

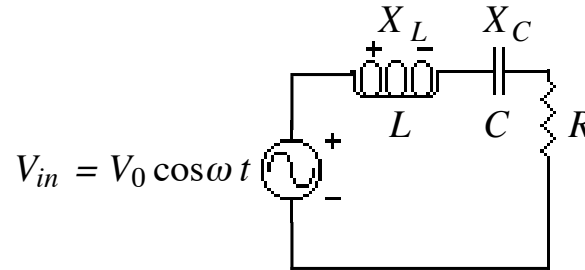
Lecture 4: R-L-C Circuits and Resonant Circuits

RLC series circuit:

- What's V_R ?
 - Simplest way to solve for V is to use voltage divider equation in complex notation:

$$V_R = \frac{V_{in} R}{R + X_C + X_L}$$

$$= \frac{V_{in} R}{R + \frac{1}{j\omega C} + j\omega L}$$



- Using complex notation for the apply voltage $V_{in} = V_0 \cos \omega t = \text{Real}(V_0 e^{j\omega t})$:

$$V_R = \frac{V_0 e^{j\omega t} R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

- We are interested in the both the magnitude of V_R and its phase with respect to V_{in} .
- First the magnitude:

$$|V_R| = \frac{|V_0 e^{j\omega t}| |R|}{\left| R + j\left(\omega L - \frac{1}{\omega C}\right) \right|}$$

$$= \frac{V_0 R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

- The *phase* of V_R with respect to V_{in} can be found by writing V_R in purely polar notation.
- For the denominator we have:

$$R + j\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \exp\left\{j \tan^{-1}\left[\frac{\omega L - \frac{1}{\omega C}}{R}\right]\right\}$$

- Define the phase angle ϕ :

$$\begin{aligned} \tan \phi &= \frac{\text{Imaginary } X}{\text{Real } X} \\ &= \frac{\omega L - \frac{1}{\omega C}}{R} \end{aligned}$$

- We can now write for V_R in complex form:

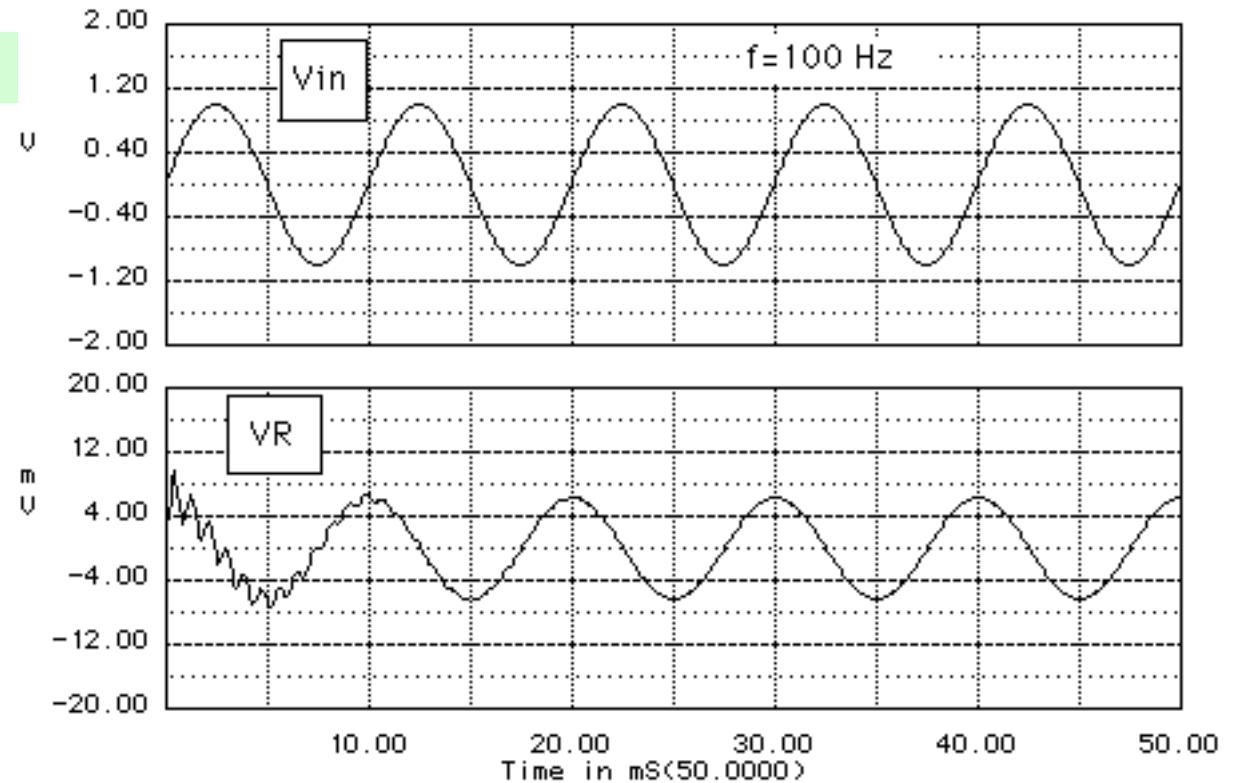
$$\begin{aligned} V_R &= \frac{V_o R e^{j\omega t}}{e^{j\phi} \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \\ &= |V_R| e^{j(\omega t - \phi)} \end{aligned}$$

