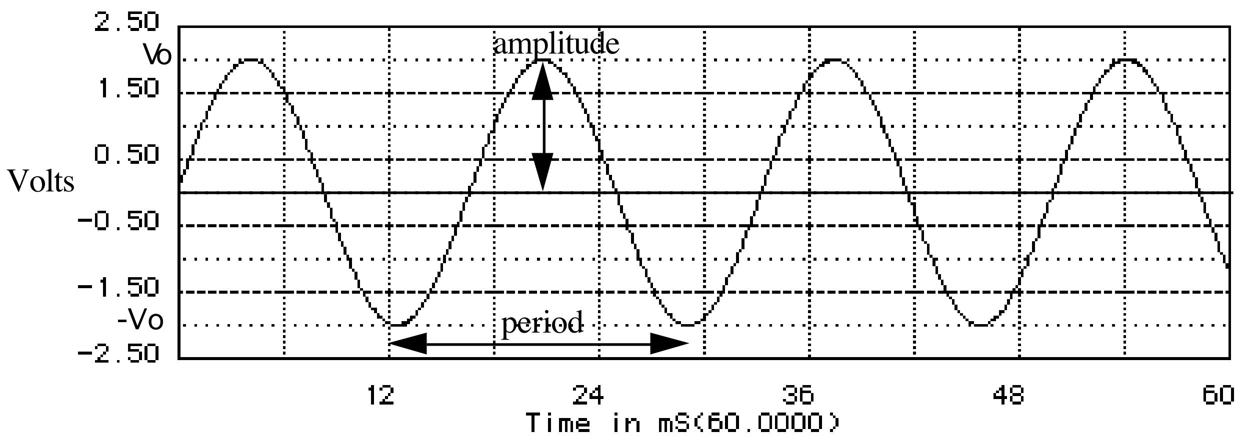
Lecture 3: R-L-C AC Circuits

AC (Alternative Current):

- Most of the time, we are interested in the voltage at a point in the circuit
  - ☞ will concentrate on voltages here rather than currents.
  - ◆ We encounter AC circuits whenever a periodic voltage is applied to a circuit.
  - ◆ The most common periodic voltage is in the form of a sine (or cosine) wave:

$$V(t) = V_0 \cos \omega t \quad \text{or} \quad V(t) = V_0 \sin \omega t$$



- V_0 is the *amplitude*:
  - V_0 = Peak Voltage (V_P)
  - V_0 = 1/2 Peak-to-Peak Voltage (V_{PP})
    - V_{PP}: easiest to read off scope
  - $V_0 = \sqrt{2}\, V_{RMS} = 1.41\, V_{RMS}$
    - V_{RMS}: what multimeters usually read
    - multimeters also usually measure the RMS current

- ω is the *angular frequency*:
  - $\omega = 2\pi f$, with f = frequency of the waveform.
  - frequency (f) and period (T) are related by:

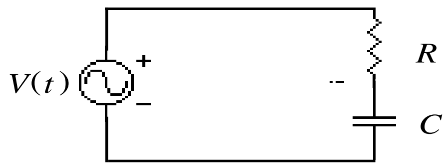
    T (sec) = $1/f$ (sec^{-1})
- *Household line voltage* is usually 110-120 V_{RMS} (156-170 V_P), f = 60 Hz.

- It is extremely important to be able to analyze circuits (systems) with sine or cosine inputs
  - Almost any waveform can be constructed from a sum of sines and cosines.
  - This is the "heart" of *Fourier analysis* (Simpson, Chapter 3).
  - The response of a circuit to a complicated waveform (e.g. a square wave) can be understood by analyzing individual sine or cosine components that make up the complicated waveform.
  - Usually only the first few components are important in determining the circuit's response to the input waveform.

