$\equiv L_{tot} \frac{dI}{dt}$$
$$L_{tot} = L_1 + L_2$$
$$= \sum L_i$$

☞ Inductors in series add like resistors in series.

The *total* inductance is *greater* than the individual inductances.

- Two inductors in parallel:



  - Since the inductors are in parallel,

    $V_1 = V_2 = V$

  - The total current in the circuit is

$$I = I_1 + I_2$$

$$\frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

$$= \frac{V}{L_1} + \frac{V}{L_2}$$

$$\equiv \frac{V}{L_{tot}}$$

$$\frac{1}{L_{tot}} = \frac{1}{L_1} + \frac{1}{L_2}$$

$$L_{tot} = \frac{L_1 L_2}{L_1 + L_2}$$

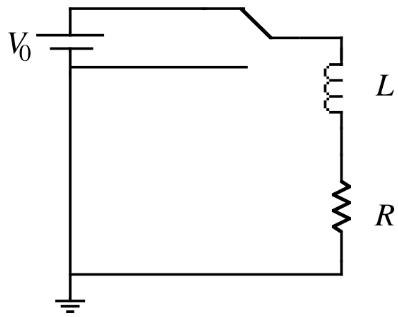
    - ☞ If we have more than 2 inductors in parallel, they combine like:

$$\frac{1}{L_{tot}} = \sum \frac{1}{L_i}$$

    - ❑ Inductors in parallel add like resistors in parallel.

The *total* inductance is *less* than the individual inductances.

- Resistors and Inductors
  - Examine voltage and current versus time for a circuit with one R and one L.
    - Assume that at $t < 0$ all voltages are zero, $V_R = V_L = 0$.
    - At $t \geq 0$ the switch is closed and the battery (V_0) is connected.



    - Like the capacitor case, apply Kirchhoff's voltage rule:

$$V_0 = V_R + V_L$$
$$= IR + L\frac{dI}{dt}$$

    - Solving the differential equation, assuming at $t = 0$, $I = 0$:

$$I(t) = \frac{V_0}{R}\left(1 - e^{-tR/L}\right)$$

      - ☞ This is just an exponential decay equation with time constant L/R (seconds).
  - What's $V_R(t)$?
    - By Ohm's law $V_R = I_R R$ at any time:

$$V_R = I(t)R = V_0\left(1 - e^{-tR/L}\right)$$

    - At $t = 0$, none of the voltage appears across the resistor, $V_R(0) = 0$.
    - At $t = \infty$, $V_R(\infty) = V_0$.

