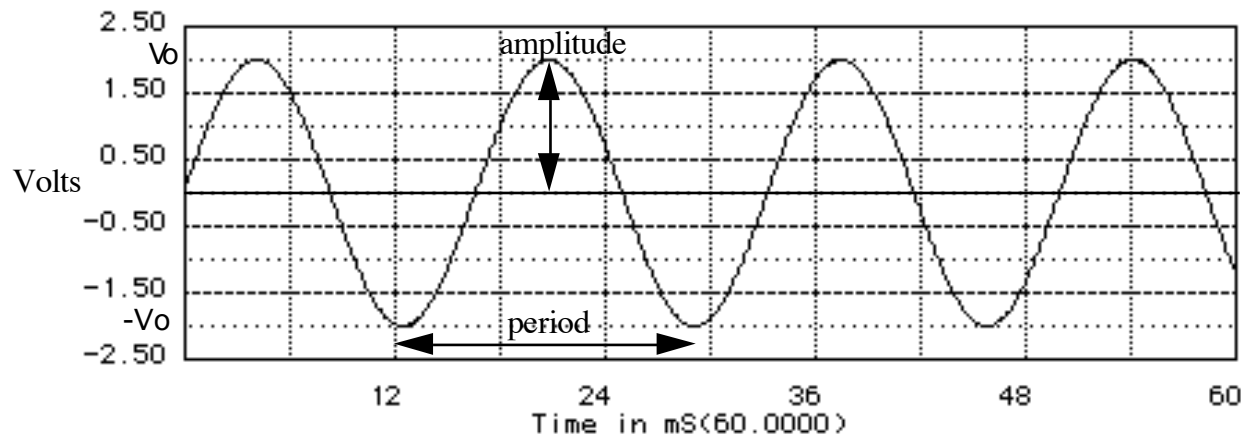


Lecture 3: R-L-C AC Circuits

AC (Alternative Current):

- Most of the time, we are interested in the voltage at a point in the circuit
 - ✎ will concentrate on voltages here rather than currents.
- ◆ We encounter AC circuits whenever a periodic voltage is applied to a circuit.
- ◆ The most common periodic voltage is in the form of a sine (or cosine) wave:
$$V(t) = V_0 \cos \omega t \quad \text{or} \quad V(t) = V_0 \sin \omega t$$



- V_0 is the *amplitude*:
 - $V_0 = \text{Peak Voltage } (V_p)$
 - $V_0 = 1/2 \text{ Peak-to-Peak Voltage } (V_{pp})$
 - V_{pp} : easiest to read off scope
 - $V_0 = \sqrt{2} V_{RMS} = 1.41 V_{RMS}$
 - V_{RMS} : what multimeters usually read

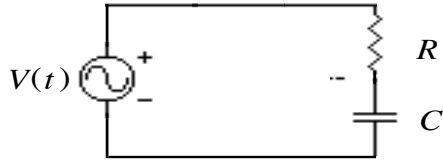
- ω is the *angular frequency*:
 - $\omega = 2\pi f$, with f = frequency of the waveform.
 - frequency (f) and period (T) are related by:

$$T \text{ (sec)} = 1/f \text{ (sec}^{-1}\text{)}$$
- *Household line voltage* is usually 110-120 V_{RMS} (156-170 V_P), $f = 60 \text{ Hz}$.
- ◆ It is extremely important to be able to analyze circuits (systems) with sine or cosine inputs
 - Almost any waveform can be constructed from a sum of sines and cosines.
 - This is the “heart” of *Fourier analysis* (Simpson, Chapter 3).
 - The response of a circuit to a complicated waveform (e.g. a square wave) can be understood by analyzing individual sine or cosine components that make up the complicated waveform.
 - Usually only the first few components are important in determining the circuit’s response to the input waveform.

R-C Circuits and AC waveforms

- There are many different techniques for solving AC circuits
 - ◆ All are based on Kirchhoff's laws.
 - ◆ When solving for voltage and/or current in an AC circuit we are really solving a differential eq.
 - ◆ Different circuit techniques are really just different ways of solving the same differential eq:
 - brute force solution to differential equation
 - complex numbers (algebra)
 - Laplace transforms (integrals)

- We will solve the following RC circuit using the brute force method and complex numbers method.
- ◆ Let the input (driving) voltage be $V(t) = V_0 \cos \omega t$ and we want to find $V_R(t)$ and $V_C(t)$.