R-C Circuits and AC waveforms

- There are many different techniques for solving AC circuits
  - All are based on Kirchhoff's laws.
  - When solving for voltage and/or current in an AC circuit we are really solving a differential eq.
  - Different circuit techniques are really just different ways of solving the same differential eq:
    - brute force solution to differential equation
    - complex numbers (algebra)
    - Laplace transforms (integrals)

- We will solve the following RC circuit using the brute force method and complex numbers method.
  - Let the input (driving) voltage be $V(t) = V_0\cos\omega t$ and we want to find $V_R(t)$ and $V_C(t)$.



  - *Brute Force Method:* Start with Kirchhoff's loop law:

$$V(t) = V_R(t) + V_C(t)$$
$$V_0\cos\omega t = IR + Q/C$$
$$= RdQ(t)/dt + Q(t)/C$$

    - We have to solve an inhomogeneous D.E.
    - The usual way to solve such a D.E. is to assume the solution has the same form as the input:

$$Q(t) = \alpha\sin\omega t + \beta\cos\omega t$$

    - Plug our trial solution $Q(t)$ back into the D.E.:

$$V_0\cos\omega t = \alpha R\omega\cos\omega t - \beta R\omega\sin\omega t + (\alpha/C)\sin\omega t + (\beta/C)\cos\omega t$$
$$= (\alpha R\omega + \beta/C)\cos\omega t + (\alpha/C - \beta R\omega)\sin\omega t$$
$$V_0 = \alpha R\omega + \beta/C$$
$$\alpha/C = \beta R\omega$$
$$\alpha = \frac{RC^2\omega V_0}{1+(RC\omega)^2}$$
$$\beta = \frac{CV_0}{1+(RC\omega)^2}$$

- We can now write the solution for $V_C(t)$:

$$V_C(t) = Q/C$$
$$= (\alpha \sin \omega t + \beta \cos \omega t)/C$$
$$= \frac{RC\omega V_0}{1+(RC\omega)^2} \sin \omega t + \frac{V_0}{1+(RC\omega)^2} \cos \omega t$$

- We would like to rewrite the above solution in such a way that only a cosine term appears.
  - In this form we can compare it to the input voltage.

$$V_C(t) = \frac{V_0}{\sqrt{1+(RC\omega)^2}} \left[\frac{RC\omega}{\sqrt{1+(RC\omega)^2}} \sin \omega t + \frac{1}{\sqrt{1+(RC\omega)^2}} \cos \omega t \right]$$

  - We get the above equation in terms of cosine only using the following basic trig:

$$\cos(\theta_1 - \theta_2) = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2$$

  - We can now define an angle such that:

$$\cos \phi = \frac{1}{\sqrt{1+(RC\omega)^2}}$$
$$\sin \phi = \frac{RC\omega}{\sqrt{1+(RC\omega)^2}}$$
$$\tan \phi = RC\omega$$
$$V_C(t) = \frac{V_0}{\sqrt{1+(RC\omega)^2}} \cos(\omega t - \phi)$$

☞ $V_C(t)$ and $V_0(t)$ are out of phase.

- Using the above expression for $V_C(t)$, we obtain:

$$V_R(t) = IR$$
$$= R\frac{dQ}{dt}$$
$$= RC\frac{dV_C}{dt}$$
$$= \frac{-RC\omega V_o}{\sqrt{1+(RC\omega)^2}}\sin(\omega t-\phi)$$

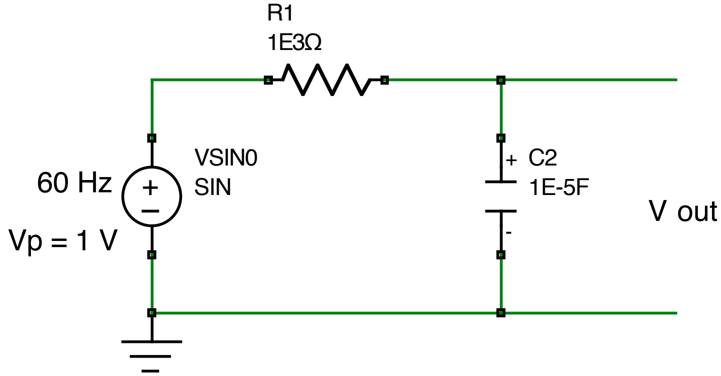
- We would like to have cosines instead of sines by using:

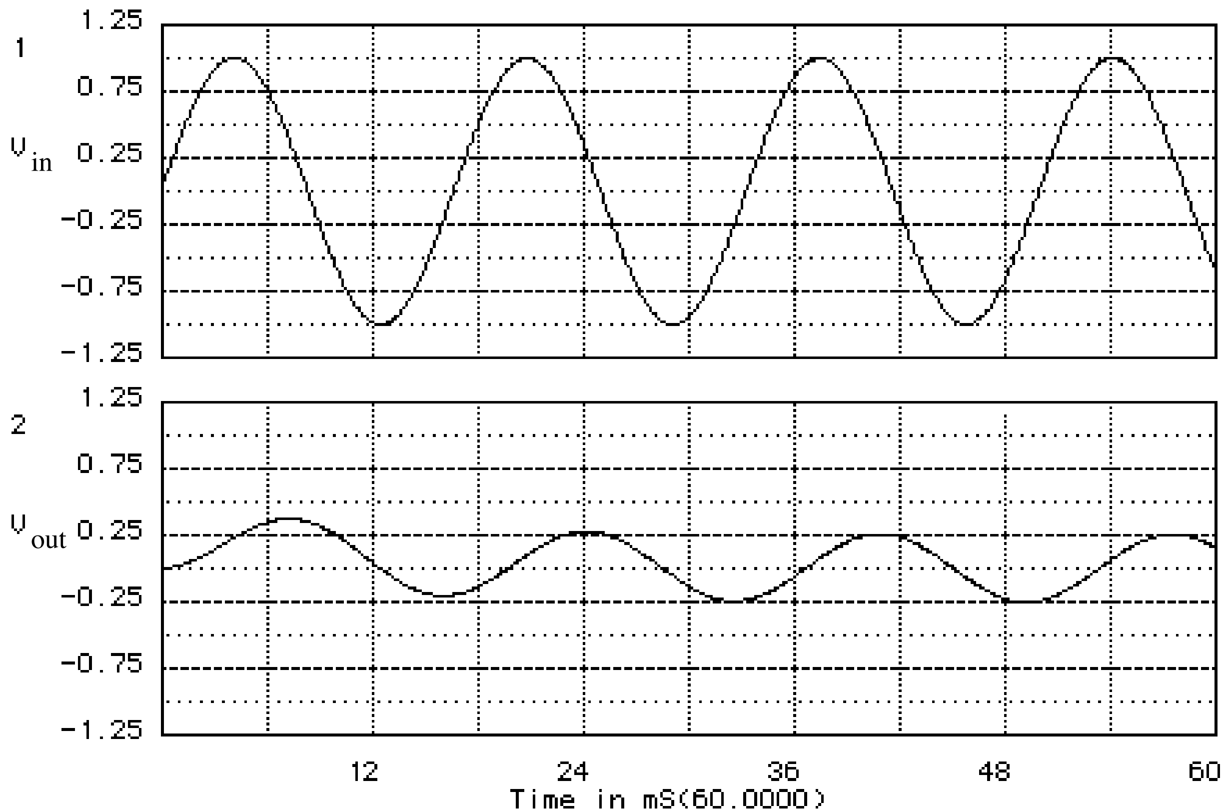
$$-\sin\theta = \cos(\theta+\tfrac{\pi}{2})$$

☞ $$V_R(t) = \frac{RC\omega V_o}{\sqrt{1+(RC\omega)^2}}\cos(\omega t-\phi+\tfrac{\pi}{2})$$

  - $V_C(t)$, $V_R(t)$, and $I(t)$ are all out of phase with the applied voltage.
  - $I(t)$ and $V_R(t)$ are in phase with each other.
  - $V_C(t)$ and $V_R(t)$ are out of phase by 90^0.
  - The amplitude of $V_C(t)$ and $V_R(t)$ depend on ω.