Depending on L , C , and ω , the phase angle can be positive or negative! In this example, if $\omega L > 1/\omega C$, then $V_R(t)$ **lags** $V_{in}(t)$.

- Finally, we can write down the solution for V by taking the real part of the above equation:

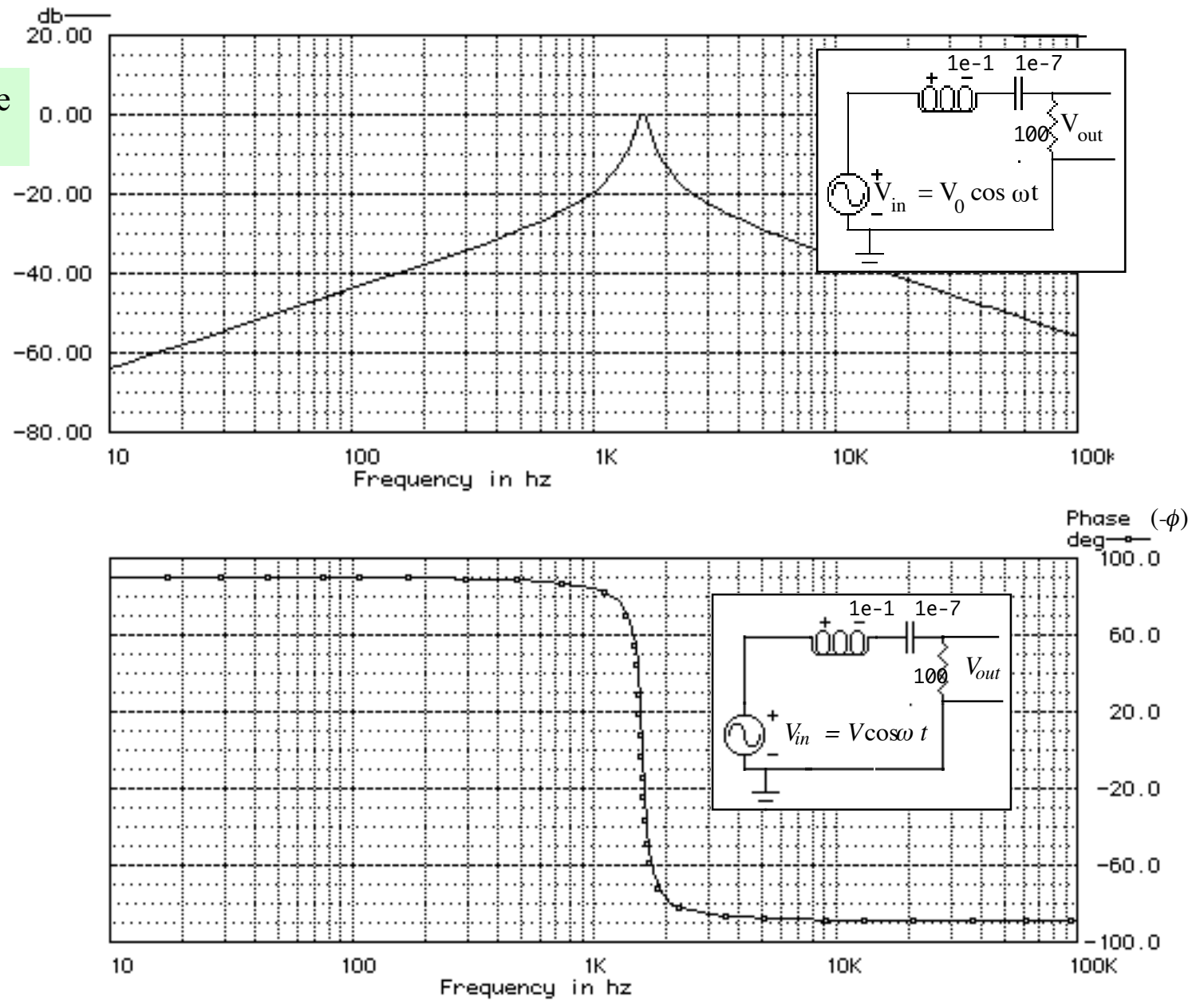
$$V_R = \text{Real} \frac{V_o R e^{j(\omega t - \phi)}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_o R \cos(\omega t - \phi)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$R = 100 \, \Omega, L = 0.1 \, \text{H}, C = 0.1 \, \mu\text{F}$$



