Lecture 4: R-L-C Circuits and Resonant Circuits

RLC series circuit:

- What's V_R ?
 - Simplest way to solve for V is to use voltage divider equation in complex notation:

$$V_{R} = \frac{V_{in}R}{R + X_{C} + X_{L}}$$

$$= \frac{V_{in}R}{R + \frac{1}{j\omega C} + j\omega L}$$

$$V_{in} = V_{0}\cos\omega t \bigcirc_{-}^{+} \bigcup_{-}^{+} \bigcup_{-}^{$$

• Using complex notation for the apply voltage $V_{in} = V_0 \cos \omega t = \text{Real}(V_0 e^{j\omega t})$:

$$V_R = \frac{V_0 e^{j\omega r} R}{R + j \left(\omega L - \frac{1}{\omega C}\right)}$$

- We are interested in the both the magnitude of V_R and its phase with respect to V_{in} .
- First the magnitude:

$$|V_R| = \frac{\left|V_0 e^{j\omega t}\right| |R|}{\left|R + j\left(\omega L - \frac{1}{\omega C}\right)\right|}$$
$$= \frac{V_0 R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

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1

- The *phase* of V_R with respect to V_{in} can be found by writing V_R in purely polar notation.
 - For the denominator we have:

the denominator we have:

$$R + j\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \exp\left\{j \tan^{-1}\left[\frac{\omega L - \frac{1}{\omega C}}{R}\right]\right\}$$

• Define the phase angle ϕ :

$$\tan \phi = \frac{\text{Imaginary } X}{\text{Real } X}$$
$$= \frac{\omega L - \frac{1}{\omega C}}{R}$$

• We can now write for V_R in complex form:

$$V_{R} = \frac{V_{o}R e^{j\omega t}}{e^{j\phi} \sqrt{R^{2} + \left(\omega L - \frac{1}{\omega C}\right)^{2}}}$$
$$= |V_{R}|e^{j(\omega t - \phi)}$$

Depending on *L*, *C*, and ω , the phase angle can be positive or negative! In this example, if $\omega L > 1/\omega C$, then $V_R(t)$ lags $V_{in}(t)$.

Finally, we can write down the solution for V by taking the real part of the above equation:

$$V_R = \operatorname{Real} \frac{V_0 R \ e^{j(\omega t - \phi)}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_0 R \cos(\omega t - \phi)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

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- $V_R \ll V_{in}$ at 100 Hz.
- V_R and V_{in} are not in phase at this frequency.
- The little wiggles on V_R are real!
 - Transient solution (homogeneous solution) to the differential eq. describing the circuit.
 - After a few cycles this contribution to V_R die out.



- In general $V_C(t)$, $V_R(t)$, and $V_L(t)$ are all out of phase with the applied voltage.
- I(t) and $V_R(t)$ are in phase in a <u>series</u> RLC circuit.
- The amplitude of V_C , V_R , and V_L depend on ω .
- The table below summarizes the 3 cases with the following definitions:

$$Z = \left[R^{2} + (\omega L - 1/\omega C)^{2} \right]^{1/2}$$

 $\tan\phi=(\omega L-1/\omega C)/R$

Gain	Magnitude	Phase
V_R/V_{in}	R/Z	- <i>φ</i>
V_L/V_{in}	$\omega L/Z$	π/2 - φ
V_C/V_{in}	$1/\omega CZ$	-π/2 - φ

- RLC circuits are resonant circuits
 - energy in the system "resonates" between the inductor and capacitor
 - "ideal" capacitors and inductors do not dissipate energy
 - resistors dissipate energy i.e. resistors do not store energy

- Resonant Frequency:
 - At the resonant frequency the imaginary part of the impedance vanishes.
 - For the series RLC circuit the impedance (Z) is: $Z = R + X_L + X_C = R + j(\omega L - 1/\omega C)$

$$|Z| = \left[R^2 + (\omega L - 1/\omega C)^2\right]^{1/2}$$

• At resonance (series, parallel etc): $\omega L = 1/\omega C$

$$\omega_R = \frac{1}{\sqrt{LC}}$$

• At the <u>resonant frequency</u> the following are true for a series RLC circuit:

$$|V_R| \text{ is maximum (ideally} = V_{in})$$

$$\bullet \phi = 0$$

$$\frac{|V_C|}{|V_{in}|} = \frac{|V_L|}{|V_{in}|} = \frac{\sqrt{L}}{R\sqrt{C}} \quad (V_C \text{ or } V_L \text{ can be } > V_{in}!)$$

