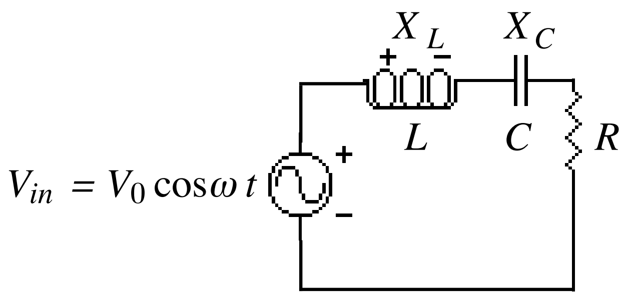
Lecture 4: R-L-C Circuits and Resonant Circuits

RLC series circuit:

- What's V_R?
  - Simplest way to solve for V is to use voltage divider equation in complex notation:

$$V_R = \frac{V_{in} R}{R + X_C + X_L} = \frac{V_{in} R}{R + \frac{1}{j\omega C} + j\omega L}$$



  - Using complex notation for the apply voltage $V_{in} = V_0 \cos\omega t = \text{Real}(V_0 e^{j\omega t})$:

$$V_R = \frac{V_0 e^{j\omega t} R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

    - We are interested in the both the magnitude of V_R and its phase with respect to V_{in}.
    - First the magnitude:

$$|V_R| = \frac{\left|V_0 e^{j\omega t}\right| |R|}{\left|R + j\left(\omega L - \frac{1}{\omega C}\right)\right|} = \frac{V_0 R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

- The *phase* of V_R with respect to V_{in} can be found by writing V_R in purely polar notation.
  - For the denominator we have:

$$R + j\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \exp\left\{ j \tan^{-1}\left[\frac{\omega L - \frac{1}{\omega C}}{R}\right]\right\}$$

  - Define the phase angle ϕ:

$$\tan\phi = \frac{\text{Imaginary } X}{\text{Real } X}$$

$$= \frac{\omega L - \frac{1}{\omega C}}{R}$$

  - We can now write for V_R in complex form:

$$V_R = \frac{V_o R\, e^{j\omega t}}{e^{j\phi}\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

