

# Lecture 1

## Probability and Statistics

### Introduction:

- Understanding of many physical phenomena depend on statistical and probabilistic concepts:
  - ★ Statistical Mechanics (physics of systems composed of many parts: gases, liquids, solids.)
    - ◆ 1 mole of anything contains  $6 \times 10^{23}$  particles (Avogadro's number)
    - ◆ impossible to keep track of all  $6 \times 10^{23}$  particles even with the fastest computer imaginable
      - ☞ resort to learning about the group properties of all the particles
      - ☞ partition function: calculate energy, entropy, pressure... of a system
  - ★ Quantum Mechanics (physics at the atomic or smaller scale)
    - ◆ wavefunction = probability amplitude
      - ☞ probability of an electron being located at (x,y,z) at a certain time.
- Understanding/interpretation of experimental data depend on statistical and probabilistic concepts:
  - ★ how do we extract the best value of a quantity from a set of measurements?
  - ★ how do we decide if our experiment is consistent/inconsistent with a given theory?
  - ★ how do we decide if our experiment is internally consistent?
  - ★ how do we decide if our experiment is consistent with other experiments?
    - ☞ In this course we will concentrate on the above experimental issues!

## Definition of probability:

- Suppose we have  $N$  trials and a specified event occurs  $r$  times.
  - ★ example: rolling a dice and the event could be rolling a 6.
- ◆ define probability ( $P$ ) of an event ( $E$ ) occurring as:  
 $P(E) = r/N$  when  $N \rightarrow \infty$
- ★ examples:
  - six sided dice:  $P(6) = 1/6$
  - coin toss:  $P(\text{heads}) = 0.5$ 
    - ☞  $P(\text{heads})$  should approach 0.5 the more times you toss the coin.
    - ☞ for a single coin toss we can never get  $P(\text{heads}) = 0.5!$
- ◆ by definition probability is a non-negative real number bounded by  $0 \leq P \leq 1$ 
  - ★ if  $P = 0$  then the event never occurs
  - ★ if  $P = 1$  then the event always occurs
  - ★ sum (or integral) of all probabilities if they are mutually exclusive must = 1.
    - events are independent if:  $P(A \cap B) = P(A)P(B)$
    - events are mutually exclusive (disjoint) if:  $P(A \cap B) = 0$  or  $P(A \cup B) = P(A) + P(B)$

$\cap$   $\equiv$  intersection,  $\cup$   $\equiv$  union

- Probability can be a discrete or a continuous variable.

- ◆ Discrete probability:  $P$  can have certain values only.

- ★ examples:

- tossing a six-sided dice:  $P(x_i) = P_i$  here  $x_i = 1, 2, 3, 4, 5, 6$  and  $P_i = 1/6$  for all  $x_i$ .

- tossing a coin: only 2 choices, heads or tails.

- ★ for both of the above discrete examples (and in general)

when we sum over all mutually exclusive possibilities:

$$\sum_i P(x_i) = 1$$

- ◆ Continuous probability:  $P$  can be any number between 0 and 1.

- ★ define a “probability density function”, pdf,  $f(x)$

$$f(x)dx = dP(x \in [x, x+dx]) \quad \text{with } x \text{ a continuous variable}$$

- ★ probability for  $x$  to be in the range  $a \leq x \leq b$  is:

$$P(a \leq x \leq b) = \int_a^b f(x)dx$$

- ★ just like the discrete case the sum of all probabilities must equal 1.

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

⇒  $f(x)$  is **normalized** to one.

- ★ probability for  $x$  to be **exactly** some number is zero since:

$$\int_{x=a}^{x=a} f(x)dx = 0$$

□

Notation:  
 $x_i$  is called a  
random variable

- Examples of some common  $P(x)$ 's and  $f(x)$ 's:

<u>Discrete = <math>P(x)</math></u>	<u>Continuous = <math>f(x)</math></u>
binomial	uniform, i.e. constant
Poisson	Gaussian
	exponential
	chi square

- How do we describe a probability distribution?
  - mean, mode, median, and variance
  - for a continuous distribution, these quantities are defined by:

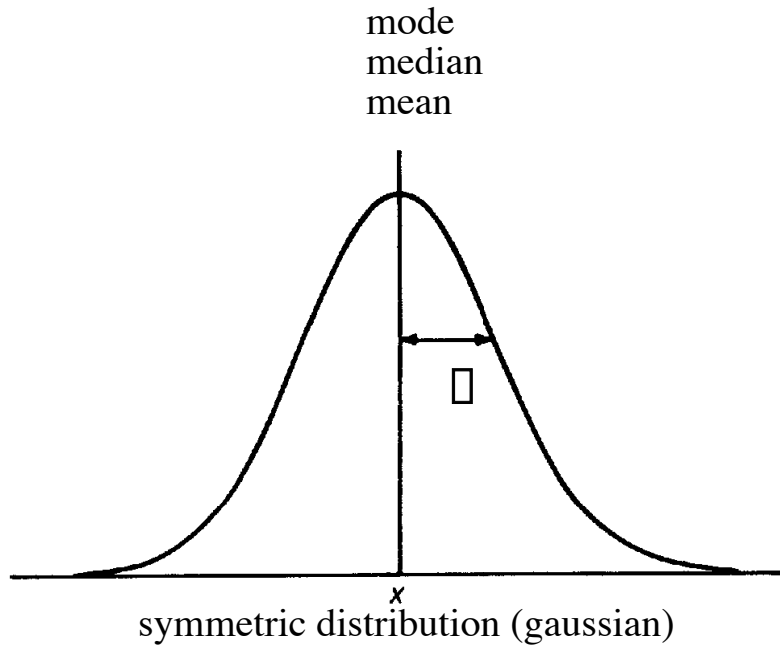
Mean	Mode	Median	Variance
average	most probable	50% point	width of distribution
$\bar{x} = \int_{-\infty}^{+\infty} x f(x) dx$	$\left. \frac{\partial f(x)}{\partial x} \right _{x=a} = 0$	$0.5 = \int_{-\infty}^a f(x) dx$	$\sigma^2 = \int_{-\infty}^{+\infty} f(x) (x - \bar{x})^2 dx$

- for a discrete distribution, the mean and variance are defined by:

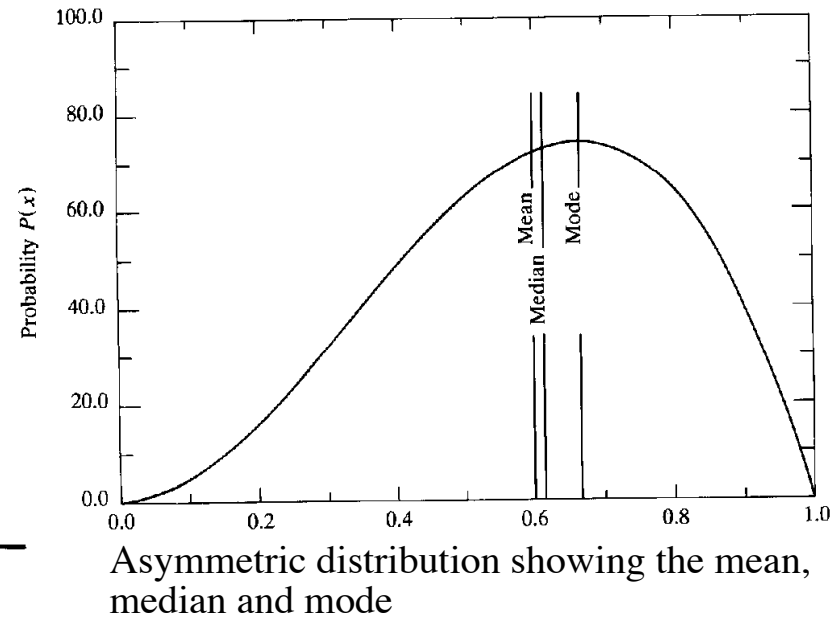
$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

- Some continuous *pdf*:



For a Gaussian pdf,  
the mean, mode,  
and median are  
all at the same  $x$ .



For most pdfs,  
the mean, mode,  
and median are  
at different locations.

- Calculation of mean and variance:

- ◆ example: a discrete data set consisting of three numbers: {1, 2, 3}

- ★ average ( $\bar{x}$ ) is just:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i = \frac{1+2+3}{3} = 2$$

- ★ complication: suppose some measurement are more precise than others.

- ☞ if each measurement  $x_i$  have a weight  $w_i$  associated with it:

$$\bar{x} = \frac{\sum_{i=1}^n x_i w_i}{\sum_{i=1}^n w_i}$$

“weighted average”

- ★ **variance** ( $\sigma^2$ ) or average squared deviation from the mean is just:

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

variance describes the width of the pdf!

