Lecture 2
Binomial and Poisson Probability Distributions

Binomial Probability Distribution

- Consider a situation where there are only two possible outcomes (a Bernoulli trial)
  - Example:
    - flipping a coin
      - head or tail
    - rolling a dice
      - 6 or not 6 (i.e. 1, 2, 3, 4, 5)
  - Label the possible outcomes by the variable k
    - find the probability $P(k)$ for event k to occur
  - Since k can take on only 2 values we define those values as:
    $k = 0$ or $k = 1$
    - let $P(k = 0) = q$ (remember $0 \le q \le 1$)
    - *something* must happen
      - $P(k = 0) + P(k = 1) = 1$
      - $P(k = 1) = p = 1 - q$
    - write the probability distribution $P(k)$ as:
      $P(k) = p^k q^{1-k}$ (Bernoulli distribution)
    - coin toss: define probability for a head as $P(1)$
      - $P(1) = 0.5$
    - dice rolling: define probability for a six to be rolled as $P(1)$
      - $P(1) = 1/6$
      - $P(0) = 5/6$ (not a six)

- What is the mean (μ) of $P(k)$?

$$\mu = \frac{\sum_{k=0}^{1} kP(k)}{\sum_{k=0}^{1} P(k)} = \frac{0 \cdot q + 1 \cdot p}{q + p} = p$$

- What is the Variance (σ^2) of $P(k)$?

$$\sigma^2 = \frac{\sum_{k=0}^{1} k^2 P(k)}{\sum_{k=0}^{1} P(k)} - \mu^2 = 0^2 \cdot P(0) + 1^2 \cdot P(1) - \mu^2 = p - p^2 = p(1 - p) = pq$$

- Suppose we have N trials (e.g. we flip a coin N times)
  - what is the probability to get m successes (= heads)?
- Consider tossing a coin twice. The possible outcomes are:
  - ★ no heads: $P(m = 0) = q^2$
  - ★ one head: $P(m = 1) = qp + pq$ (toss 1 is a tail, toss 2 is a head or toss 1 is head, toss 2 is a tail)
    $= 2pq$ ← two outcomes because we don't care which of the tosses is a head
  - ★ two heads: $P(m = 2) = p^2$
  - ★ $P(0) + P(1) + P(2) = q^2 + 2pq + p^2 = (q + p)^2 = 1$
- We want the probability distribution function $P(m, N, p)$ where:
  - m = number of success (e.g. number of heads in a coin toss)
  - N = number of trials (e.g. number of coin tosses)
  - p = probability for a success (e.g. 0.5 for a head)

- If we look at the three choices for the coin flip example, each term is of the form:
  $C_m p^m q^{N-m}$ $m = 0, 1, 2$, $N = 2$ for our example, $q = 1 - p$ always!
  - coefficient C_m takes into account the number of ways an outcome can occur regardless of order
  - for $m = 0$ or 2 there is only one way for the outcome (both tosses give heads or tails): $C_0 = C_2 = 1$
  - for $m = 1$ (one head, two tosses) there are two ways that this can occur: $C_1 = 2$.
- Binomial coefficients: number of ways of taking N things m at time

$$C_{N,m} = \binom{N}{m} = \frac{N!}{m!(N-m)!}$$

  - $0! = 1! = 1$, $2! = 1\cdot 2 = 2$, $3! = 1\cdot 2\cdot 3 = 6$, $m! = 1\cdot 2\cdot 3 \cdots m$
  - Order of things is not important
    - e.g. 2 tosses, one head case ($m = 1$)
      - we don't care if toss 1 produced the head or if toss 2 produced the head
  - Unordered groups such as our example are called *combinations*
  - Ordered arrangements are called *permutations*
  - For N distinguishable objects, if we want to group them m at a time, the number of permutations:

$$P_{N,m} = \frac{N!}{(N-m)!}$$

    - example: If we tossed a coin twice ($N = 2$), there are two ways for getting one head ($m = 1$)
    - example: Suppose we have 3 balls, one white, one red, and one blue.
      - Number of possible pairs we could have, keeping track of order is 6 (rw, wr, rb, br, wb, bw):