- Example: RC Circuit





- Solving circuits with complex numbers:
  - PROS:
    - don't explicitly solve differential equations (lots of algebra).
    - can find magnitude and phase of voltage separately.
  - CONS:
    - have to use complex numbers!
    - No "physics" in complex numbers.
  - What's a complex number? (see Simpson, Appendix E, P835)
    - Start with $j \equiv \sqrt{-1}$ (solution to $x^2 + 1 = 0$).
    - A complex number can be written in two forms:
      - $X = A + jB$
        - A and B are *real* numbers
      - $X = R\, e^{j\phi}$
        - $R = (A^2 + B^2)^{1/2}$ and $\tan\phi = B/A$ (remember $e^{j\phi} = \cos\phi + j\sin\phi$)
    - Define the complex conjugate of X as:

      $X^* = A - jB$ or $X^* = R\, e^{-j\phi}$
    - The magnitude of X can be found from:

      $|X| = (XX^*)^{1/2} = (X^*X)^{1/2} = (A^2 + B^2)^{1/2}$
    - Suppose we have 2 complex numbers, X and Y with phases α and β respectively,

      $$Z = \frac{X}{Y} = \frac{|X|e^{j\alpha}}{|Y|e^{j\beta}} = \frac{|X|}{|Y|}e^{j(\alpha-\beta)}$$
        - magnitude of Z: $|X|/|Y|$
        - phase of Z: $\alpha - \beta$
  - So why is this useful?

- Consider the case of the capacitor and AC voltage:

$$V(t) = V_0 \cos\omega t$$
$$= \mathrm{Real}\left(V_0 e^{j\omega t}\right)$$
$$Q = CV$$
$$I(t) = C\frac{dV}{dt}$$
$$= -C\omega V_0 \sin\omega t$$
$$= \mathrm{Real}\left(j\omega C V_0 e^{j\omega t}\right)$$
$$= \mathrm{Real}\left(\frac{V_0 e^{j\omega t}}{1/j\omega C}\right)$$
$$= \mathrm{Real}\left(\frac{V}{X_C}\right)$$

  - V and X_C are complex numbers
  - ☞ We now have Ohm's law for capacitors using the capacitive reactance X_C:

$$X_C = \frac{1}{j\omega C}$$

- We can make a similar case for the inductor:

$$V = L\frac{dI}{dt}$$

$$I(t) = \frac{1}{L}\int V\,dt$$

$$= \frac{1}{L}\int V_0 \cos\omega t\,dt$$

$$= \frac{V_0 \sin\omega t}{L\omega}$$

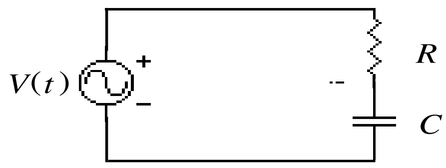
$$= \mathrm{Real}\left(\frac{V_0 e^{j\omega t}}{j\omega L}\right)$$

$$= \mathrm{Real}\left(\frac{V}{X_L}\right)$$

  - V and X_L are complex numbers
  - ☞ We now have Ohm's law for inductors using the inductive reactance X_L:

    $X_L = j\omega L$
- X_C and X_L act like frequency dependent resistors.
  - They also have a *phase* associated with them due to their complex nature.
  - $X_L \Rightarrow 0$ as $\omega \Rightarrow 0$ (short circuit, DC)
  - $X_L \Rightarrow \infty$ as $\omega \Rightarrow \infty$ (open circuit)
  - $X_C \Rightarrow 0$ as $\omega \Rightarrow \infty$ (short circuit)
  - $X_C \Rightarrow \infty$ as $\omega \Rightarrow 0$ (open circuit, DC)

- Back to the RC circuit.
  - Allow voltages, currents, and charge to be complex:



$$V_{in} = V_0 \cos \omega t$$
$$= \mathrm{Real}\left(V_0 e^{j\omega t}\right)$$
$$= \mathrm{Real}\left(V_R + V_C\right)$$