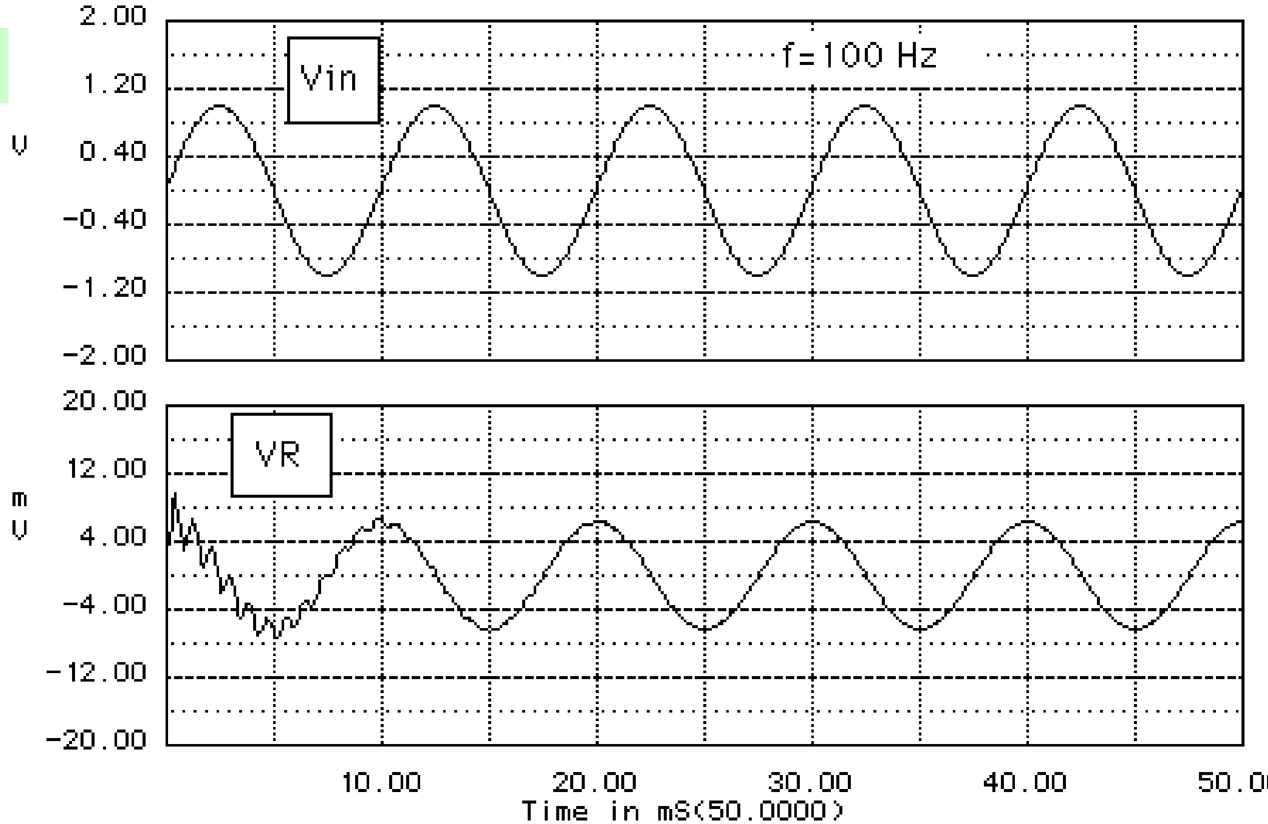
$$= |V_R| e^{j(\omega t - \phi)}$$

Depending on L, C, and ω, the phase angle can be positive or negative! In this example, if $\omega L > 1/\omega C$, then $V_R(t)$ **lags** $V_{in}(t)$.

- Finally, we can write down the solution for V by taking the real part of the above equation:

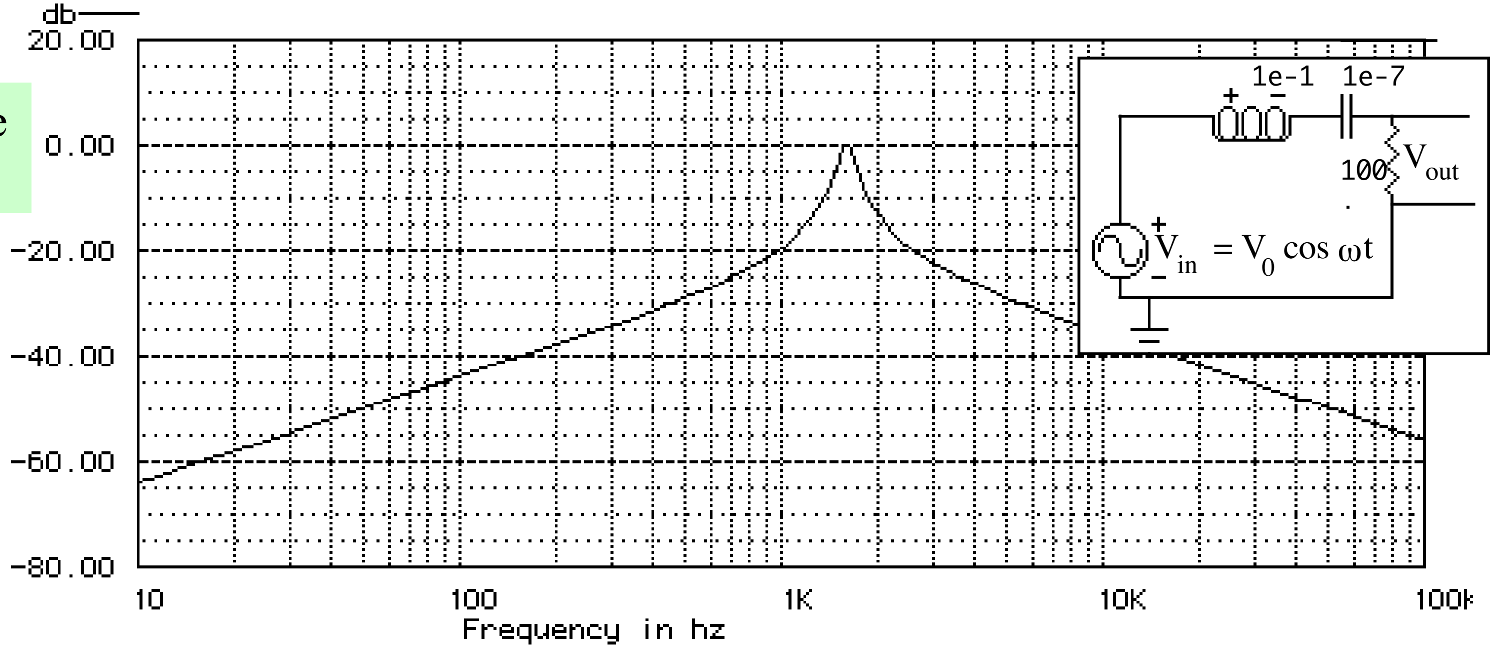
$$V_R = \text{Real}\, \frac{V_0 R\, e^{j(\omega t - \phi)}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_0 R \cos(\omega t - \phi)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

