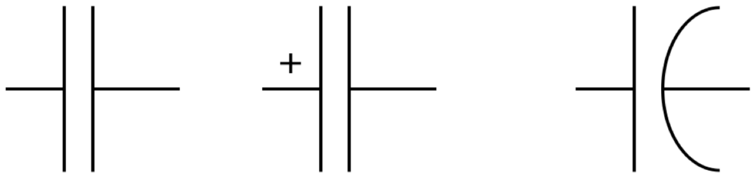
Lecture 2: Capacitors and Inductors

Capacitance:

- Capacitance (C) is defined as the ratio of charge (Q) to voltage (V) on an object:
  - $C = Q/V$ = Coulombs/Volt = Farad
    - ⇨ Capacitance of an object depends on geometry and its dielectric constant.
    - ⇨ For same voltage on two devices, device that can hold more charge has higher capacitance
  - Symbol(s) for capacitors:



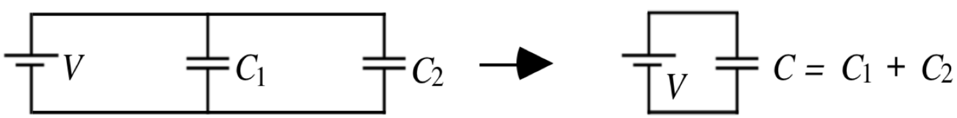
polarized

  - A capacitor is a device that stores electric charge (memory devices).
  - A capacitor is a device that stores energy

$$E = \frac{Q^2}{2C} = \frac{CV^2}{2}$$

  - Capacitors are easy to fabricate in small sizes (μm), use in chips

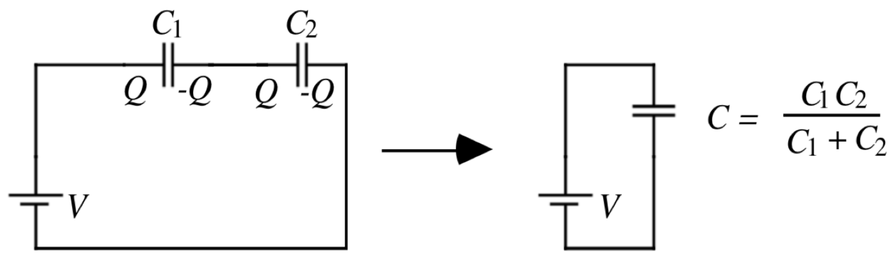
- How to combine capacitance:
  - capacitors in parallel adds like resistors in series:



$$C_{tot} = C_1 + C_2 = \sum C_i$$

Total capacitance is *more* than individual capacitance!

- capacitors in series add like resistors in parallel:



$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} = \sum \frac{1}{C_i}$$

Total capacitance is *less* than individual capacitance!

- Energy and Power in Capacitors
  - How much energy can a "typical" capacitor store?
    - Pick a 4 μF Cap (it would read 4 mF) rated at 3 kV
      $$E = \tfrac{1}{2}CV^2 = \tfrac{1}{2}4\times10^{-6} \bullet 3000^2 = 18 \text{ J}$$
    - This is the same as dropping a 2 kg weight (about 4 pounds) 1 meter
  - How much power is dissipated in a capacitor?

$$Power = \frac{dE}{dt}$$

$$= \frac{d}{dt}\left(\frac{CV^2}{2}\right)$$

$$P = CV\frac{dV}{dt}$$

    - dV/dt must be finite otherwise we source (or sink) an infinite amount of power!

THIS WOULD BE UNPHYSICAL.

- the voltage across a capacitor cannot change instantaneously
  - ⇨ a useful fact when trying to guess the transient (short term) behavior of a circuit
- the voltage across a resistor can change instantaneously
  - ❑ the power dissipated in a resistor does not depend on dV/dt:

    $P = I^2R$ or V^2/R

◆ Why do capacitors come in such small values?

- Example: Calculate the size of a 1 Farad parallel capacitor with air between the plates.

$$C = \frac{k\varepsilon_0 A}{d}$$

k = dielectric constant (= 1 for air)

$\varepsilon_0 = 8.85 \times 10^{-12} N^{-1} m^{-2}$

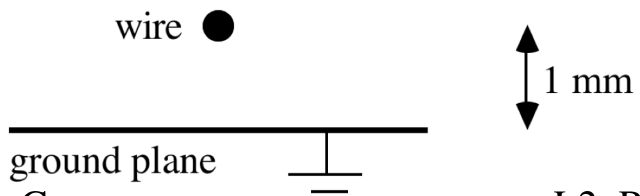
d = distance between plates (assumed 1 mm)

A = area of plates $= 1.1 \times 10^8 m^2$!!!