- ◆ $V_R \ll V_{in}$ at 100 Hz.
- ◆ V_R and V_{in} are not in phase at this frequency.
- ◆ The little wiggles on V_R are real!
 - Transient solution (homogeneous solution) to the differential eq. describing the circuit.
 - After a few cycles this contribution to V_R die out.

Bode plot of magnitude of V_R/V_{in} vs. frequency



- ◆ In general $V_C(t)$, $V_R(t)$, and $V_L(t)$ are all out of phase with the applied voltage.
- ◆ $I(t)$ and $V_R(t)$ are in phase in a series RLC circuit.
- ◆ The amplitude of V_C , V_R , and V_L depend on ω .
- ◆ The table below summarizes the 3 cases with the following definitions:

$$Z = \left[R^2 + (\omega L - 1/\omega C)^2 \right]^{1/2}$$

$$\tan \phi = (\omega L - 1/\omega C) / R$$

Gain	Magnitude	Phase
V_R/V_{in}	R/Z	$-\phi$
V_L/V_{in}	$\omega L/Z$	$\pi/2 - \phi$
V_C/V_{in}	$1/\omega CZ$	$-\pi/2 - \phi$

- RLC circuits are resonant circuits
 - ◆ energy in the system “resonates” between the inductor and capacitor
 - ◆ “ideal” capacitors and inductors do not dissipate energy
 - ◆ resistors dissipate energy i.e. resistors do not store energy

- Resonant Frequency:
 - ◆ At the resonant frequency the imaginary part of the impedance vanishes.
 - ◆ For the series RLC circuit the impedance (Z) is:

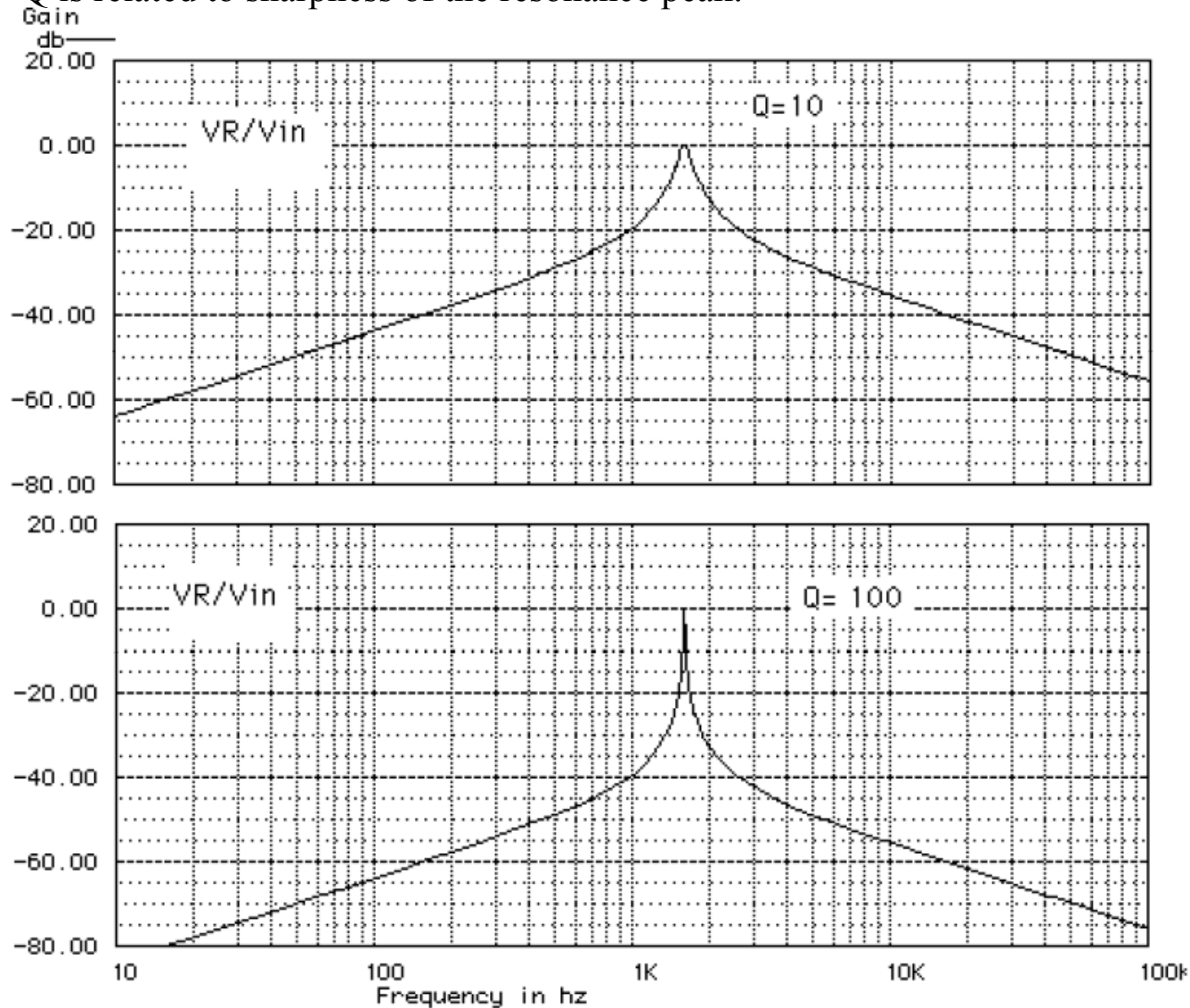
$$Z = R + X_L + X_C = R + j(\omega L - 1/\omega C)$$

$$|Z| = \left[R^2 + (\omega L - 1/\omega C)^2 \right]^{1/2}$$
 - ◆ At resonance (series, parallel etc):

$$\omega L = 1/\omega C$$

$$\omega_R = \frac{1}{\sqrt{LC}}$$
 - ◆ At the resonant frequency the following are true for a series RLC circuit:
 - $|V_R|$ is maximum (ideally $= V_{in}$)
 - $\phi = 0$
 - $\frac{|V_C|}{|V_{in}|} = \frac{|V_L|}{|V_{in}|} = \frac{\sqrt{L}}{R\sqrt{C}}$ (V_C or V_L can be $> V_{in}$!)
 - ☞ *The circuit acts like a narrow band pass filter.*
- There is an exact analogy between an RLC circuit and a harmonic oscillator (mass attached to spring):
 - ◆ $m \frac{d^2 x}{dt^2} + B \frac{dx}{dt} + kx = 0$ damped harmonic oscillator
 - ◆ $L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$ undriven RLC circuit
 - ◆ $x \Leftrightarrow q$ (electric charge), $L \Leftrightarrow m$, $k \Leftrightarrow 1/C$
 - ◆ B (coefficient of damping) $\Leftrightarrow R$

- Q (quality factor) of a circuit: determines how well the RLC circuit stores energy
 - ◆ $Q = 2\pi (\text{max energy stored})/(\text{energy lost}) \text{ per cycle}$
 - ◆ Q is related to sharpness of the resonance peak:



- ◆ The maximum energy stored in the inductor is $LI^2/2$ with $I = I_{MAX}$.
- no energy is stored in the capacitor at this instant because I and V_C are 90° out of phase.
- The energy lost in one cycle:

$$\text{power} \times (\text{time for cycle}) = I_{RMS}^2 R \times \frac{2\pi}{\omega_R} = \frac{1}{2} I_{\max}^2 R \times \frac{2\pi}{\omega_R}$$

$$Q = \frac{2\pi \left(\frac{LI_{\max}^2}{2} \right)}{\frac{2\pi}{\omega_R} \left(\frac{RI_{\max}^2}{2} \right)} = \frac{\omega_R L}{R}$$

- There is another popular, equivalent expression for Q

$$Q = \frac{\omega_R}{\omega_U - \omega_L}$$

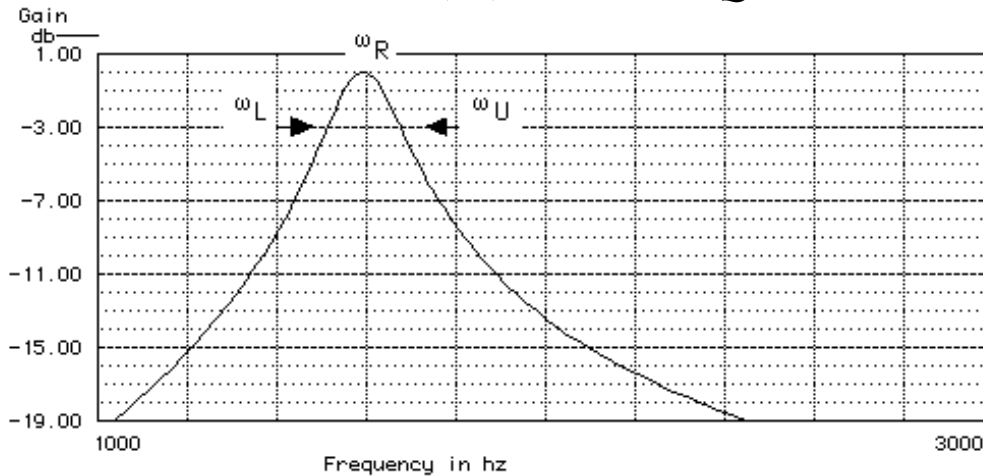
- ω_U (ω_L) is the upper (lower) 3 dB frequency of the resonance curve.
- Q is related to sharpness of the resonance peak.
- Will skip the derivation here as it involves a bit of algebra.
- two crucial points of the derivation:

$$\frac{V_R}{V_{in}} = \frac{1}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right)^2}}$$

$$Q \left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega} \right) = \pm 1$$