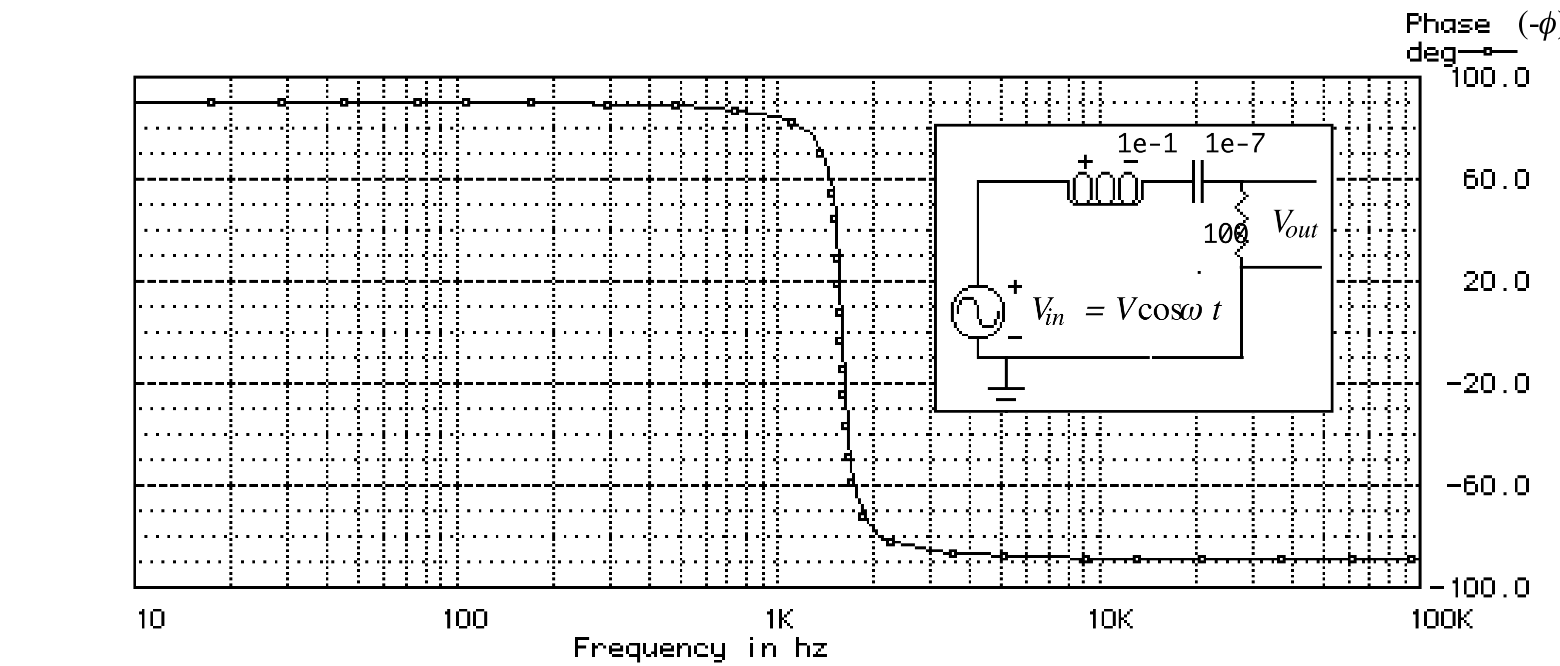
$R = 100\ \Omega$, $L = 0.1$ H, $C = 0.1\ \mu$F