- ⇨ square plate of 6.5 miles per side
- ❑ breakthroughs in capacitor technologies (driven by laptop/cell phone industries)
- ⇨ 10 μF capacitor in 0402 package (1.0 mm × 0.5 mm × 0.5 mm) costs only 3 pennies

◆ How small can we make capacitors?

- A wire near a ground plane has $C \sim 0.1$ pf $= 10^{-13}$ F.

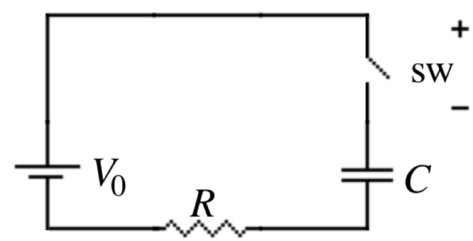


“Stray capacitance” slow down signal or induces cross talk by capacitive coupling!

- Some words to the wise on capacitors and their labeling.
  - Typical capacitors are multiples of micro Farads (10^{-6} F) or pico Farads (10^{-12} F).
    - ⇨ Whenever you see mF it almost always is micro, *not* milli F and *never* mega F.
    - ⇨ picoFarad (10^{-12} F) is sometimes written as pf and pronounced *puff*.
  - no *single* convention for labeling capacitors
    - Many manufacturers have their own labeling scheme (See Horowitz and Hill lab manual).

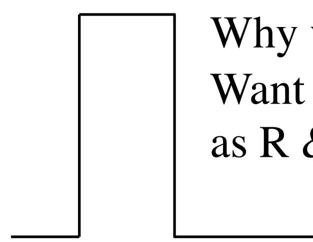
- Resistors and Capacitors
  - Examine voltage and current vs. time for a circuit with one R and one C.
    - Assume that at $t < 0$ all voltages are zero, $V_R = V_C = 0$.
    - At $t \geq 0$ the switch is closed and the battery (V_0) is connected.
    - Apply Kirchhoff's voltage rule:

$$V_0 = V_R + V_C$$
$$= IR + \frac{Q}{C}$$
$$= R\frac{dQ}{dt} + \frac{Q}{C}$$



- Solve the differential equation by differentiating both sides of above equation:

$$\frac{dV_0}{dt} = \frac{1}{C}\frac{dQ}{dt} + R\frac{d^2Q}{dt^2}$$
$$0 = \frac{I}{C} + R\frac{dI}{dt}$$
$$\frac{dI}{dt} = -\frac{I}{RC}$$



Why we care about the DC behavior?
Want to know how a circuit (characterize as R & C) responses to a pulse!