- $\sigma$  is called the **standard deviation**

- ☞ rewrite the above expression by expanding the summations:

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \bar{x}^2 - 2 \bar{x} \sum_{i=1}^n x_i$$

$$= \frac{1}{n} \sum_{i=1}^n x_i^2 + \bar{x}^2 - 2 \bar{x}^2$$

$$= \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

$\langle \rangle \equiv \text{average}$

- $n$  in the denominator would be  $n - 1$  if we determined the average ( $\bar{x}$ ) from the data itself.

- ★ using the definition of  $\bar{x}$  from above we have for our example of  $\{1,2,3\}$ :

$$\bar{x}^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2 = 4.67 - 2^2 = 0.67$$

- ★ the case where the measurements have different weights is more complicated:

$$\bar{x}^2 = \frac{\sum_{i=1}^n w_i (x_i - \bar{x})^2}{\sum_{i=1}^n w_i} = \frac{\sum_{i=1}^n w_i x_i^2}{\sum_{i=1}^n w_i} - \bar{x}^2$$

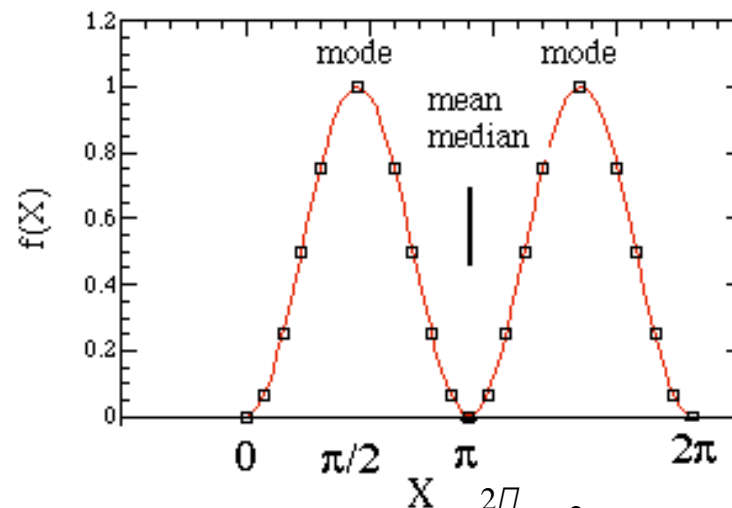
- $\bar{x}$  is the **weighted** mean

- if we calculated  $\bar{x}$  from the data,  $\bar{x}^2$  gets multiplied by a factor  $n/(n-1)$ .

- ◆ example: a continuous probability distribution,  $f(x) = \sin^2 x$  for  $0 \leq x \leq 2\pi$

- ★ has two modes!

- ★ has same mean and median, but differ from the mode(s).



- ★  $f(x)$  is not properly normalized:  $\int_0^{2\pi} \sin^2 x dx = \pi \neq 1$

- ☞ normalized pdf:  $f(x) = \sin^2 x / \int_0^{2\pi} \sin^2 x dx = \frac{1}{\pi} \sin^2 x$

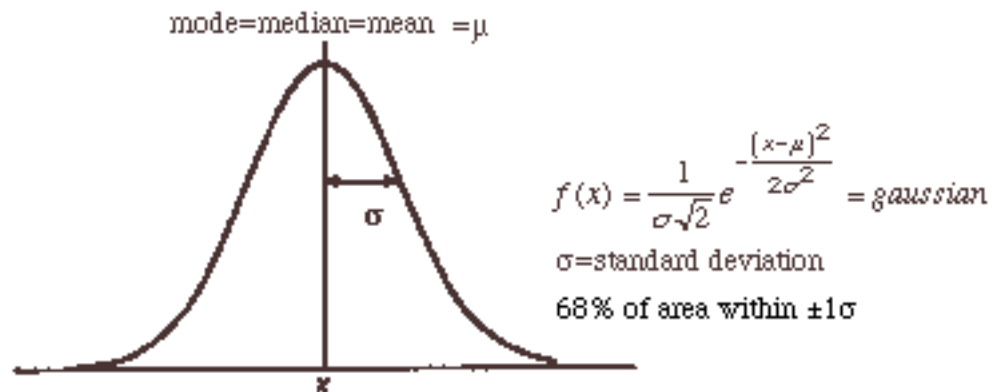
- ★ for continuous probability distributions, the mean, mode, and median are calculated using either integrals or derivatives:

$$\bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x \sin^2 x dx = \pi$$

$$\text{mode} : \frac{\partial}{\partial x} \sin^2 x = 0 \Rightarrow \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{median} : \frac{1}{2\pi} \int_0^{2\pi} \sin^2 x dx = \frac{1}{2} \Rightarrow \pi = \bar{x}$$

