Lecture 4
Propagation of errors

Introduction

- Example: we measure the current (I) and resistance (R) of a resistor.
  - Ohm's law:

    V = IR
  - If we know the uncertainties (e.g. standard deviations) in I and R, what is the uncertainty in V?
- Given a functional relationship between several measured variables (x, y, z),

  $$Q = f(x, y, z)$$
  - What is the uncertainty in Q if the uncertainties in x, y, and z are known?
    - To answer this question we use a technique called Propagation of Errors.
  - Usually when we talk about uncertainties in a measured variable such as x, we assume:
    - the value of x represents the mean of a Gaussian distribution
    - the uncertainty in x is the standard deviation (σ) of the Gaussian distribution
    - not all measurements can be represented by Gaussian distributions (more on that later)

Propagation of Error Formula

- To calculate the variance in Q as a function of the variances in x and y we use the following:

  $$\sigma_Q^2 = \sigma_x^2 \left(\frac{\partial Q}{\partial x}\right)^2 + \sigma_y^2 \left(\frac{\partial Q}{\partial y}\right)^2 + 2\sigma_{xy} \left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial Q}{\partial y}\right)$$
  - If the variables x and y are uncorrelated ($\sigma_{xy} = 0$), the last term in the above equation is zero.
  - Assume we have several measurement of the quantities x (e.g. $x_1, x_2 ... x_N$) and y (e.g. $y_1, y_2 ... y_N$).
    - The average of x and y:

      $$\mu_x = \frac{1}{N}\sum_{i=1}^{N} x_i \text{ and } \mu_y = \frac{1}{N}\sum_{i=1}^{N} y_i$$

- define: $Q_i \equiv f(x_i, y_i)$

  $Q \equiv f(\mu_x, \mu_y)$ evaluated at the average values
- expand Q_i about the average values:

$$Q_i = f(\mu_x, \mu_y) + (x_i - \mu_x)\left(\frac{\partial Q}{\partial x}\right)_{\mu_x} + (y_i - \mu_y)\left(\frac{\partial Q}{\partial y}\right)_{\mu_y} + \text{higher order terms}$$

- assume the measured values are close to the average values
  - ☞ neglect the higher order terms:

$$Q_i - Q = (x_i - \mu_x)\left(\frac{\partial Q}{\partial x}\right)_{\mu_x} + (y_i - \mu_y)\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}$$

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}(Q_i - Q)^2$$

$$= \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_x)^2\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}^2 + \frac{1}{N}\sum_{i=1}^{N}(y_i - \mu_y)^2\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}^2 + \frac{2}{N}\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y)\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}$$

$$= \sigma_x^2\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}^2 + \sigma_y^2\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}^2 + 2\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y)$$

- If the measurements are uncorrelated
  - ☞ the summation in the above equation is zero

$$\sigma_Q^2 = \sigma_x^2\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}^2 + \sigma_y^2\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}^2$$

uncorrelated errors

- If x and y are correlated, define σ_{xy} as:

$$\sigma_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_x)(y_i - \mu_y)$$

$$\sigma_Q^2 = \sigma_x^2\left(\frac{\partial Q}{\partial x}\right)^2_{\mu_x} + \sigma_y^2\left(\frac{\partial Q}{\partial y}\right)^2_{\mu_y} + 2\left(\frac{\partial Q}{\partial x}\right)_{\mu_x}\left(\frac{\partial Q}{\partial y}\right)_{\mu_y}\sigma_{xy}$$

correlated errors

- Example: Power in an electric circuit.

  $P = I^2R$

  - Let $I = 1.0 \pm 0.1$ amp and $R = 10 \pm 1\ \Omega$
    - $P = 10$ watts
  - calculate the variance in the power using propagation of errors

$$\sigma_P^2 = \sigma_I^2\left(\frac{\partial P}{\partial I}\right)^2_{I=1} + \sigma_R^2\left(\frac{\partial P}{\partial R}\right)^2_{R=10} = \sigma_I^2(2IR)^2 + \sigma_R^2(I^2)^2 = (0.1)^2(2\cdot 1\cdot 10)^2 + (1)^2(1^2)^2 = 5 \text{ watts}^2$$