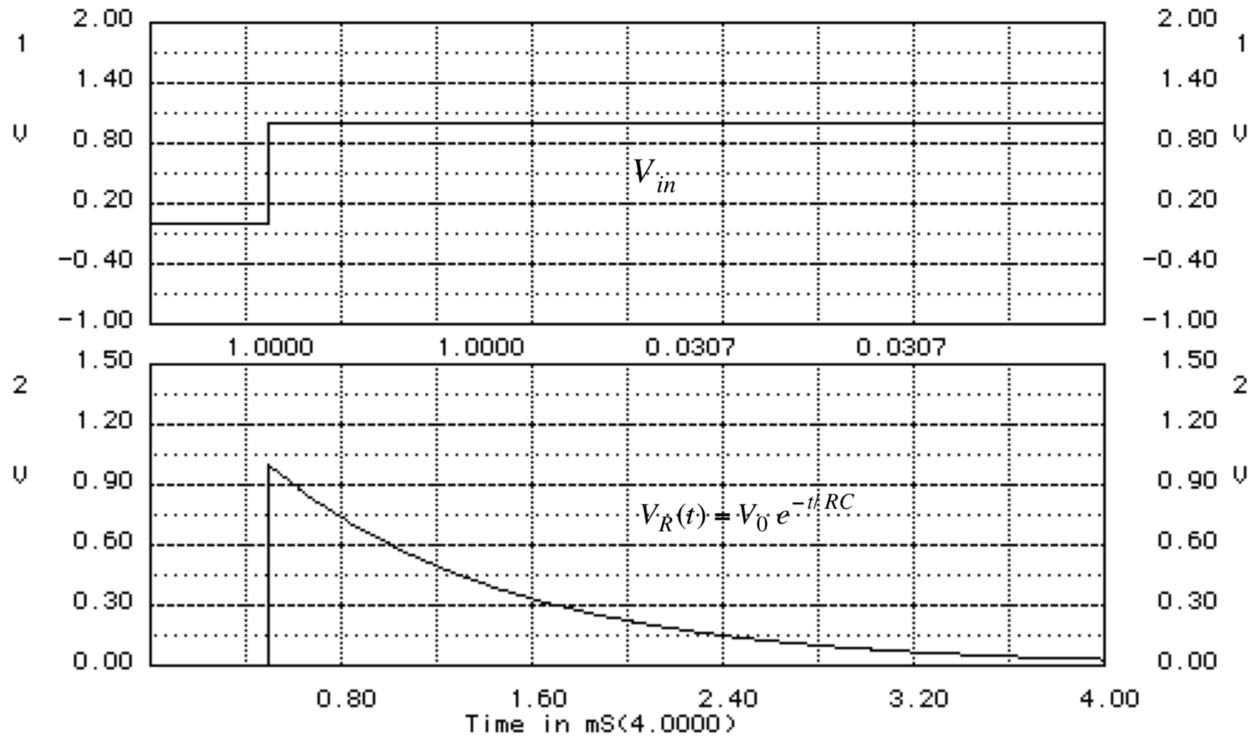
- This is just an exponential decay equation with time constant RC (sec).
- The current as a function of time through the resistor and capacitor is:

$$I(t) = I_0 e^{-t/RC}$$

- What's $V_R(t)$?
  - By Ohm's law:

$$V_R(t) = I_R \cdot R$$
$$= I_0 R\, e^{-t/RC}$$
$$= V_0\, e^{-t/RC}$$

  - At $t = 0$ all the voltage appears across the resistor, $V_R(0) = V_0$.
  - At $t = \infty$, $V_R(\infty) = 0$.



- What's $V_C(t)$?
  - Easiest way to answer is to use the fact that $V_0 = V_R + V_C$ is valid for all t.

$$V_C = V_0 - V_R$$

$$V_C = V_0\left(1 - e^{-t/RC}\right)$$

    - ⇨ At $t = 0$ all the voltage appears across the resistor so $V_C(0) = 0$.
    - ⇨ At $t = \infty$, $V_C(\infty) = V_0$.

- Suppose we wait until $I = 0$ and then short out the battery.

