Lecture 3
Gaussian Probability Distribution

Introduction

- Gaussian probability distribution is perhaps the most used distribution in all of science.
  - also called "bell shaped curve" or *normal* distribution
- Unlike the binomial and Poisson distribution, the Gaussian is a continuous distribution:

$$P(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

μ = mean of distribution (also at the same place as mode and median)

σ^2 = variance of distribution

y is a continuous variable ($-\infty \le y \le \infty$)

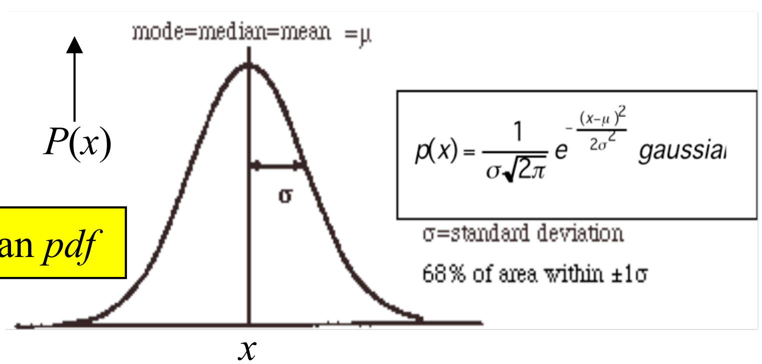
Karl Friedrich Gauss 1777-1855

- Probability (P) of y being in the range $[a, b]$ is given by an integral:

$$P(a < y < b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

  - The integral for arbitrary a and b cannot be evaluated analytically
    - ☞ can be looked up in a table (e.g. Appendixes A and B of Taylor) or use mathematical software.

Plot of Gaussian *pdf*



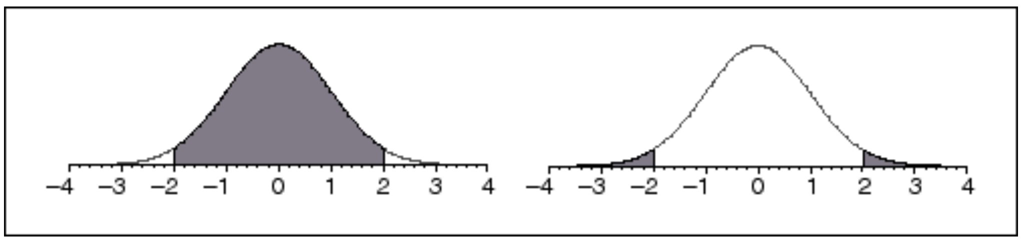
- The total area under the curve is normalized to one.
  - the probability integral:

$$P(-\infty < y < \infty) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = 1$$

- We often talk about a measurement being a certain number of standard deviations (σ) away from the mean (μ) of the Gaussian.
  - We can associate a probability for a measurement to be $|\mu - n\sigma|$ from the mean just by calculating the area outside of this region.

$n\sigma$	Prob. of exceeding $\pm n\sigma$
0.67	0.5
1	0.32
2	0.05
3	0.003
4	0.00006

It is very unlikely (< 0.3%) that a measurement taken at random from a Gaussian *pdf* will be more than ± 3σ from the true mean of the distribution.



95% of area within 2σ   Only 5% of area outside 2σ

Relationship between Gaussian and Binomial distribution

- The Gaussian distribution can be derived from the binomial (or Poisson) assuming:
  - *p* is finite
  - *N* is very large
  - we have a continuous variable rather than a discrete variable

- An example illustrating the small difference between the two distributions under the above conditions:
  - Consider tossing a coin 10,000 time.
    - $p(\text{heads}) = 0.5$
    - $N = 10{,}000$
    - For a binomial distribution:
      - mean number of heads = $\mu = Np = 5000$
      - standard deviation $\sigma = [Np(1 - p)]^{1/2} = 50$
      - The probability to be within $\pm 1\sigma$ for this binomial distribution is:

$$P = \sum_{5000-50}^{5000+50} \frac{10^4!}{(10^4-m)!m!} 0.5^m \, 0.5^{10^4-m} = 0.69$$

    - For a Gaussian distribution:

$$P(\mu - \sigma < y < \mu + \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\sigma}^{\mu+\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = 0.68$$

    - Both distributions give about the same probability!

Central Limit Theorem

- Gaussian distribution is very applicable because of the Central Limit Theorem
- A crude statement of the Central Limit Theorem:
  - Things that are the result of the addition of lots of small effects tend to become Gaussian.
- A more exact statement:
  - Let $Y_1, Y_2, \ldots Y_n$ be an infinite sequence of independent random variables each with the same probability distribution.
  - Suppose that the mean (μ) and variance (σ^2) of this distribution are both finite.