$$P_{3,2} = \frac{3!}{(3-2)!} = 6$$

      - If order is not important (rw = wr), then the binomial formula gives

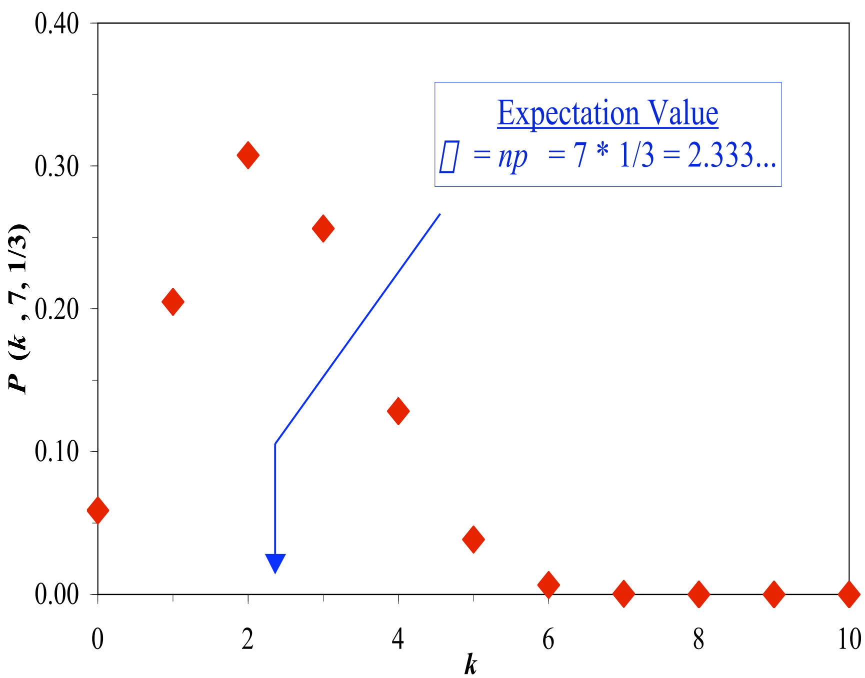
$$C_{3,2} = \frac{3!}{2!(3-2)!} = 3$$

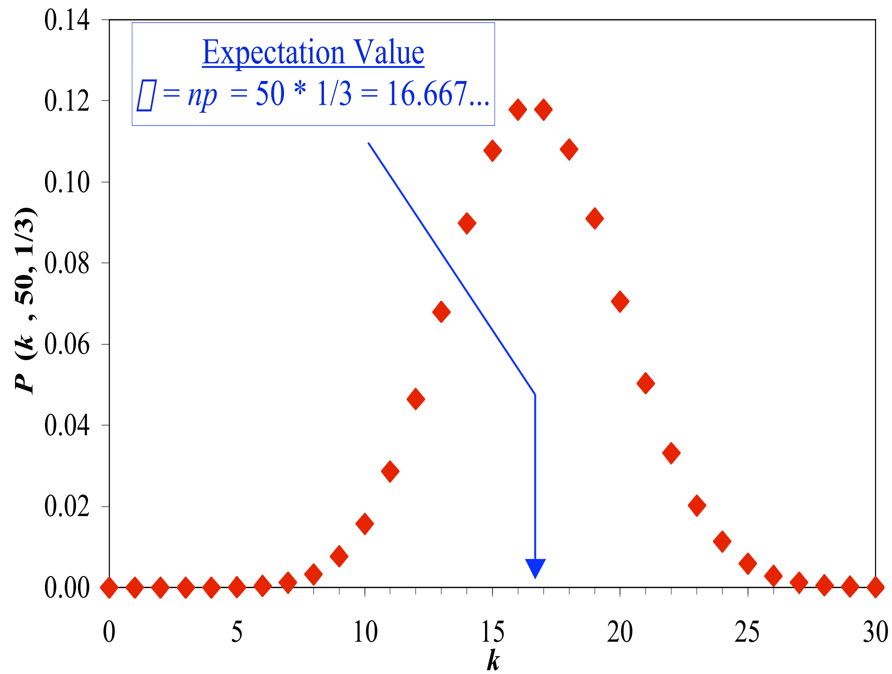
number of two-color combinations

- Binomial distribution: the probability of *m* success out of *N* trials:

$$P(m,N,p) = C_{N,m}\, p^m q^{N-m} = \binom{N}{m} p^m q^{N-m} = \frac{N!}{m!(N-m)!} p^m q^{N-m}$$

  - *p* is probability of a success and *q* = 1 - *p* is probability of a failure
  - Consider a game where the player bats 4 times:
    - probability of 0/4 = $(0.67)^4$ = 20%
    - probability of 1/4 = $[4!/(3!1!)](0.33)^1(0.67)^3$ = 40%
    - probability of 2/4 = $[4!/(2!2!)](0.33)^2(0.67)^2$ = 29%
    - probability of 3/4 = $[4!/(1!3!)](0.33)^3(0.67)^1$ = 10%
    - probability of 4/4 = $[4!/(0!4!)](0.33)^4(0.67)^0$ = 1%
    - probability of getting at least one hit = 1 - *P*(0) = 0.8





- To show that the binomial distribution is properly normalized, use Binomial Theorem:

$$(a+b)^k = \sum_{l=0}^{k} \binom{k}{l} a^{k-l} b^l$$

$$\sum_{m=0}^{N} P(m,N,p) = \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} = (p+q)^N = 1$$

  ☞ binomial distribution is properly normalized

- Mean of binomial distribution:

$$\mu = \frac{\sum_{m=0}^{N} mP(m,N,p)}{\sum_{m=0}^{N} P(m,N,p)} = \sum_{m=0}^{N} mP(m,N,p) = \sum_{m=0}^{N} m\binom{N}{m} p^m q^{N-m}$$

  ★ A cute way of evaluating the above sum is to take the derivative:

$$\frac{\partial}{\partial p}\left[\sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m}\right] = 0$$

$$\sum_{m=0}^{N} m\binom{N}{m} p^{m-1} q^{N-m} - \sum_{m=0}^{N} \binom{N}{m} p^m (N-m)(1-p)^{N-m-1} = 0$$

$$p^{-1}\sum_{m=0}^{N} m\binom{N}{m} p^m q^{N-m} = N(1-p)^{-1}\sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} - (1-p)^{-1}\sum_{m=0}^{N} m\binom{N}{m} p^m (1-p)^{N-m}$$

$$p^{-1}\mu = N(1-p)^{-1}\cdot 1 - (1-p)^{-1}\mu$$

$$\mu = Np$$

- Variance of binomial distribution (obtained using similar trick):

$$\sigma^2 = \frac{\sum_{m=0}^{N}(m-\mu)^2 P(m,N,p)}{\sum_{m=0}^{N} P(m,N,p)} = Npq$$

- Example: Suppose you observed m special events (success) in a sample of N events
  - measured probability ("efficiency") for a special event to occur:

$$\varepsilon = \frac{m}{N}$$

  - error on the probability ("error on the efficiency"):

$$\sigma_\varepsilon = \frac{\sigma_m}{N} = \frac{\sqrt{Npq}}{N} = \frac{\sqrt{N\varepsilon(1-\varepsilon)}}{N} = \sqrt{\frac{\varepsilon(1-\varepsilon)}{N}}$$

    - ☞ sample (N) should be as large as possible to reduce uncertainty in the probability measurement
- Example: Suppose a baseball player's batting average is 0.33 (1 for 3 on average).
  - Consider the case where the player either gets a hit or makes an out (forget about walks here!).

    probability for a hit: $p = 0.33$

    probability for "no hit": $q = 1 - p = 0.67$
  - On average how many hits does the player get in 100 at bats?