- ◆ *Brute Force Method:* Start with Kirchhoff's loop law:

$$V(t) = V_R(t) + V_C(t)$$

$$V_0 \cos \omega t = IR + Q/C$$

$$= R dQ(t)/dt + Q(t)/C$$

- We have to solve an inhomogeneous D.E.
- The usual way to solve such a D.E. is to assume the solution has the same form as the input:

$$Q(t) = \alpha \sin \omega t + \beta \cos \omega t$$

- Plug our trial solution $Q(t)$ back into the D.E.:

$$V_0 \cos \omega t = \alpha R \omega \cos \omega t - \beta R \omega \sin \omega t + (\alpha/C) \sin \omega t + (\beta/C) \cos \omega t$$

$$= (\alpha R \omega + \beta/C) \cos \omega t + (\alpha/C - \beta R \omega) \sin \omega t$$

$$V_0 = \alpha R \omega + \beta/C$$

$$\alpha/C = \beta R \omega$$

$$\alpha = \frac{RC^2 \omega V_0}{1 + (RC\omega)^2}$$

$$\beta = \frac{CV_0}{1 + (RC\omega)^2}$$

- We can now write the solution for $V_C(t)$:

$$\begin{aligned} V_C(t) &= Q/C \\ &= (\alpha \sin \omega t + \beta \cos \omega t) / C \\ &= \frac{RC\omega V_0}{1 + (RC\omega)^2} \sin \omega t + \frac{V_0}{1 + (RC\omega)^2} \cos \omega t \end{aligned}$$

- We would like to rewrite the above solution in such a way that only a cosine term appears.
 - In this form we can compare it to the input voltage.

$$V_C(t) = \frac{V_0}{\sqrt{1 + (RC\omega)^2}} \left[\frac{RC\omega}{\sqrt{1 + (RC\omega)^2}} \sin \omega t + \frac{1}{\sqrt{1 + (RC\omega)^2}} \cos \omega t \right]$$

- We get the above equation in terms of cosine only using the following basic trig:

$$\cos(\theta_1 - \theta_2) = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2$$
- We can now define an angle such that:

$$\cos \phi = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

$$\sin \phi = \frac{RC\omega}{\sqrt{1 + (RC\omega)^2}}$$

$$\tan \phi = RC\omega$$

$$V_C(t) = \frac{V_0}{\sqrt{1 + (RC\omega)^2}} \cos(\omega t - \phi)$$

- ☞ $V_C(t)$ and $V_o(t)$ are out of phase.

- Using the above expression for $V_C(t)$, we obtain:

$$\begin{aligned}
 V_R(t) &= IR \\
 &= R \frac{dQ}{dt} \\
 &= RC \frac{dV_C}{dt} \\
 &= \frac{-RC\omega V_o}{\sqrt{1+(RC\omega)^2}} \sin(\omega t - \phi)
 \end{aligned}$$