Actually, the *Y*'s can be from different *pdf*'s!

- For any numbers a and b:

$$\lim_{n\to\infty} P\left[a < \frac{Y_1+Y_2+\cdots Y_n - n\mu}{\sigma\sqrt{n}} < b\right] = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}y^2}\,dy$$

- C.L.T. tells us that under a wide range of circumstances the probability distribution that describes the sum of random variables tends towards a Gaussian distribution as the number of terms in the sum $\to\infty$.
- Alternatively:

$$\lim_{n\to\infty} P\left[a < \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} < b\right] = \lim_{n\to\infty} P\left[a < \frac{\bar{Y}-\mu}{\sigma_m} < b\right] = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}y^2}\,dy$$

  - σ_m is sometimes called "the error in the mean" (more on that later).

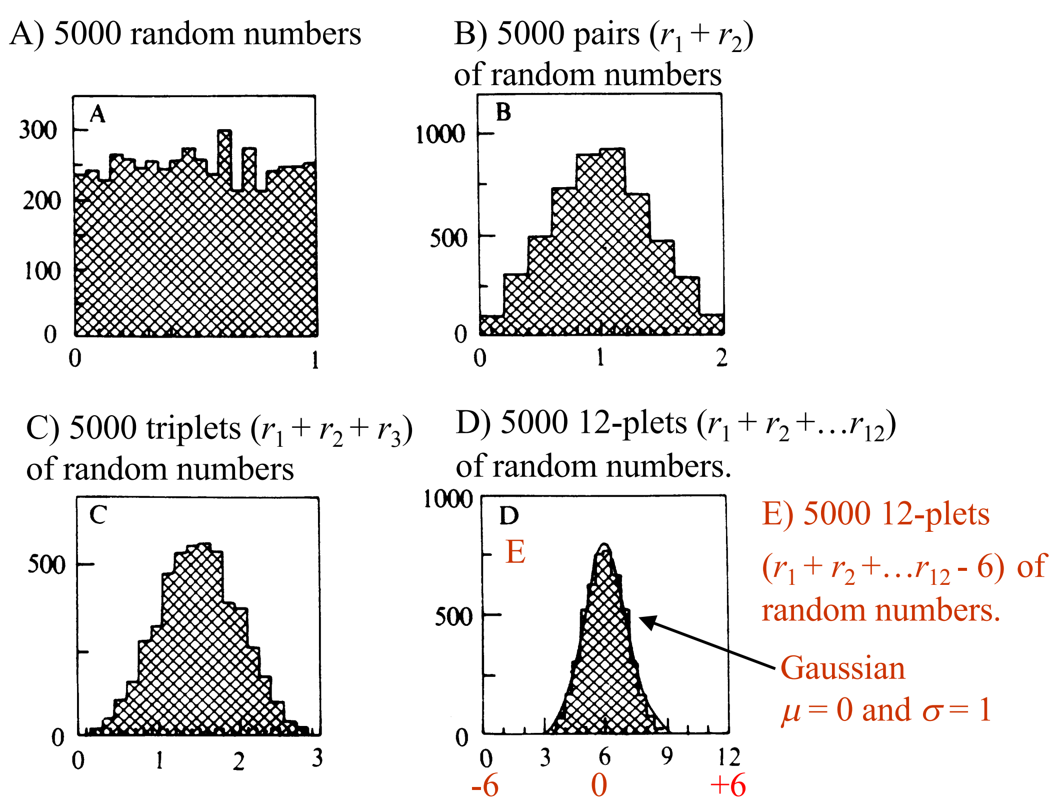
- For CLT to be valid:
  - μ and σ of *pdf* must be finite.
  - No one term in sum should dominate the sum.
- A random variable is not the same as a random number.
  - Devore: *Probability and Statistics for Engineering and the Sciences*:
    - A random variable is any rule that associates a number with each outcome in S
      - S is the set of possible outcomes.
- Recall if y is described by a Gaussian *pdf* with $\mu = 0$ and $\sigma = 1$ then the probability that $a < y < b$ is given by:

$$P(a < y < b) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}y^2}\,dy$$

- The CLT is true even if the Y's are from different *pdf*'s as long as the means and variances are defined for each *pdf*!
  - See Appendix of Barlow for a proof of the Central Limit Theorem.