- $V_R << V_{in}$ at 100 Hz.
- V_R and V_{in} are not in phase at this frequency.
- The little wiggles on V_R are real!
  - Transient solution (homogeneous solution) to the differential eq. describing the circuit.
  - After a few cycles this contribution to V_R die out.

Bode plot of magnitude of V_R/V_{in} vs. frequency





- In general $V_C(t)$, $V_R(t)$, and V_L(t) are all out of phase with the applied voltage.
- $I(t)$ and $V_R(t)$ are in phase in a <u>series</u> RLC circuit.
- The amplitude of V_C, V_R, and V_L depend on ω.
- The table below summarizes the 3 cases with the following definitions:

$$Z = \left[R^2 + (\omega L - 1/\omega C)^2\right]^{1/2}$$

$$\tan\phi = (\omega L - 1/\omega C)/R$$

Gain	Magnitude	Phase
V_R/V_{in}	R/Z	$-\phi$
V_L/V_{in}	$\omega L/Z$	$\pi/2 - \phi$
V_C/V_{in}	$1/\omega CZ$	$-\pi/2 - \phi$

- RLC circuits are resonant circuits
  - energy in the system "resonates" between the inductor and capacitor
  - "ideal" capacitors and inductors do not dissipate energy
  - resistors dissipate energy i.e. resistors do not store energy

- Resonant Frequency:
  - At the resonant frequency the imaginary part of the impedance vanishes.
  - For the series RLC circuit the impedance (Z) is:

$$Z = R + X_L + X_C = R + j(\omega L - 1/\omega C)$$

$$|Z| = \left[R^2 + (\omega L - 1/\omega C)^2\right]^{1/2}$$

  - At resonance (series, parallel etc):