  - We can write an expression for the charge (Q) taking into account the phase difference (ϕ) between applied voltage and the voltage across the capacitor (V_C).

$$Q(t) = CV_C(t)$$
$$= Ae^{j(\omega t - \phi)}$$

    - Q and V_C are complex
    - A and C are real
  - We can find the complex current by differentiating the above:

$$I(t) = dQ(t)/dt$$
$$= j\omega Ae^{j(\omega t - \phi)}$$
$$= j\omega Q(t)$$
$$= j\omega CV_C(t)$$
$$V_{in} = V_C + V_R$$
$$= V_C + IR$$
$$= V_C + j\omega CV_C R$$

$$V_C = \frac{V_{in}}{1 + j\omega RC}$$

$$= V_{in}\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

$$= V_{in}\frac{X_C}{R + X_C}$$

- *looks like a voltage divider equation!!!!!*

- We can easily find the magnitude of V_C:

$$|V_C| = |V_{in}|\frac{|X_C|}{|R + X_C|}$$

$$= \frac{V_0 \frac{1}{\omega C}}{\sqrt{R^2 + (1/\omega C)^2}}$$

$$= \frac{V_0}{\sqrt{1 + (RC\omega)^2}}$$

  - same as the result on page 4.

- Is this solution the same as what we had when we solved by brute force page 4?

$$V_C = \mathrm{Real}\left(\frac{V_{in}}{1+j\omega RC}\right)$$

$$= \mathrm{Real}\left(\frac{V_0 e^{j\omega t}}{1+j\omega RC}\right)$$

$$= \mathrm{Real}\left(\frac{V_0 e^{j\omega t}}{\sqrt{1+(\omega RC)^2}\, e^{j\phi}}\right)$$

ϕ is given by $\tan\phi = \omega RC$

$$= \mathrm{Real}\left(\frac{V_0 e^{j(\omega t-\phi)}}{\sqrt{1+(\omega RC)^2}}\right)$$

$$= \frac{V_0 \cos(\omega t - \phi)}{\sqrt{1+(\omega RC)^2}}$$

☞ YES the solutions are identical.

- We can now solve for the voltage across the resistor.
  - Start with the voltage divider equation in complex form:

$$V_R = \frac{V_{in} R}{R + X_C}$$

$$|V_R| = \frac{|V_{in}| R}{|R + X_C|}$$

$$= \frac{V_0 R}{\sqrt{R^2 + (1/\omega C)^2}}$$

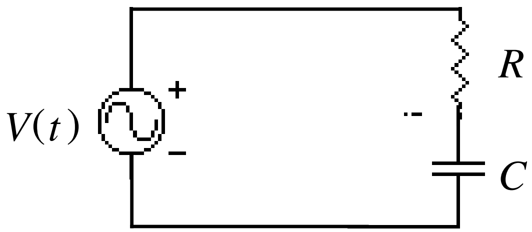
$$= \frac{V_0 \omega R C}{\sqrt{1 + (\omega R C)^2}}$$

  - ☞ This amplitude is the same as the brute force differential equation case!
- In adding complex voltages, we must take into account the phase difference between them.
  - the sum of the voltages at a given time satisfy:
    - ✔ $V_0^2 = |V_R|^2 + |V_C|^2$
    - ✘ $V_0 = |V_R| + |V_C|$

R-C Filters

- Allow us to select (reject) wanted (unwanted) signals on the basis of their frequency structure.
- Allow us to change the phase of the voltage or current in a circuit.
- Define the gain (G) or transfer (H) function of a circuit:
  - $G(j\omega) = H(j\omega) = V_{out}/V_{in}$ ($j\omega$ is often denoted by s).
  - G is independent of time, but can depend on ω, R, L, C.