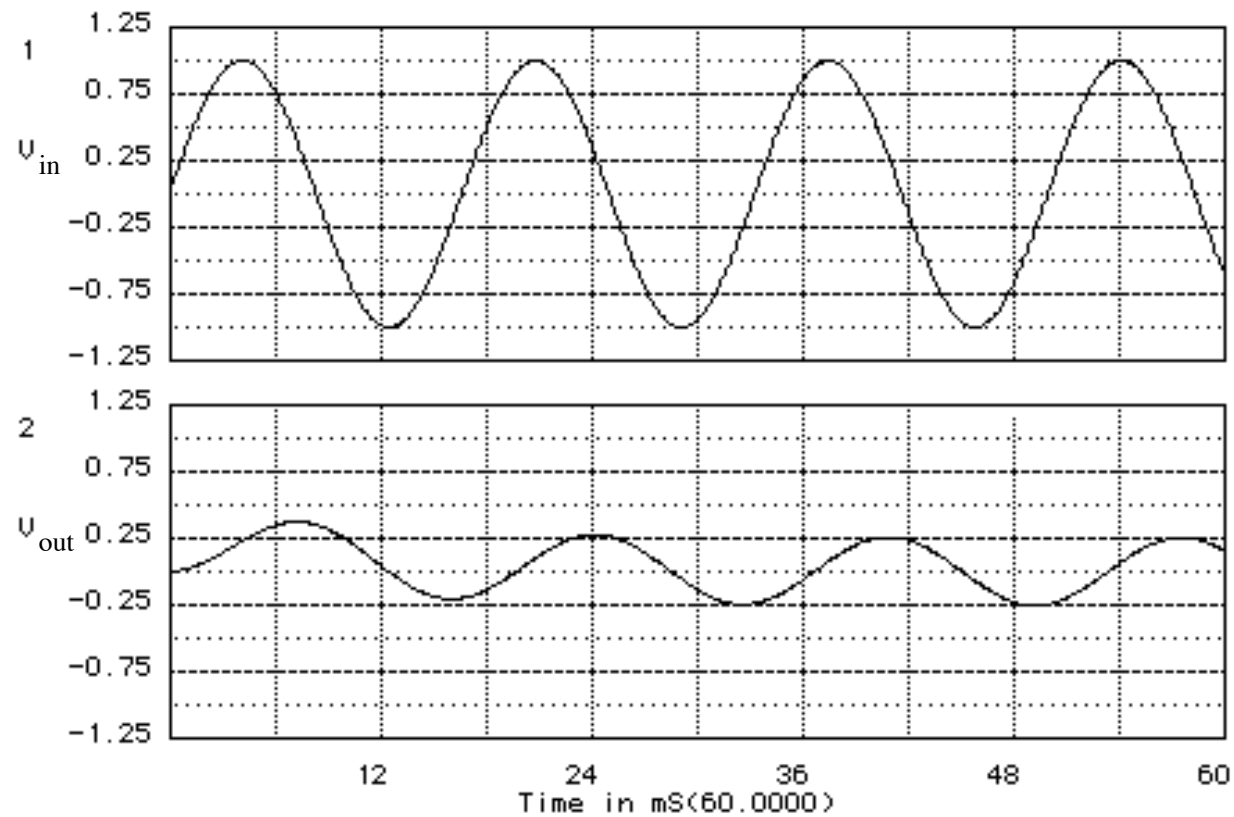
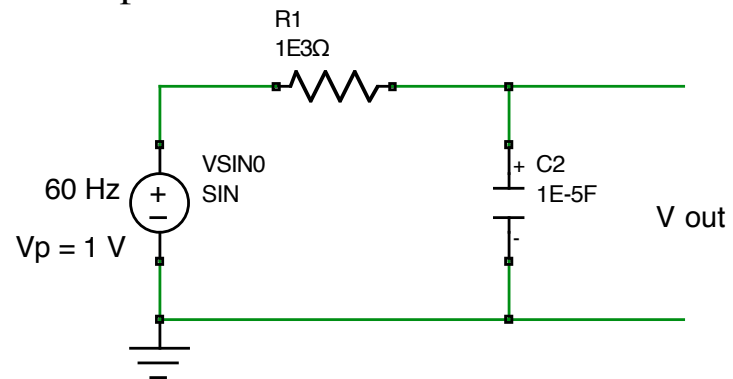
- We would like to have cosines instead of sines by using:

$$-\sin \theta = \cos(\theta + \frac{\pi}{2})$$

$$\Rightarrow V_R(t) = \frac{RC\omega V_o}{\sqrt{1+(RC\omega)^2}} \cos(\omega t - \phi + \frac{\pi}{2})$$

- $V_C(t)$, $V_R(t)$, and $I(t)$ are all out of phase with the applied voltage.
- $I(t)$ and $V_R(t)$ are in phase with each other.
- $V_C(t)$ and $V_R(t)$ are out of phase by 90° .
- The amplitude of $V_C(t)$ and $V_R(t)$ depend on ω .

■ Example: RC Circuit



- ◆ Solving circuits with complex numbers:
 - PROS:
 - don't explicitly solve differential equations (lots of algebra).
 - can find magnitude and phase of voltage separately.
 - CONS:
 - have to use complex numbers!
 - No “physics” in complex numbers.
 - What's a complex number? (see Simpson, Appendix E, P835)
 - Start with $j \equiv \sqrt{-1}$ (solution to $x^2 + 1 = 0$).
 - A complex number can be written in two forms:
 - ✱ $X = A + jB$
 - A and B are *real* numbers
 - ✱ $X = R e^{j\phi}$
 - $R = (A^2 + B^2)^{1/2}$ and $\tan\phi = B/A$ (remember $e^{j\phi} = \cos\phi + j \sin\phi$)
 - Define the complex conjugate of X as:

$$X^* = A - jB \quad \text{or} \quad X^* = R e^{-j\phi}$$
 - The magnitude of X can be found from:

$$|X| = (XX^*)^{1/2} = (X^* X)^{1/2} = (A^2 + B^2)^{1/2}$$
 - Suppose we have 2 complex numbers, X and Y with phases α and β respectively,

$$Z = \frac{X}{Y} = \frac{|X|e^{j\alpha}}{|Y|e^{j\beta}} = \frac{|X|}{|Y|} e^{j(\alpha-\beta)}$$
 - magnitude of Z : $|X|/|Y|$
 - phase of Z : $\alpha - \beta$
 - So why is this useful?