- Example: Generate a Gaussian distribution using random numbers.
  - Random number generator gives numbers distributed uniformly in the interval [0,1]
    - $\mu = 1/2$ and $\sigma^2 = 1/12$
  - Procedure:
    - Take 12 numbers (r_i) from your computer's random number generator
    - Add them together
    - Subtract 6
    - ☞ Get a number that looks as if it is from a Gaussian *pdf*!

$$P\left[a < \frac{Y_1+Y_2+\cdots Y_n - n\mu}{\sigma\sqrt{n}} < b\right]$$
$$= P\left[a < \frac{\sum_{i=1}^{12} r_i - 12\cdot\frac{1}{2}}{\sqrt{\frac{1}{12}}\cdot\sqrt{12}} < b\right]$$
$$= P\left[-6 < \sum_{i=1}^{12} r_i - 6 < 6\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-6}^{6} e^{-\frac{1}{2}y^2}\,dy$$



Thus the sum of 12 uniform random numbers minus 6 is distributed as if it came from a Gaussian *pdf* with $\mu = 0$ and $\sigma = 1$.

12 is close to ∞!

- Example: A watch makes an error of at most ±1/2 minute per day.
  After one year, what's the probability that the watch is accurate to within ±25 minutes?
  - Assume that the daily errors are uniform in [-1/2, 1/2].
    - For each day, the average error is zero and the standard deviation $1/\sqrt{12}$ minutes.
    - The error over the course of a year is just the addition of the daily error.
    - Since the daily errors come from a uniform distribution with a well defined mean and variance
      - Central Limit Theorem is applicable:

$$\lim_{n\to\infty} P\left[a < \frac{Y_1+Y_2+\cdots Y_n - n\mu}{\sigma\sqrt{n}} < b\right] = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}y^2}\, dy$$

      - The upper limit corresponds to +25 minutes:

$$b = \frac{Y_1+Y_2+\cdots Y_n - n\mu}{\sigma\sqrt{n}} = \frac{25-365\times 0}{\sqrt{\frac{1}{12}}\cdot\sqrt{365}} = 4.5$$

      - The lower limit corresponds to -25 minutes:

$$a = \frac{Y_1+Y_2+\cdots Y_n - n\mu}{\sigma\sqrt{n}} = \frac{-25-365\times 0}{\sqrt{\frac{1}{12}}\cdot\sqrt{365}} = -4.5$$

      - The probability to be within ± 25 minutes:

$$P = \frac{1}{\sqrt{2\pi}}\int_{-4.5}^{4.5} e^{-\frac{1}{2}y^2}\, dy = 0.999997 = 1 - 3\times 10^{-6}$$

      - <u>less than three in a million chance</u> that the watch will be off by more than 25 minutes in a year!

- Example: The daily income of a "card shark" has a uniform distribution in the interval [-\$40,\$50]. What is the probability that s/he wins more than \$500 in 60 days?
  - Lets use the CLT to estimate this probability:

$$\lim_{n\to\infty} P\left[a < \frac{Y_1+Y_2+\cdots Y_n - n\mu}{\sigma\sqrt{n}} < b\right] = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}y^2}\,dy$$

  - The probability distribution of daily income is uniform, $p(y) = 1$.
    - ☞ need to be normalized in computing the average daily winning (μ) and its standard deviation (σ).

$$\mu = \frac{\int_{-40}^{50} y p(y) dy}{\int_{-40}^{50} p(y) dy} = \frac{\frac{1}{2}[50^2-(-40)^2]}{50-(-40)} = 5$$

$$\sigma^2 = \frac{\int_{-40}^{50} y^2 p(y) dy}{\int_{-40}^{50} p(y) dy} - \mu^2 = \frac{\frac{1}{3}[50^3-(-40)^3]}{50-(-40)} - 25 = 675$$

  - The lower limit of the winning is \$500:

$$a = \frac{Y_1+Y_2+\cdots Y_n - n\mu}{\sigma\sqrt{n}} = \frac{500-60\times5}{\sqrt{675}\cdot\sqrt{60}} = \frac{200}{201} = 1$$

  - The upper limit is the maximum that the shark could win (50\$/day for 60 days):

$$b = \frac{Y_1+Y_2+\cdots Y_n - n\mu}{\sigma\sqrt{n}} = \frac{3000-60\times5}{\sqrt{675}\cdot\sqrt{60}} = \frac{2700}{201} = 134$$

$$P = \frac{1}{\sqrt{2\pi}}\int_1^{134} e^{-\frac{1}{2}y^2}\,dy = \frac{1}{\sqrt{2\pi}}\int_1^{\infty} e^{-\frac{1}{2}y^2}\,dy = 0.16$$

  ☞ 16% chance to win > \$500 in 60 days