- For an RC circuit we can define G_R and G_C:



$$G_R \equiv \frac{V_R}{V_{in}} = \left|\frac{R}{R + X_C}\right| = \left|\frac{R}{R + 1/j\omega C}\right|$$

$$G_C \equiv \frac{V_C}{V_{in}} = \left|\frac{X_C}{R + X_C}\right| = \left|\frac{1/j\omega C}{R + 1/j\omega C}\right|$$

- We can categorize the G's as follows:

	G_R	G_C
High Frequencies	≈ 1, no phase shift **high pass filter**	$\approx 1/j\omega CR \approx 0$, phase shift
Low Frequencies	$\approx j\omega CR \approx 0$, phase shift	≈ 1, no phase shift **low pass filter**

- Decibels and Bode Plots:
  - Decibel (dB) describes voltage or power gain:

$$\text{dB} = 20\log(V_{out}/V_{in})$$
$$= 10\log(P_{out}/P_{in})$$

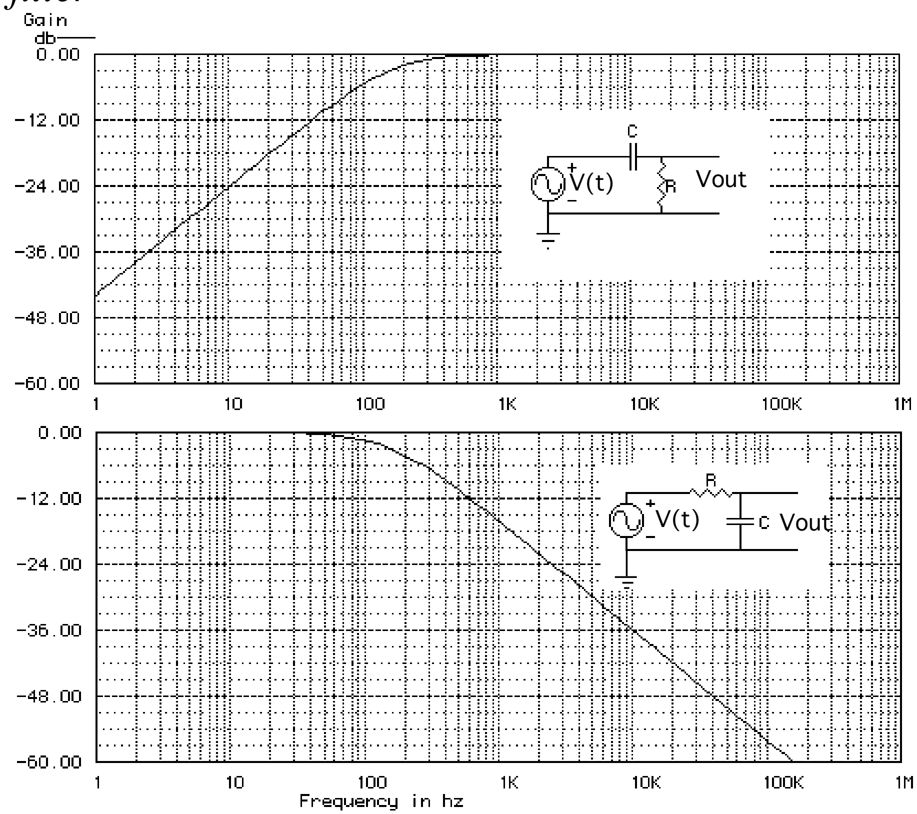
  - *Bode Plot* is a log-log plot with dB on the y axis and log(ω) or log(f) on the x axis.

- 3 dB point or 3 dB frequency:
  - also called break frequency, corner frequency, 1/2 power point
  - At the 3 dB point:

$$\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{2}} \quad \text{since } 3 = 20\log(V_{out}/V_{in})$$

$$\frac{P_{out}}{P_{in}} = \frac{1}{2} \quad \text{since } 3 = 10\log(P_{out}/P_{in})$$

☞ *$\omega RC = 1$ for high or low pass filter*



Phase vs frequency for capacitor