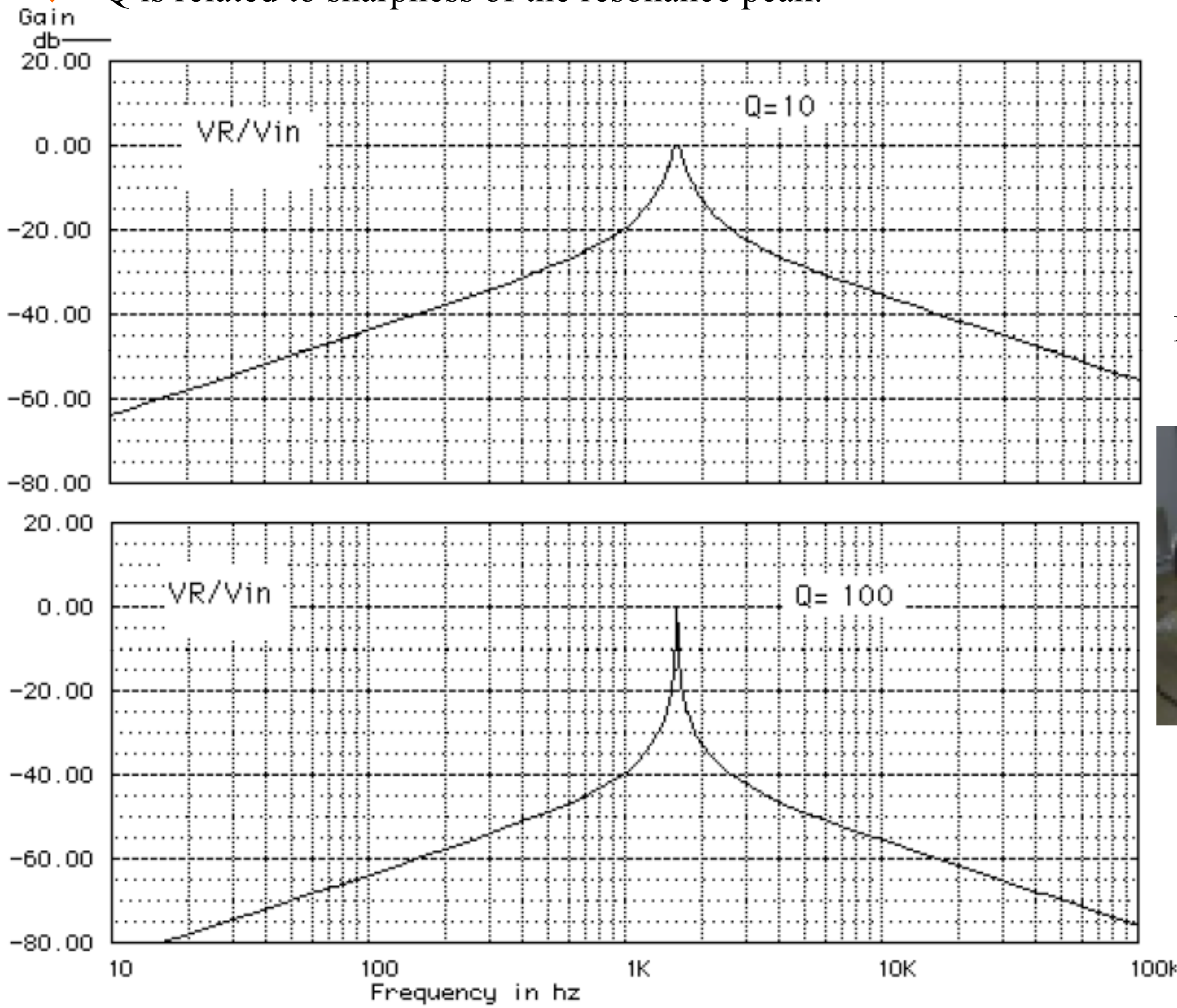
$$\omega L = 1/\omega C$$

$$\omega_R = \frac{1}{\sqrt{LC}}$$

  - At the <u>resonant frequency</u> the following are true for a series RLC circuit:
    - $|V_R|$ is maximum (ideally = V_{in})
    - $\phi = 0$
    - $\frac{|V_C|}{|V_{in}|} = \frac{|V_L|}{|V_{in}|} = \frac{\sqrt{L}}{R\sqrt{C}}$ (V_C or V_L can be $> V_{in}$!)
    - ⇨ *The circuit acts like a narrow band pass filter.*

- There is an exact analogy between an RLC circuit and a harmonic oscillator (mass attached to spring):
  - $m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = 0$ damped harmonic oscillator
  - $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ undriven RLC circuit
  - $x \Leftrightarrow q$ (electric charge), $L \Leftrightarrow m$, $k \Leftrightarrow 1/C$
  - B (coefficient of damping) $\Leftrightarrow R$

- Q (quality factor) of a circuit: determines how well the RLC circuit stores energy
  - Q = 2π (max energy stored)/(energy lost) per cycle
  - Q is related to sharpness of the resonance peak:



Superconducting
Radio Frequency Cavity
Q = 3.3×10^{10}

- The maximum energy stored in the inductor is $LI^2/2$ with $I = I_{MAX}$.
  - no energy is stored in the capacitor at this instant because I and V_C are 90^0 out of phase.
  - The energy lost in one cycle:

$$\text{power} \times (\text{time for cycle}) = I_{RMS}^2 R \times \frac{2\pi}{\omega_R} = \tfrac{1}{2} I_{\max}^2 R \times \frac{2\pi}{\omega_R}$$

$$Q = \frac{2\pi\left(\frac{LI_{Max}^2}{2}\right)}{\frac{2\pi}{\omega_R}\left(\frac{RI_{Max}^2}{2}\right)} = \frac{\omega_R L}{R}$$