- ◆ Consider the case of the capacitor and AC voltage:

$$V(t) = V_0 \cos \omega t$$

$$= \text{Re al} \left(V_0 e^{j\omega t} \right)$$

$$Q = CV$$

$$I(t) = C \frac{dV}{dt}$$

$$= -C\omega V_0 \sin \omega t$$

$$= \text{Re al} \left(j\omega C V_0 e^{j\omega t} \right)$$

$$= \text{Re al} \left(\frac{V_0 e^{j\omega t}}{1/j\omega C} \right)$$

$$= \text{Re al} \left(\frac{V}{X_C} \right)$$

- V and X_C are complex numbers

- 👉 We now have Ohm's law for capacitors using the capacitive reactance X_C :

$$X_C = \frac{1}{j\omega C}$$

- ◆ We can make a similar case for the inductor:

$$\begin{aligned}
 V &= L \frac{dI}{dt} \\
 I(t) &= \frac{1}{L} \int V dt \\
 &= \frac{1}{L} \int V_0 \cos \omega t dt \\
 &= \frac{V_0 \sin \omega t}{L\omega} \\
 &= \text{Re al} \left(\frac{V_0 e^{j\omega t}}{j\omega L} \right) \\
 &= \text{Re al} \left(\frac{V}{X_L} \right)
 \end{aligned}$$

- V and X_L are complex numbers

- 👉 We now have Ohm's law for inductors using the inductive reactance X_L :

$$X_L = j\omega L$$

- ◆ X_C and X_L act like frequency dependent resistors.

- They also have a *phase* associated with them due to their complex nature.
- $X_L \Rightarrow 0$ as $\omega \Rightarrow 0$ (short circuit, DC)
- $X_L \Rightarrow \infty$ as $\omega \Rightarrow \infty$ (open circuit)
- $X_C \Rightarrow 0$ as $\omega \Rightarrow \infty$ (short circuit)
- $X_C \Rightarrow \infty$ as $\omega \Rightarrow 0$ (open circuit, DC)

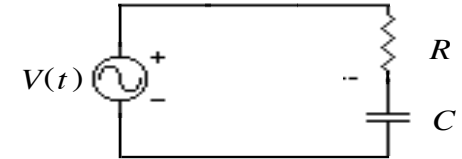
◆ Back to the RC circuit.

- Allow voltages, currents, and charge to be complex:

$$V_{in} = V_0 \cos \omega t$$

$$= \text{Re al} \left(V_0 e^{j\omega t} \right)$$

$$= \text{Re al} (V_R + V_C)$$



- We can write an expression for the charge (Q) taking into account the phase difference (ϕ) between applied voltage and the voltage across the capacitor (V_C).

$$Q(t) = CV_C(t)$$

$$= Ae^{j(\omega t - \phi)}$$

- Q and V_C are complex

- A and C are real

- We can find the complex current by differentiating the above:

$$I(t) = dQ(t)/dt$$

$$= j\omega Ae^{j(\omega t - \phi)}$$

$$= j\omega Q(t)$$

$$= j\omega CV_C(t)$$

$$V_{in} = V_C + V_R$$

$$= V_C + IR$$

$$= V_C + j\omega CV_C R$$

$$\begin{aligned}
 V_C &= \frac{V_{in}}{1 + j\omega RC} \\
 &= V_{in} \frac{1}{R + \frac{1}{j\omega C}} \\
 &= V_{in} \frac{X_C}{R + X_C}
 \end{aligned}$$

□ *looks like a voltage divider equation!!!!*

■ We can easily find the magnitude of V_C :

$$\begin{aligned}
 |V_C| &= |V_{in}| \frac{|X_C|}{|R + X_C|} \\
 &= \frac{V_0 \frac{1}{\omega C}}{\sqrt{R^2 + (1/\omega C)^2}} \\
 &= \frac{V_0}{\sqrt{1 + (RC\omega)^2}}
 \end{aligned}$$

□ same as the result on page 4.