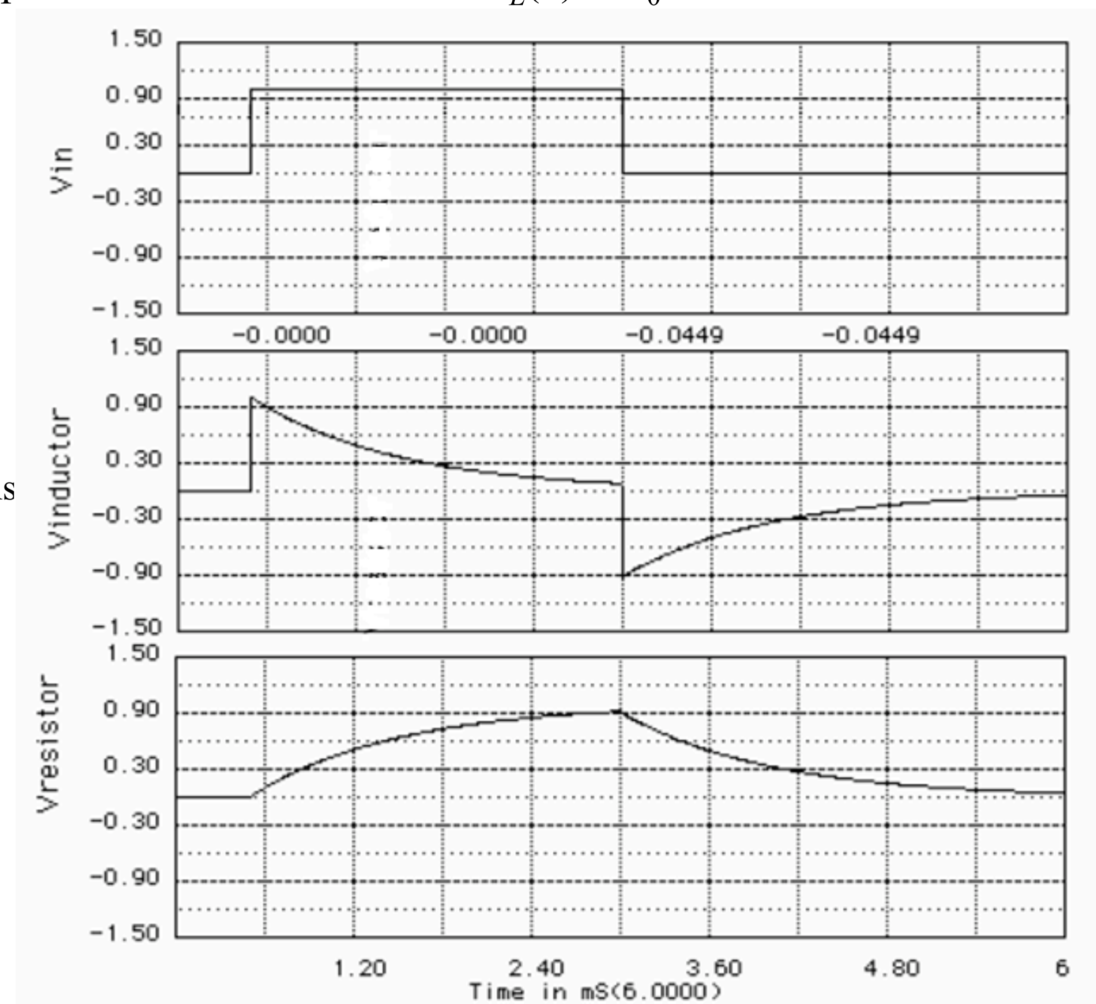
- What's $V_L(t)$?
  - Easiest way to answer is to use the fact that $V_0 = V_R + V_L$ is valid for all t.

$$V_L = V_0 - V_R$$

$$V_L(t) = V_0 e^{-tR/L}$$

  - At $t = 0$, all the voltage appears across the inductor so $V_L(0) = V_0$.
  - At $t = \infty$, $V_L(\infty) = 0$.

Pick $L/R = 1$ ms



- Suppose $V_C(t) = V_0 \sin\omega t$ instead of DC, what happens to V_L and I_L?

$$V = L\frac{dI_L}{dt}$$

$$I_L = \frac{1}{L}\int_0^t V dt$$

$$= -\frac{V_0}{\omega L}\cos\omega t$$

$$I_L(t) = \frac{V_0}{\omega L}\sin(\omega t - \pi/2)$$

☞ The current in an inductor varies like a sine wave too, but 90^0 out of phase with the voltage.

- We can write an equation that looks like Ohm's law by defining V^*:

  $V^* = V_0 \sin(\omega t - \pi/2)$

  ☞ $V^* = I_L \omega L = I_L R^*$

  - ωL can be identified as a kind of resistance, <u>inductive reactance</u>:

    $X_L \equiv \omega L$ (Ohms)

    - $X_L = 0$ if $\omega = 0$

      ☞ low frequencies: an inductor looks like a short circuit (low resistance).

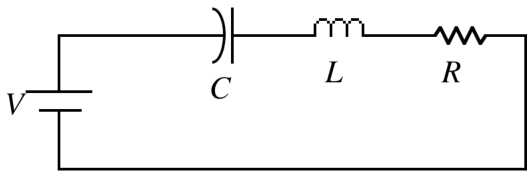
    - $X_L = \infty$ if $\omega = \infty$