  - There is another popular, equivalent expression for Q

$$Q = \frac{\omega_R}{\omega_U - \omega_L}$$

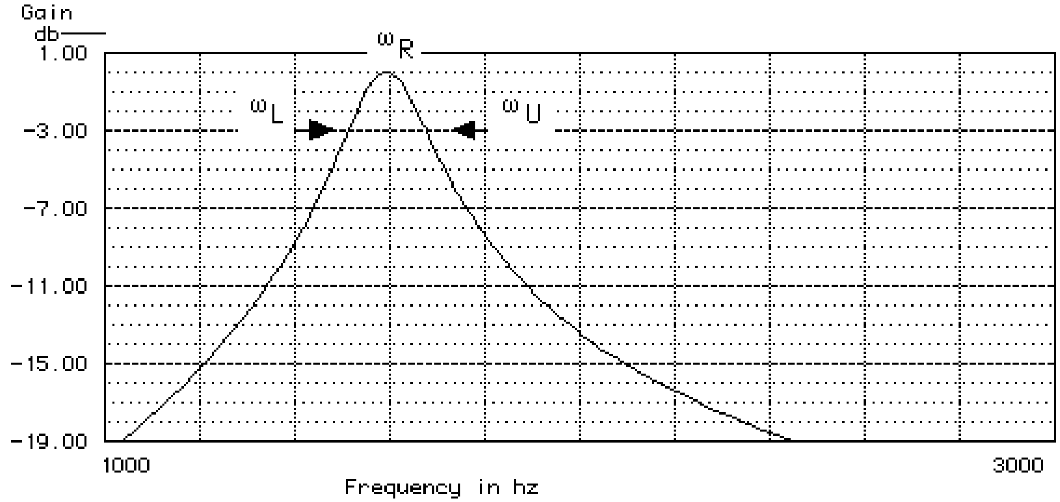
    - ω_U (ω_L) is the upper (lower) 3 dB frequency of the resonance curve.
      - Q is related to sharpness of the resonance peak.
    - Will skip the derivation here as it involves a bit of algebra.
      - two crucial points of the derivation:

$$\frac{V_R}{V_{in}} = \frac{1}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega}\right)^2}}$$

$$Q\left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega}\right) = \pm 1$$

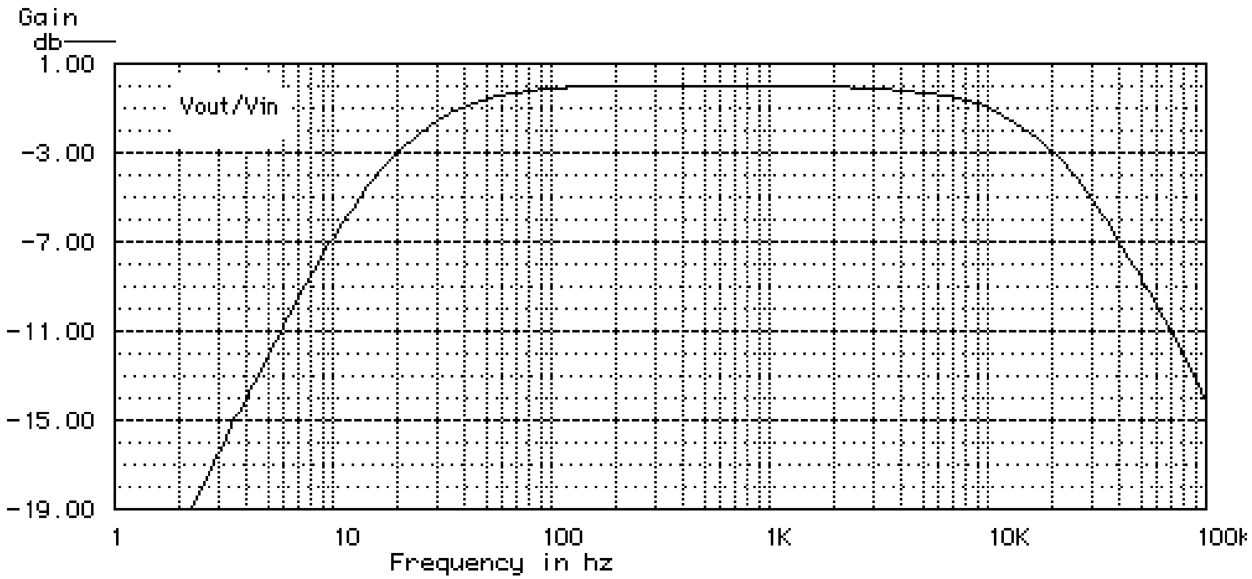
at the upper and lower 3 dB points

- Q can be measured from the shape of the resonance curve
  - one does not need to know R, L, or C to find Q!