  - $P = 10 \pm 2$ watts
    - If the true value of the power was 10 W and we measured it many times with an uncertainty (σ) of ± 2 W and Gaussian statistics apply
      - 68% of the measurements would lie in the range [8,12] W
  - Sometimes its convenient to put the above calculation in terms of relative errors:

$$\frac{\sigma_P^2}{P^2} = \frac{\sigma_I^2}{P^2}\left(\frac{\partial P}{\partial I}\right)^2 + \frac{\sigma_R^2}{P^2}\left(\frac{\partial P}{\partial R}\right)^2 = \frac{4\sigma_I^2}{I^2} + \frac{\sigma_R^2}{R^2} = 4\left(\frac{0.1}{1}\right)^2 + \left(\frac{1}{10}\right)^2 = 0.1^2(4+1)$$

    - the uncertainty in the *current* dominates the uncertainty in the power
      - current must be measured more precisely to greatly reduce the uncertainty in the power

- Example: The error in the average.
  - The average of several measurements each with the same uncertainty (σ) is given by:

$$\mu = \frac{1}{n}(x_1 + x_2 + \ldots x_n)$$

$$\sigma_\mu^2 = \sigma_{x_1}^2\left(\frac{\partial\mu}{\partial x_1}\right)^2 + \sigma_{x_2}^2\left(\frac{\partial\mu}{\partial x_2}\right)^2 + \ldots\sigma_{x_n}^2\left(\frac{\partial\mu}{\partial x_n}\right)^2 = \sigma^2\left(\frac{1}{n}\right)^2 + \sigma^2\left(\frac{1}{n}\right)^2 + \ldots\sigma\left(\frac{1}{n}\right)^2 = n\sigma^2\left(\frac{1}{n}\right)^2$$

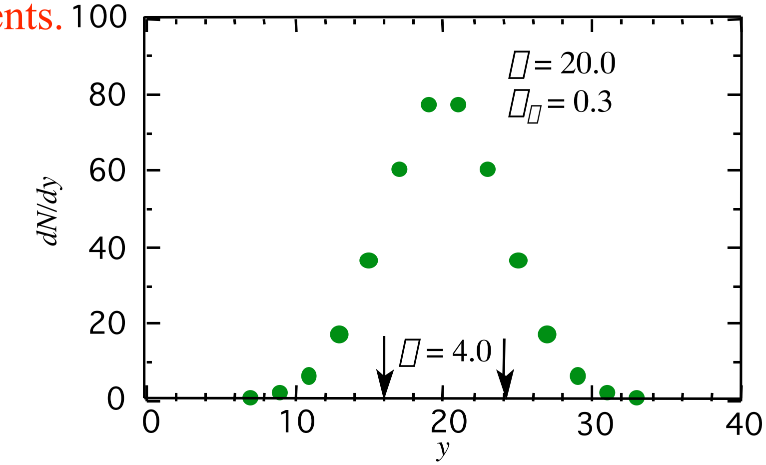
$$\sigma_\mu = \frac{\sigma}{\sqrt{n}}$$

"error in the mean"

☞ We can determine the mean better by combining measurements.

☞ The precision only increases as the square root of the number of measurements.

- Do not confuse σ_μ with σ!
- σ is related to the width of the *pdf* (e.g. Gaussian) that the measurements come from.
- σ does not get smaller as we combine measurements.



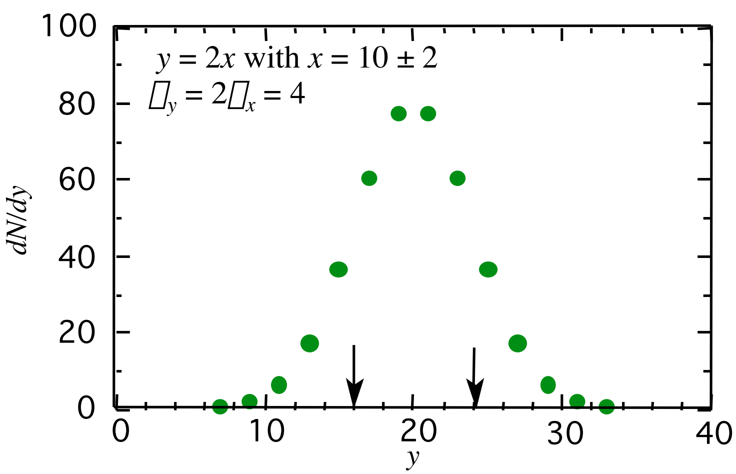
Problem in the Propagation of Errors

- In calculating the variance using propagation of errors
  - we usually assume the error in measured variable (e.g. x) is Gaussian
- If x is described by a Gaussian distribution
  - $f(x)$ may not be described by a Gaussian distribution!