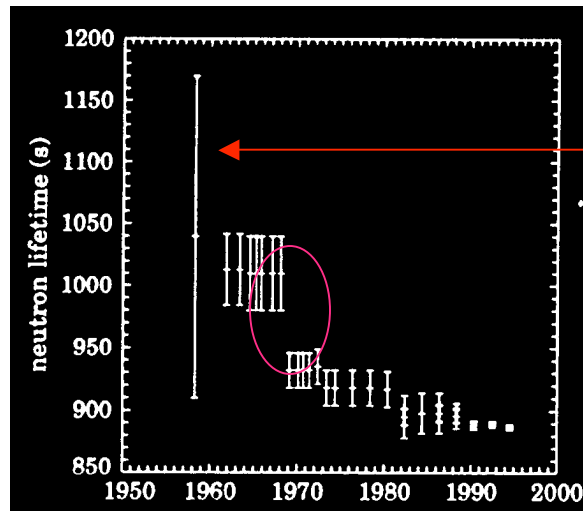
- ◆ example: Gaussian distribution function, a continuous probability distribution





## Accuracy and Precision:

- Accuracy: The accuracy of an experiment refers to how close the experimental measurement is to the true value of the quantity being measured.
- Precision: This refers to how well the experimental result has been determined, without regard to the true value of the quantity being measured.
  - ◆ just because an experiment is precise it does not mean it is accurate!!
  - ◆ measurements of the neutron lifetime over the years:



The size of bar reflects the precision of the experiment

- ★ steady increase in precision but any of these measurements accurate?

## Measurement Errors (Uncertainties)

- Use results from probability and statistics as a way of indicating how “good” a measurement is.
  - ◆ most common quality indicator:  
relative precision = [uncertainty of measurement]/measurement
    - ★ example: we measure a table to be 10 inches with uncertainty of 1 inch.  
relative precision =  $1/10 = 0.1$  or 10% (% relative precision)
  - ◆ uncertainty in measurement is usually square root of variance:  
 $\sigma$  = standard deviation
    - ★ usually calculated using the technique of “propagation of errors”.

## Statistics and Systematic Errors

- Results from experiments are often presented as:

$$N \pm XX \pm YY$$

$N$ : value of quantity measured (or determined) by experiment.

$XX$ : statistical error, usually assumed to be from a Gaussian distribution.

With the assumption of Gaussian statistics we can say (calculate) something about how well our experiment agrees with other experiments and/or theories.

Expect an 68% chance that the true value is between  $N - XX$  and  $N + XX$ .

$YY$ : systematic error. Hard to estimate, distribution of errors usually not known.

- ◆ examples: mass of proton =  $0.9382769 \pm 0.0000027$  GeV

$$\text{mass of W boson} = 80.8 \pm 1.5 \pm 2.4 \text{ GeV}$$

- What's the difference between statistical and systematic errors?
  - ◆ statistical errors are “random” in the sense that if we repeat the measurement enough times:
 
$$\overline{XX} \rightarrow 0$$
  - ◆ systematic errors do **not**  $\rightarrow 0$  with repetition.
    - ★ examples of sources of systematic errors:
      - voltmeter not calibrated properly
      - a ruler not the length we think is (meter stick might really be < meter!)
  - ◆ because of systematic errors, an experimental result can be precise, but not accurate!
- How do we combine systematic and statistical errors to get one estimate of precision?
  - ☞ **big problem!**
  - ◆ two choices:
    - ★  $\Delta_{\text{tot}} = \overline{XX} + \overline{YY}$  add them linearly
    - ★  $\Delta_{\text{tot}} = (\overline{XX}^2 + \overline{YY}^2)^{1/2}$  add them in quadrature
- Some other ways of quoting experimental results
  - ◆ lower limit: “the mass of particle  $X$  is  $> 100 \text{ GeV}$ ”
  - ◆ upper limit: “the mass of particle  $X$  is  $< 100 \text{ GeV}$ ”
  - ◆ asymmetric errors: mass of particle  $X = 100^{+4}_{-3} \text{ GeV}$