    $\mu = Np = 100 \cdot 0.33 = 33$ hits
  - What's the standard deviation for the number of hits in 100 at bats?

    $\sigma = (Npq)^{1/2} = (100 \cdot 0.33 \cdot 0.67)^{1/2} \approx 4.7$ hits
    - ☞ we expect ≈ 33 ± 5 hits per 100 at bats

Poisson Probability Distribution

- A widely used discrete probability distribution
- Consider the following conditions:
  - ★ p is very small and approaches 0
    - ◆ example: a 100 sided dice instead of a 6 sided dice, $p = 1/100$ instead of 1/6
    - ◆ example: a 1000 sided dice, $p = 1/1000$
  - ★ N is very large and approaches ∞
    - ◆ example: throwing 100 or 1000 dice instead of 2 dice
  - ★ product Np is finite
- Example: radioactive decay
  - ★ Suppose we have 25 mg of an element
    - ☞ very large number of atoms: $N \approx 10^{20}$
  - ★ Suppose the lifetime of this element $\tau = 10^{12}$ years $\approx 5\text{x}10^{19}$ seconds
    - ☞ probability of a given nucleus to decay in one second is very small: $p = 1/\tau = 2\text{x}10^{-20}$/sec
    - ☞ $Np = 2$/sec **finite!**
    - ☞ number of counts in a time interval is a Poisson process
- Poisson distribution can be derived by taking the appropriate limits of the binomial distribution

$$P(m,N,p) = \frac{N!}{m!(N-m)!} p^m q^{N-m}$$

$$\frac{N!}{(N-m)!} = \frac{N(N-1)\cdots(N-m+1)(N-m)!}{(N-m)!} = N^m$$

$$q^{N-m} = (1-p)^{N-m} = 1 - p(N-m) + \frac{p^2(N-m)(N-m-1)}{2!} + \cdots \approx 1 - pN + \frac{(pN)^2}{2!} + \cdots \approx e^{-pN}$$

$$P(m,N,p) = \frac{N^m}{m!} p^m e^{-pN}$$

Let $\lambda = Np$

$$P(m,\lambda) = \frac{e^{-\lambda}\lambda^m}{m!}$$

$$\sum_{m=0}^{m=\infty} \frac{e^{-\lambda}\lambda^m}{m!} = e^{-\lambda}\sum_{m=0}^{m=\infty} \frac{\lambda^m}{m!} = e^{-\lambda}e^{\lambda} = 1$$

Poisson distribution is normalized

- m is always an integer ≥ 0
- λ does not have to be an integer

★ It is easy to show that:

$\mu = Np$ = mean of a Poisson distribution

$\sigma^2 = Np = \lambda$ = variance of a Poisson distribution

mean and variance are the same number

- Radioactivity example with an average of 2 decays/sec:

★ What's the probability of zero decays in one second?

$$p(0,2) = \frac{e^{-2}2^0}{0!} = \frac{e^{-2}\cdot 1}{1} = e^{-2} = 0.135 \rightarrow 13.5\%$$

★ What's the probability of more than one decay in one second?

$$p(>1,2) = 1 - p(0,2) - p(1,2) = 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} = 1 - e^{-2} - 2e^{-2} = 0.594 \rightarrow 59.4\%$$

★ Estimate the most probable number of decays/sec?

$$\left.\frac{\partial}{\partial m} P(m,\lambda)\right|_{m^*} = 0$$

- To solve this problem its convenient to maximize $\ln P(m, \mu)$ instead of $P(m, \mu)$.

$$\ln P(m,\mu) = \ln\left[\frac{e^{-\mu}\mu^m}{m!}\right] = -\mu + m\ln\mu - \ln m!$$