- *What does the standard deviation that we calculate from propagation of errors mean?*
  - Example: The new distribution is Gaussian.
    - Let $y = Ax$, with A = a constant and x a Gaussian variable.
      - ☞ $\mu_y = A\mu_x$ and $\sigma_y = A\sigma_x$
    - Let the probability distribution for x be Gaussian:

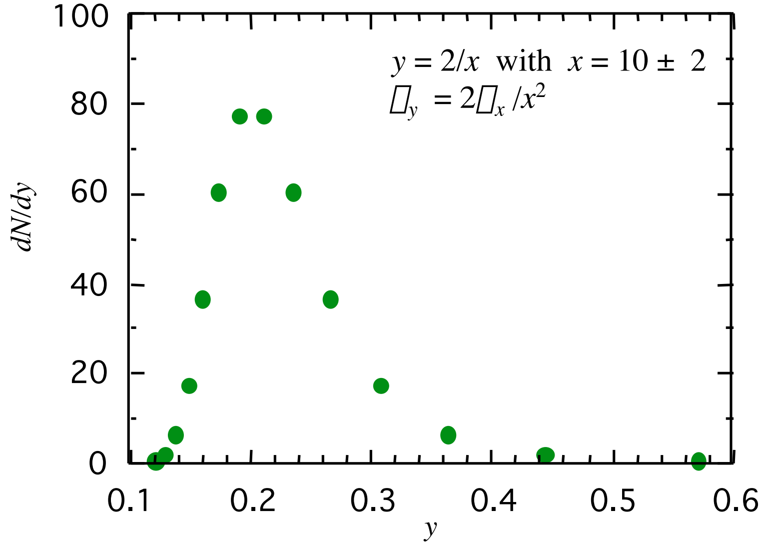
$$p(x,\mu_x,\sigma_x)dx = \frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}dx = \frac{1}{\frac{\sigma_y}{A}\sqrt{2\pi}}e^{-\frac{\left(\frac{y}{A}-\frac{\mu_y}{A}\right)^2}{2\left(\frac{\sigma_y}{A}\right)^2}}\frac{1}{A}dy = \frac{1}{\sigma_y\sqrt{2\pi}}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}dy = p(y,\mu_y,\sigma_y)dy$$

  - ☞ The new probability distribution for y, $p(y, \mu_y, \sigma_y)$, is also described by a Gaussian.



Start with a Gaussian with $\mu = 10$, $\sigma = 2$
Get another Gaussian with $\mu = 20$, $\sigma = 4$

- Example: When the new distribution is non-Gaussian: $y = 2/x$.
  - The transformed probability distribution function for y does not have the form of a Gaussian.



Start with a Gaussian with $\mu = 10$, $\sigma = 2$.
DO NOT get another Gaussian!
Get a *pdf* with $\mu = 0.2$, $\sigma = 0.04$.
This new *pdf* has longer tails than a Gaussian *pdf*:
Prob($y > \mu_y + 5\sigma_y$) = 5×10^{-3}, for Gaussian $\approx 3\times10^{-7}$

- *Unphysical situations can arise if we use the propagation of errors results blindly!*
  - Example: Suppose we measure the volume of a cylinder: $V = \pi R^2 L$.
    - Let R = 1 cm exact, and L = 1.0 ± 0.5 cm.
    - Using propagation of errors:

      $\sigma_V = \pi R^2 \sigma_L = \pi/2$ cm^3.

      $V = \pi \pm \pi/2$ cm^3
    - If the error on V (σ_V) is to be interpreted in the Gaussian sense
      - finite probability (≈ 3%) that the volume (V) is < 0 since V is only 2σ away from than 0!
      - Clearly this is unphysical!
      - Care must be taken in interpreting the meaning of σ_V.