- The circuit acts like a narrow band pass filter.
- There is an exact analogy between an RLC circuit and a harmonic oscillator (mass attached to spring):
 - $m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = 0$ damped harmonic oscillator
 - $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ undriven RLC circuit
 - $x \Leftrightarrow q$ (electric charge), $L \Leftrightarrow m$, $k \Leftrightarrow 1/C$
 - B (coefficient of damping) $\Leftrightarrow R$ K.K. Gan L4: RLC and Resonance Circuits

- Q (quality factor) of a circuit: determines how well the RLC circuit stores energy
 - $Q = 2\pi$ (max energy stored)/(energy lost) per cycle
 - Q is related to sharpness of the resonance peak:



- The maximum energy stored in the inductor is $LI^2/2$ with $I = I_{MAX}$.
 - no energy is stored in the capacitor at this instant because I and V_C are 90⁰ out of phase.
 - The energy lost in one cycle:

power × (time for cycle) =
$$I_{RMS}^2 R \times \frac{2\pi}{\omega_R} = \frac{1}{2} I_{max}^2 R \times \frac{2\pi}{\omega_R}$$

$$Q = \frac{2\pi \left(\frac{LI_{Max}^2}{2}\right)}{\frac{2\pi}{\omega_R} \left(\frac{RI_{Max}^2}{2}\right)} = \frac{\omega_R L}{R}$$

• There is another popular, equivalent expression for Q

$$Q = \frac{\omega_R}{\omega_U - \omega_L}$$

- $\omega_U(\omega_L)$ is the upper (lower) 3 dB frequency of the resonance curve.
 - \circ Q is related to sharpness of the resonance peak.
- Will skip the derivation here as it involves a bit of algebra.
 - two crucial points of the derivation:

$$\frac{V_R}{V_{in}} = \frac{1}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega}\right)^2}}$$
$$Q\left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega}\right) = \pm 1$$
 at the upper and lower 3 dB points

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- Q can be measured from the shape of the resonance curve
 - one does not need to know R, L, or C to find Q!





- Example: Audio filter (band pass filter)
 - Audio filter is matched to the frequency range of the ear (20-20,000 Hz).



• Let's design an audio filter using low and high pass RC circuits.



- Ideally, the frequency response is flat over 20-20,000 Hz, and rolls off sharply at frequencies below 20 Hz and above 20,000 Hz.
 - Set 3 dB points as follows:
 - lower 3 dB point : 20 Hz = $1/2\pi R_1 C_1$
 - upper 3 dB point: $2 \times 10^4 \text{ Hz} = 1/2 \pi R_2 C_2$
 - If we put these two filters together we don't want the 2nd stage to affect the 1st stage.
 - can accomplish this by making the impedance of the $2^{nd}(Z_2)$ stage much larger than R_1 .
 - Remember R_1 is in parallel with Z_2 .

$$Z_1 = R_1 + 1/j\omega C_1$$

$$Z_2 = R_2 + 1/j\omega C_2$$

In order to insure that the second stage does not "load" down the first stage we need:

 $R_2 >> R_1$ since at high frequencies $Z_2 \Rightarrow R_2$

• We can now pick and calculate values for the *R*'s and *C*'s in the problem.