- Is this solution the same as what we had when we solved by brute force page 3?

$$\begin{aligned}
 V_C &= \text{Re al} \left(\frac{V_{in}}{1 + j\omega RC} \right) \\
 &= \text{Re al} \left(\frac{V_0 e^{j\omega t}}{1 + j\omega RC} \right) \\
 &= \text{Re al} \left(\frac{V_0 e^{j\omega t}}{\sqrt{1 + (\omega RC)^2} e^{j\phi}} \right) \\
 &= \text{Re al} \left(\frac{V_0 e^{j(\omega t - \phi)}}{\sqrt{1 + (\omega RC)^2}} \right) \\
 &= \frac{V_0 \cos(\omega t - \phi)}{\sqrt{1 + (\omega RC)^2}}
 \end{aligned}$$

ϕ is given by $\tan \phi = \omega RC$

 YES the solutions are identical.

- We can now solve for the voltage across the resistor.
 - Start with the voltage divider equation in complex form:

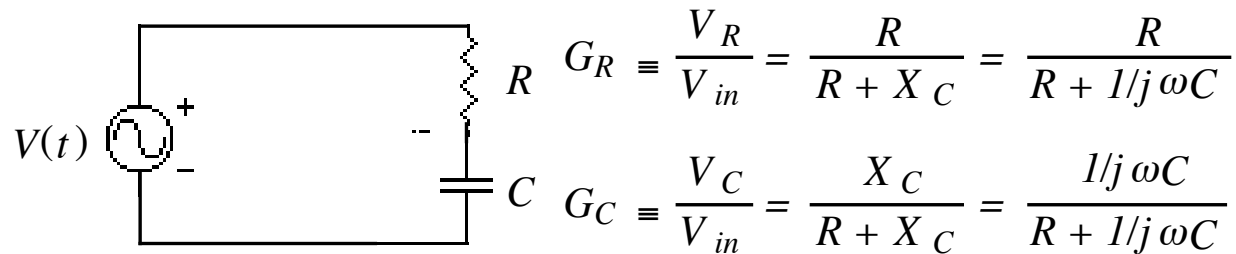
$$\begin{aligned}
 V_R &= \frac{V_{in} R}{R + X_C} \\
 |V_R| &= \frac{|V_{in}| R}{|R + X_C|} \\
 &= \frac{V_0 R}{\sqrt{R^2 + (1/\omega C)^2}} \\
 &= \frac{V_0 \omega RC}{\sqrt{1 + (\omega RC)^2}}
 \end{aligned}$$

- 👉 This amplitude is the same as the brute force differential equation case!
- In adding complex voltages, we must take into account the phase difference between them.
 - the sum of the voltages at a given time satisfy:
 - ✓ $V_0^2 = |V_R|^2 + |V_C|^2$
 - ✗ $V_0 = |V_R| + |V_C|$

R-C Filters

- ◆ Allow us to select (reject) wanted (unwanted) signals on the basis of their frequency structure.
- ◆ Allow us to change the phase of the voltage or current in a circuit.
- ◆ Define the gain (G) or transfer (H) function of a circuit:
 - $G(j\omega) = H(j\omega) = V_{out}/V_{in}$ ($j\omega$ is often denoted by s).
 - G is independent of time, but can depend on ω, R, L, C .

- ◆ For an RC circuit we can define G_R and G_C :



- ◆ We can categorize the G 's as follows:

	G_R	G_C
High Frequencies	≈ 1 , no phase shift high pass filter	$\approx 1/j\omega CR \approx 0$, phase shift
Low Frequencies	$\approx j\omega CR \approx 0$, phase shift	≈ 1 , no phase shift low pass filter

- Decibels and Bode Plots:

- ◆ Decibel (dB) describes voltage or power gain:

$$\text{dB} = 20 \log(V_{out} / V_{in})$$

$$= 10 \log(P_{out} / P_{in})$$

- ◆ *Bode Plot* is a log-log plot with dB on the y axis and $\log(\omega)$ or $\log(f)$ on the x axis.

- ◆ 3 dB point or 3 dB frequency:
 - also called break frequency, corner frequency, 1/2 power point
 - At the 3 dB point:

$$\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{2}} \quad \text{since } 3 = 20 \log(V_{out} / V_{in})$$

$$\frac{P_{out}}{P_{in}} = \frac{1}{2} \quad \text{since } 3 = 10 \log(P_{out} / P_{in})$$

✏ $\omega RC = 1$ for high or low pass filter