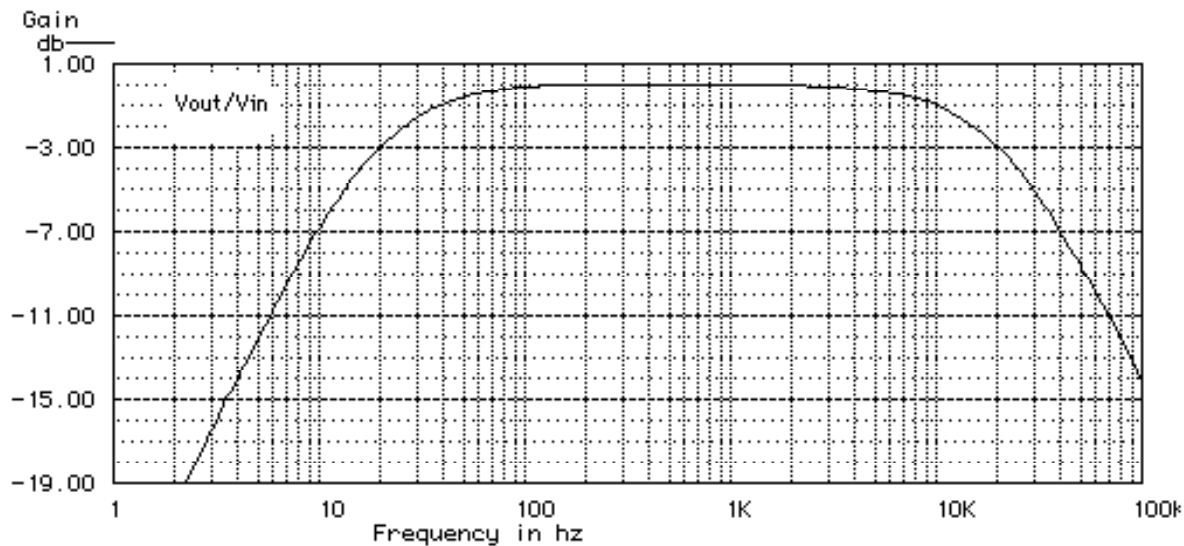
at the upper and lower 3 dB points

- Q can be measured from the shape of the resonance curve
- one does not need to know R , L , or C to find Q !

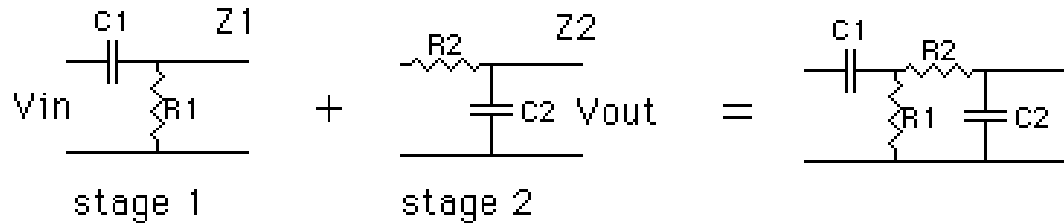


$$Q = \frac{\omega_R}{\omega_U - \omega_L}$$

- Example: Audio filter (band pass filter)
- Audio filter is matched to the frequency range of the ear (20-20,000 Hz).



- Let's design an audio filter using low and high pass RC circuits.



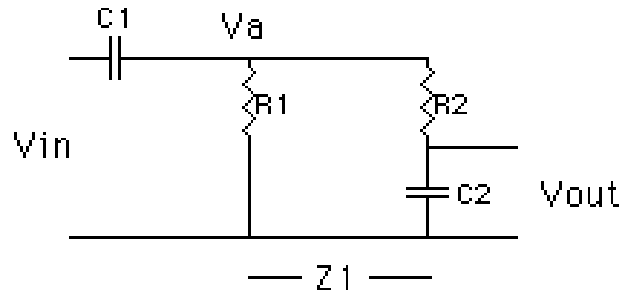
- Ideally, the frequency response is flat over 20-20,000 Hz, and rolls off sharply at frequencies below 20 Hz and above 20,000 Hz.
 - Set 3 dB points as follows:
 - lower 3 dB point : $20 \text{ Hz} = 1/2\pi R_1 C_1$
 - upper 3 dB point: $2 \times 10^4 \text{ Hz} = 1/2\pi R_2 C_2$
 - If we put these two filters together we don't want the 2nd stage to affect the 1st stage.
 - can accomplish this by making the impedance of the 2nd (Z_2) stage much larger than R_1 .
 - Remember R_1 is in parallel with Z_2 .

$$Z_1 = R_1 + 1/j\omega C_1$$

$$Z_2 = R_2 + 1/j\omega C_2$$
 - In order to insure that the second stage does not “load” down the first stage we need:

$$R_2 \gg R_1 \text{ since at high frequencies } Z_2 \Rightarrow R_2$$
 - We can now pick and calculate values for the R 's and C 's in the problem.
 - Let $C_1 = 1 \mu\text{F} \Rightarrow R_1 = 1/(20\text{Hz } 2\pi C_1) = 8 \text{ k}\Omega$
 - Let $R_2 > 100R_1 \Rightarrow R_2 = 1 \text{ M}\Omega$, and $C_2 = 1/(2 \times 10^4 \text{ Hz } 2\pi R_2) = 8 \text{ pf}$
 - $R_1 = 8 \text{ k}\Omega$, $C_1 = 1 \mu\text{F}$
 $R_2 = 1 \text{ M}\Omega$, $C_2 = 8 \text{ pf}$

- ◆ Exact derivation for above filter:
 - ◆ In the above circuit we treated the two RC filters as independent.
 - ◆ *Why did this work?*
 - ◆ We want to calculate the gain ($|V_{out}/V_{in}|$) of the following circuit:



- Working from right to left, we have:

$$V_{out} = V_a X_2 / (X_2 + R_2)$$

$$V_a = V_{in} Z_1 / Z_T$$

- Z_T is the total impedance of the circuit as seen from the input.
- Z_1 is the parallel impedance of R_1 and R_2 , in series with C_2 .

$$Z_1 = \frac{R_1(R_2 + X_2)}{R_1 + R_2 + X_2}$$

$$Z_T = X_1 + Z_1$$

$$V_a = \frac{V_{in} R_1 (R_2 + X_2)}{X_1 (R_1 + R_2 + X_2) + R_1 (R_2 + X_2)}$$

- Finally we can solve for the gain $G = |V_{out}/V_{in}|$:

$$\frac{V_{out}}{V_{in}} = \frac{R_1 X_2}{X_1 (R_1 + R_2 + X_2) + R_1 (R_2 + X_2)}$$

- We can relate this to our previous result by rewriting the above as:

$$\frac{V_{out}}{V_{in}} = \frac{R_1 \frac{X_2}{R_2 + X_2}}{X_1 \left(\frac{R_1}{R_2 + X_2} + 1 \right) + R_1}$$

- If we now remember the approximation ($R_1 \ll R_2 + X_2$) made on the previous page to insure that the second stage did not load down the first then we get the following:

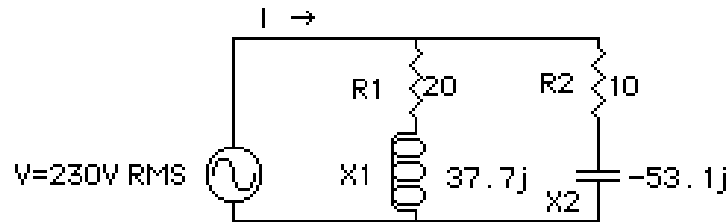
$$\frac{V_{out}}{V_{in}} = \frac{R_1}{R_1 + X_1} \frac{X_2}{R_2 + X_2}$$

- The gain of the circuit looks like the *product* of two filters, one high pass and one low pass!
- If we calculate the gain of this circuit in dB, the total gain is the sum of the gain of each piece:

$$\begin{aligned} \text{Gain in dB} &= 20 \log \left(\frac{V_{out}}{V_{in}} \right) \\ &= 20 \log \left(\frac{R_1}{R_1 + X_1} \right) + 20 \log \left(\frac{X_2}{R_2 + X_2} \right) \end{aligned}$$

- 👉 The gain of successive filters measured in dB's add!

- Another Example: Calculate $|I|$ and the phase angle between V_{in} and I for the following circuit:



- First calculate $|I|$.
 - The total current out of the input source (I) is related to V_{in} and the total impedance (Z_T) of the circuit by Ohm's law:

$$I = V_{in} / Z_T$$
 - The total impedance of the circuit is given by the parallel impedance of the two branches:

$$1/Z_T = 1/Z_1 + 1/Z_2$$

$$Z_1 = R_1 + X_1$$

$$Z_2 = R_2 + X_2$$
 - Putting in numerical values for the R 's and X 's we have:

$$Z_1 = 20 + j37.7 \, \Omega$$

$$Z_2 = 10 - j53.1 \, \Omega$$

$$Z_T = 67.4 + j11.8 \, \Omega$$
 - We can now find the magnitude of the current:

$$|I| = |V_{in}| / |Z_T|$$

$$= 230 \, \text{V} / 68.4 \, \Omega$$

$$= 3.36 \, \text{A}$$

This is RMS value since $|V_{in}|$ is given as RMS

- Calculate the phase angle between V_{in} and I :
 - ◆ It's easiest to solve this by writing V and Z in polar form:

$$V_{in} = (230)e^{j\omega t}$$

$$Z_T = (68.4)e^{j\phi}$$

$$\begin{aligned}\tan \phi &= \text{Im } Z_T / \text{Re } Z_T \\ &= 11.8 / 67.4\end{aligned}$$

$$\phi = 9.9^\circ$$

- ◆ Finally we can write for the current:

$$I = 3.36e^{j(\omega t - \phi)}$$

- ◆ Taking the real part of I :

$$I = 3.36 \cos(\omega t - 9.9^\circ) \text{ A}$$

👉 The current lags the voltage by 9.9° .