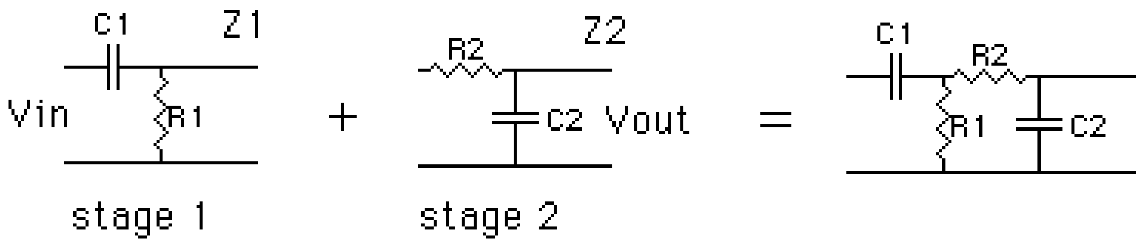


$$Q = \frac{\omega_R}{\omega_U - \omega_L}$$

- Example: Audio filter (band pass filter)
  - Audio filter is matched to the frequency range of the ear (20-20,000 Hz).



- Let's design an audio filter using low and high pass RC circuits.



- Ideally, the frequency response is flat over 20-20,000 Hz, and rolls off sharply at frequencies below 20 Hz and above 20,000 Hz.
  - Set 3 dB points as follows:
    - lower 3 dB point : 20 Hz = $1/2\pi R_1 C_1$
    - upper 3 dB point: $2\text{x}10^4$ Hz = $1/2\pi R_2 C_2$
  - If we put these two filters together we don't want the 2nd stage to affect the 1st stage.
    - can accomplish this by making the impedance of the 2nd (Z_2) stage much larger than R_1.
    - Remember R_1 is in parallel with Z_2.

$$Z_1 = R_1 + 1/j\omega C_1$$

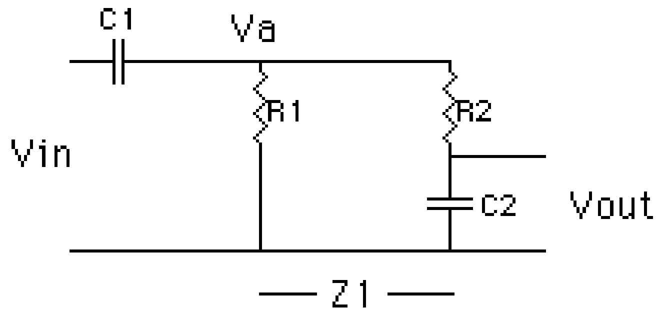
$$Z_2 = R_2 + 1/j\omega C_2$$

  - In order to ensure that the second stage does not "load" down the first stage we need:

    $R_2 >> R_1$ since at high frequencies $Z_2 \Rightarrow R_2$
  - We can now pick and calculate values for the R' s and C' s in the problem.
    - Let C_1 = 1 μF $\Rightarrow R_1 = 1/(20\text{Hz}\ 2\pi C_1)$ = 8 kΩ
    - Let $R_2 > 100 R_1 \Rightarrow R_2$ = 1 MΩ, and $C_2 = 1/(2\text{x}10^4\ \text{Hz}\ 2\pi R_2)$ = 8 pf
    - ⇨ R_1 = 8 kΩ, C_1 = 1 μF

      R_2 = 1 MΩ, C_2 = 8 pf

- Exact derivation for above filter:
  - In the above circuit we treated the two RC filters as independent.
  - *Why did this work?*
  - We want to calculate the gain ($|V_{out}/V_{in}|$) of the following circuit:



- Working from right to left, we have:

$$V_{out} = V_a X_2 /(X_2 + R_2)$$

$$V_a = V_{in} Z_1 / Z_T$$

  - Z_T is the total impedance of the circuit as seen from the input.
  - Z_1 is the parallel impedance of R_1 and R_2, in series with C_2.