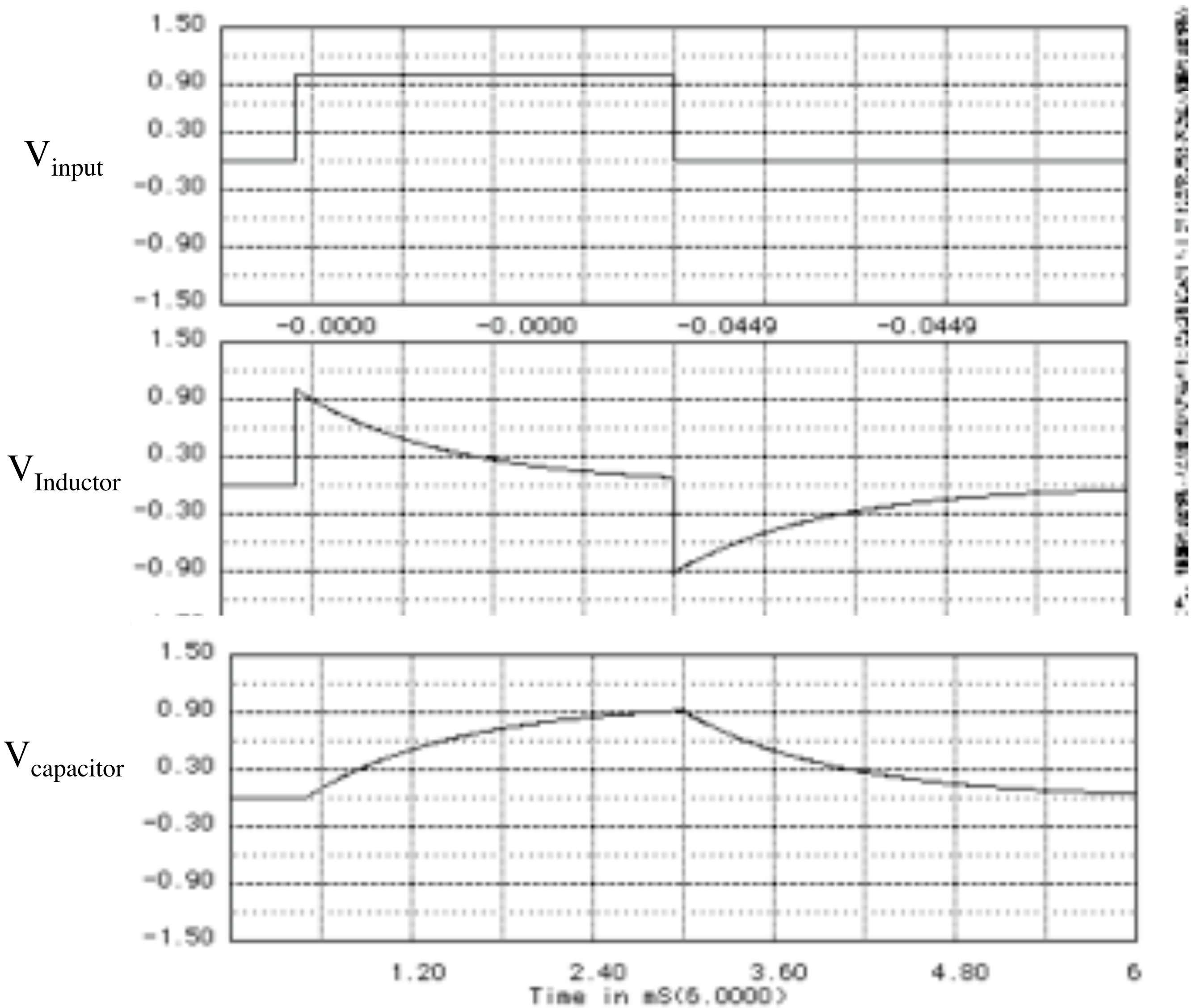
      ☞ high frequencies: an inductor looks like an open circuit.

- Some things to remember about *R*, *L*, and *C*'s.
  - For DC circuits, after many time constants (*L*/*R* or *RC*):
    - Inductor acts like a wire (0 Ω).
    - Capacitor acts like an open circuit (∞ Ω).

  - For circuits where the voltage changes very rapidly or transient behavior:
    - Capacitor acts like a wire (0 Ω).
    - Inductor acts like an open circuit (∞ Ω).

  - Example, RLC circuit with DC supply:
    - At $t = 0$, voltages on *R*, *C* are zero and $V_L = V_0$.
    - At $t = \infty$, voltages on *R*, *L* are zero and $V_C = V_0$.




V_input
V_Inductor
V_capacitor
1.50
0.90
0.30
-0.30
-0.90
-1.50
-0.0000
-0.0000
-0.0449
-0.0449
1.20
2.40
3.60
4.80
6
Time in mS(6.0000)