- In order to handle the factorial when take the derivative we use *Stirling's Approximation*:

$$\ln m! \approx m\ln m - m$$

$$\frac{\partial}{\partial m}\ln P(m,\mu) = \frac{\partial}{\partial m}(-\mu + m\ln\mu - \ln m!)$$

$$\approx \frac{\partial}{\partial m}(-\mu + m\ln\mu - m\ln m + m)$$

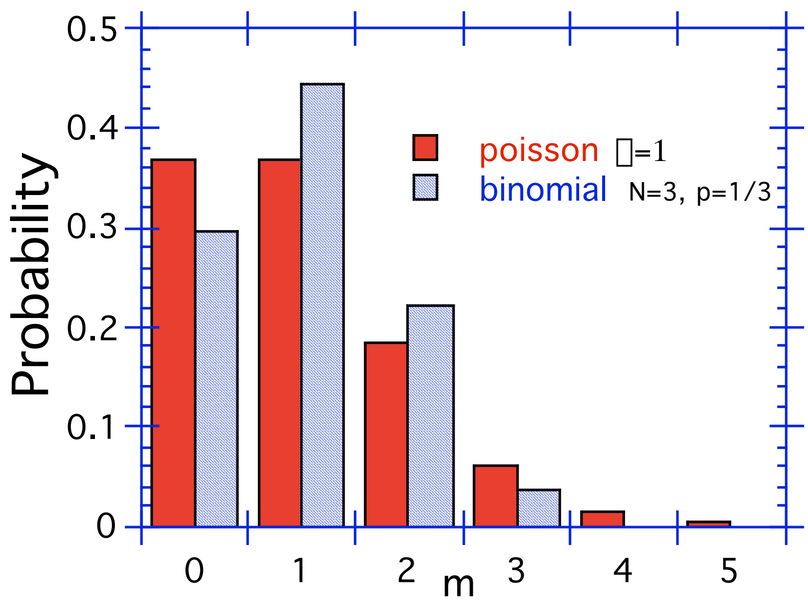
$$= \ln\mu - \ln m - m\frac{1}{m} + 1$$

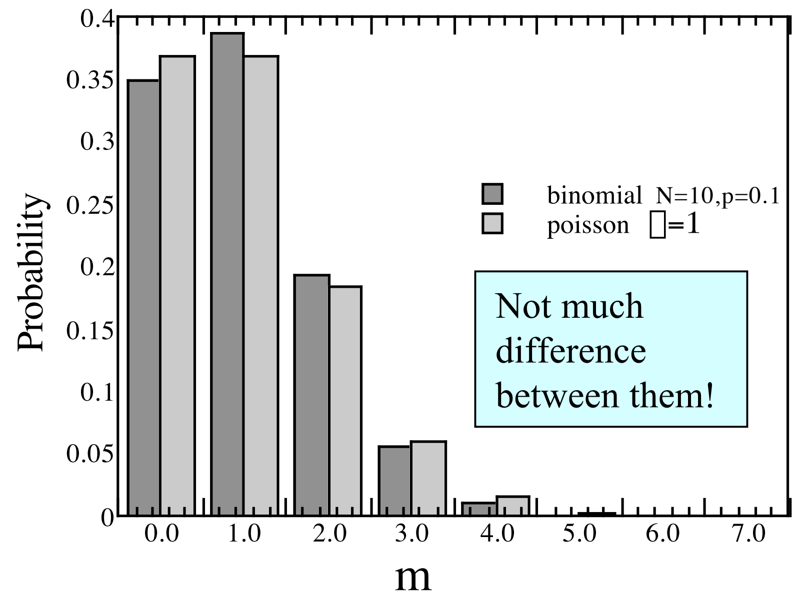
$$= 0$$

$$m^* = \mu$$

- ☞ The most probable value for m is just the average of the distribution
- ☞ If you observed m events in an experiment, the error on m is $\boxed{\sigma = \sqrt{\mu} = \sqrt{m}}$
- This is only approximate since Stirlings Approximation is only valid for large m.
- Strictly speaking m can only take on integer values while μ is not restricted to be an integer.

Comparison of Binomial and Poisson distributions with mean $\mu = 1$





For large N: Binomial distribution looks like a Poisson of the same mean