• Let
$$C_1 = 1 \ \mu F \Rightarrow R_1 = 1/(20 \text{Hz} \ 2\pi C_1) = 8 \ \text{k}\Omega$$

• Let
$$R_2 > 100R_1 \Rightarrow R_2 = 1 \text{ M}\Omega$$
, and $C_2 = 1/(2 \times 10^4 \text{ Hz } 2\pi R_2) = 8 \text{ pf}$

^{III} $R_1 = 8 \text{ k}\Omega, C_1 = 1 \text{ μF}$ $R_2 = 1 \text{ M}\Omega, C_2 = 8 \text{ pf}$

- Exact derivation for above filter:
 - In the above circuit we treated the two RC filters as independent.
 - Why did this work?
 - We want to calculate the gain $(|V_{out}/V_{in}|)$ of the following circuit:



• Working from right to left, we have:

$$V_{out} = V_a X_2 / (X_2 + R_2)$$
$$V_a = V_{in} Z_1 / Z_T$$

• Z_T is the total impedance of the circuit as seen from the input.

•
$$Z_1$$
 is the parallel impedance of R_1 and R_2 , in series with C_2 .

$$Z_{1} = \frac{R_{1}(R_{2} + X_{2})}{R_{1} + R_{2} + X_{2}}$$

$$Z_{T} = X_{1} + Z_{1}$$

$$V_{a} = \frac{V_{in}R_{1}(R_{2} + X_{2})}{X_{1}(R_{1} + R_{2} + X_{2}) + R_{1}(R_{2} + X_{2})}$$
Finally we can solve for the gain $G = |V_{out}/V_{in}|$:
$$\frac{V_{out}}{V_{in}} = \frac{R_{1}X_{2}}{X_{1}(R_{1} + R_{2} + X_{2}) + R_{1}(R_{2} + X_{2})}$$

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• We can relate this to our previous result by rewriting the above as:

$$\frac{V_{out}}{V_{in}} = \frac{R_1 \frac{X_2}{R_2 + X_2}}{X_1 \left(\frac{R_1}{R_2 + X_2} + 1\right) + R_1}$$

If we now remember the approximation $(R_1 \le R_2 + X_2)$ made on the previous page to insure that the second stage did not load down the first then we get the following:

$$\frac{V_{out}}{V_{in}} = \frac{R_1}{R_1 + X_1} \frac{X_2}{R_2 + X_2}$$

- The gain of the circuit looks like the *product* of two filters, one high pass and one low pass!
- If we calculate the gain of this circuit in dB, the total gain is the sum of the gain of each piece:

Gain in dB = 20
$$\log\left(\frac{V_{out}}{V_{in}}\right)$$

= 20 $\log\left(\frac{R_1}{R_1 + X_1}\right) + 20 \log\left(\frac{X_2}{R_2 + X_2}\right)$

The gain of successive filters measured in dB's add!

• Another Example: Calculate |I| and the phase angle between V_{in} and I for the following circuit:



First calculate |I|.

- The total current out of the input source (I) is related to V_{in} and the total impedance (Z_T) of the circuit by Ohm's law: $I = V_{in} / Z_T$
- The total impedance of the circuit is given by the parallel impedance of the two branches:

13

$$1/Z_{T} = 1/Z_{1} + 1/Z_{2}$$

$$Z_{1} = R_{1} + X_{1}$$

$$Z_{2} = R_{2} + X_{2}$$
• Putting in numerical values for the *R*'s and *X*'s we have:

$$Z_{1} = 20 + j37.7 \ \Omega$$

$$Z_{2} = 10 - j53.1 \ \Omega$$

$$Z_{T} = 67.4 + j11.8 \ \Omega$$
• We can now find the magnitude of the current:

$$|I| = |V_{in}|/|Z_{T}|$$

$$= 230 \ V/68.4 \ \Omega$$

$$= 3.36 \ A$$
This is RMS value since $|V_{in}|$ is given as RMS
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- Calculate the phase angle between V_{in} and I:
 - It's easiest to solve this by writing V and Z in polar form:

$$V_{in} = (230)e^{j\omega t}$$
$$Z_T = (68.4)e^{j\phi}$$
$$\tan \phi = \operatorname{Im} Z_T / \operatorname{Re} Z_T$$
$$= 11.8/67.4$$

$$\phi = 9.9^{\circ}$$

• Finally we can write for the current:

$$I = (230/68.4)e^{j(\omega t - \phi)}$$

• Taking the real part of *I*:

$$I = 3.36\cos(\omega t - 9.9^{\circ})$$
 A

The current lags the voltage by 9.9° .