$$Z_1 = \frac{R_1\left(R_2 + X_2\right)}{R_1 + R_2 + X_2}$$

$$Z_T = X_1 + Z_1$$

⇨ $$V_a = \frac{V_{in} R_1\left(R_2 + X_2\right)}{X_1\left(R_1 + R_2 + X_2\right) + R_1\left(R_2 + X_2\right)}$$

- Finally we can solve for the gain $G = |V_{out}/V_{in}|$:

$$\frac{V_{out}}{V_{in}} = \frac{R_1 X_2}{X_1\left(R_1 + R_2 + X_2\right) + R_1\left(R_2 + X_2\right)}$$

- We can relate this to our previous result by rewriting the above as:

$$\frac{V_{out}}{V_{in}} = \frac{R_1 \dfrac{X_2}{R_2 + X_2}}{X_1\left(\dfrac{R_1}{R_2 + X_2} + 1\right) + R_1}$$

- If we now remember the approximation ($R_1 \ll R_2 + X_2$) made on the previous page to ensure that the second stage did not load down the first then we get the following:

$$\frac{V_{out}}{V_{in}} = \frac{R_1}{R_1 + X_1}\frac{X_2}{R_2 + X_2}$$

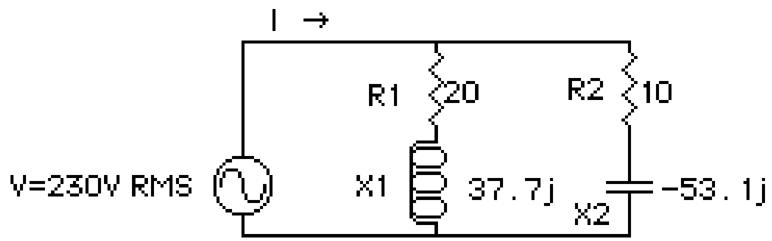
- The gain of the circuit looks like the *product* of two filters, one high pass and one low pass!
- If we calculate the gain of this circuit in dB, the total gain is the sum of the gain of each piece:

$$\text{Gain in dB} = 20\ \log\left(\frac{V_{out}}{V_{in}}\right)$$

$$= 20\ \log\left(\frac{R_1}{R_1 + X_1}\right) + 20\ \log\left(\frac{X_2}{R_2 + X_2}\right)$$

⇨ The gain of successive filters measured in dB's add!

- Another Example: Calculate $|I|$ and the phase angle between V_{in} and I for the following circuit:



  - First calculate $|I|$.
    - The total current out of the input source (I) is related to V_{in} and the total impedance (Z_T) of the circuit by Ohm's law:

      $$I = V_{in} / Z_T$$

    - The total impedance of the circuit is given by the parallel impedance of the two branches:

      $$1/Z_T = 1/Z_1 + 1/Z_2$$
      $$Z_1 = R_1 + X_1$$
      $$Z_2 = R_2 + X_2$$

    - Putting in numerical values for the R's and X's we have:

      $$Z_1 = 20 + j37.7\ \Omega$$
      $$Z_2 = 10 - j53.1\ \Omega$$
      $$Z_T = 67.4 + j11.8\ \Omega$$

    - We can now find the magnitude of the current:

      $$|I| = |V_{in}|/|Z_T|$$
      $$= 230\ \text{V}/68.4\ \Omega$$
      $$= 3.36\ \text{A}$$

This is RMS value since $|V_{in}|$ is given as RMS

- Calculate the phase angle between V_{in} and I:
  - It's easiest to solve this by writing V and Z in polar form:

$$V_{in} = (230)e^{j\omega t}$$

$$Z_T = (68.4)e^{j\phi}$$

$$\tan\phi = \operatorname{Im} Z_T / \operatorname{Re} Z_T$$

$$= 11.8/67.4$$

$$\phi = 9.9^0$$

  - Finally we can write for the current:

$$I = 3.36e^{j(\omega t - \phi)}$$

  - Taking the real part of I:

$$I = 3.36\cos(\omega t - 9.9^0) \text{ A}$$

    ⇨ The current lags the voltage by 9.9^0.