$$0 = V_R + V_C$$

$$V_R = -V_C$$

$$R\frac{dQ}{dt} = -\frac{Q}{C}$$

$$\frac{dQ}{dt} = -\frac{Q}{RC}$$

  - Solving the exponential equation yields,

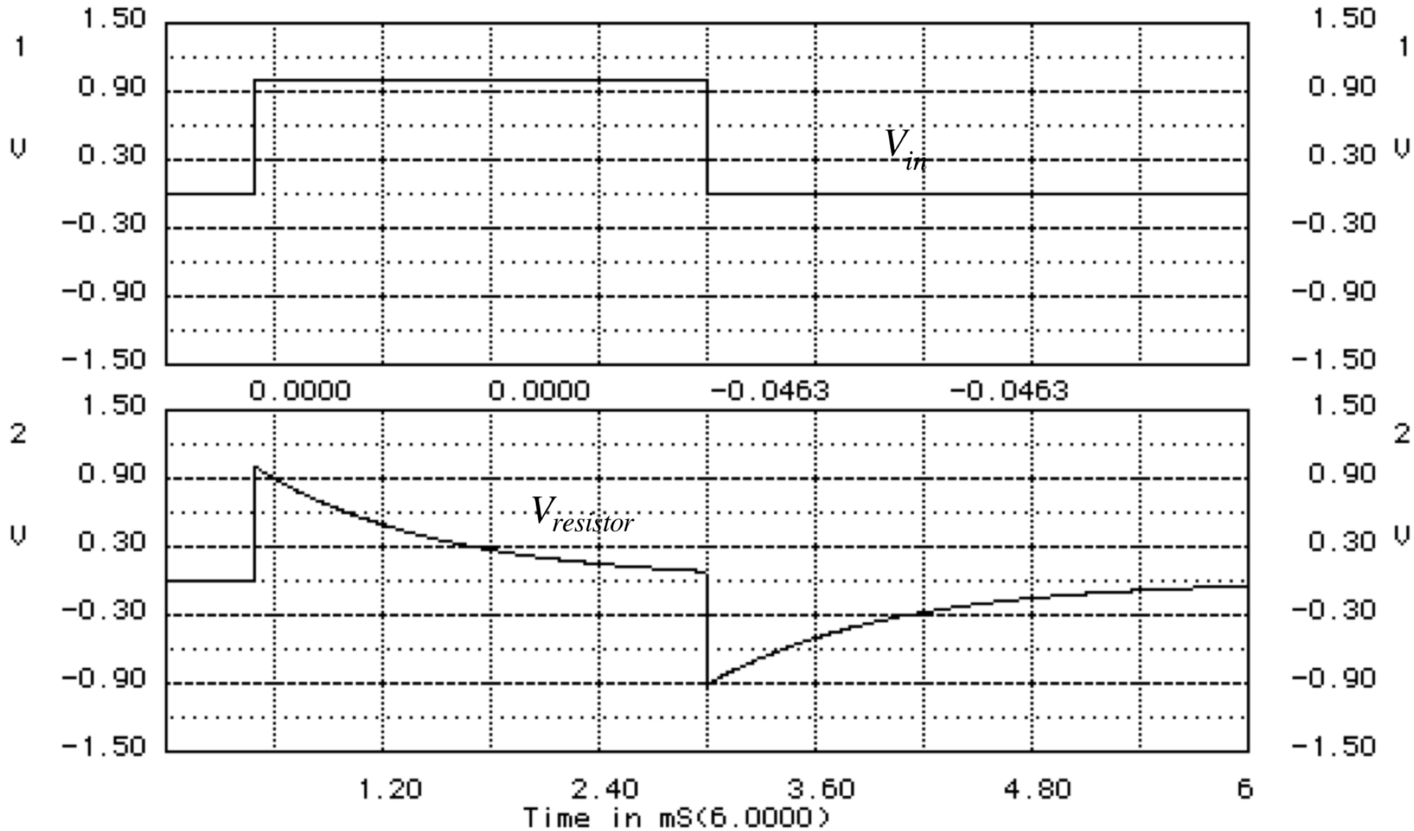
$$Q(t) = Q_0 e^{-t/RC}$$

  - We can find V_C using $V = Q/C$,

$$V_C(t) = V_0 e^{-t/RC}$$

  - Finally we can find the voltage across the resistor using $V_R = -V_C$,

$$V_R(t) = -V_0 e^{-t/RC}$$



- Suppose $V_C(t) = V_0 \sin\omega t$ instead of DC
  - ⇨ What happens to V_C and I_C?

$$Q(t) = CV(t)$$
$$= CV_0 \sin \omega t$$
$$I_C = dQ/dt$$
$$= \omega CV_0 \cos \omega t$$
$$= \omega CV_0 \sin(\omega t + \pi/2)$$

⇨ **current in capacitor varies like a sine wave too, but 90^0 out of phase with voltage.**

- We can write an equation that looks like Ohm's law by defining V^*:

  $$V^* = V_0 \sin(\omega t + \pi/2)$$

  - ⇨ the relationship between the voltage and current in C looks like:

    $$V^* = I_C/\omega C$$
    $$= I_C R^*$$

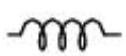
    - ⇨ $1/\omega C$ can be identified as a kind of resistance, <u>capacitive reactance</u>:

      $$X_C \equiv 1/\omega C \text{ (Ohms)}$$

      - $X_C = 0$ for $\omega = \infty$
        - ⇨ high frequencies: a capacitor looks like a short circuit
        - ⇨ "bypass capacitor" on power supply "short circuit" ripple (noise) to ground
      - $X_C = \infty$ for $\omega = 0$
        - ⇨ low frequencies: a capacitor looks like an open circuit (high resistance).
        - ⇨ look at AC output of a circuit via a capacitor (use AC coupling on scope)

Inductance:

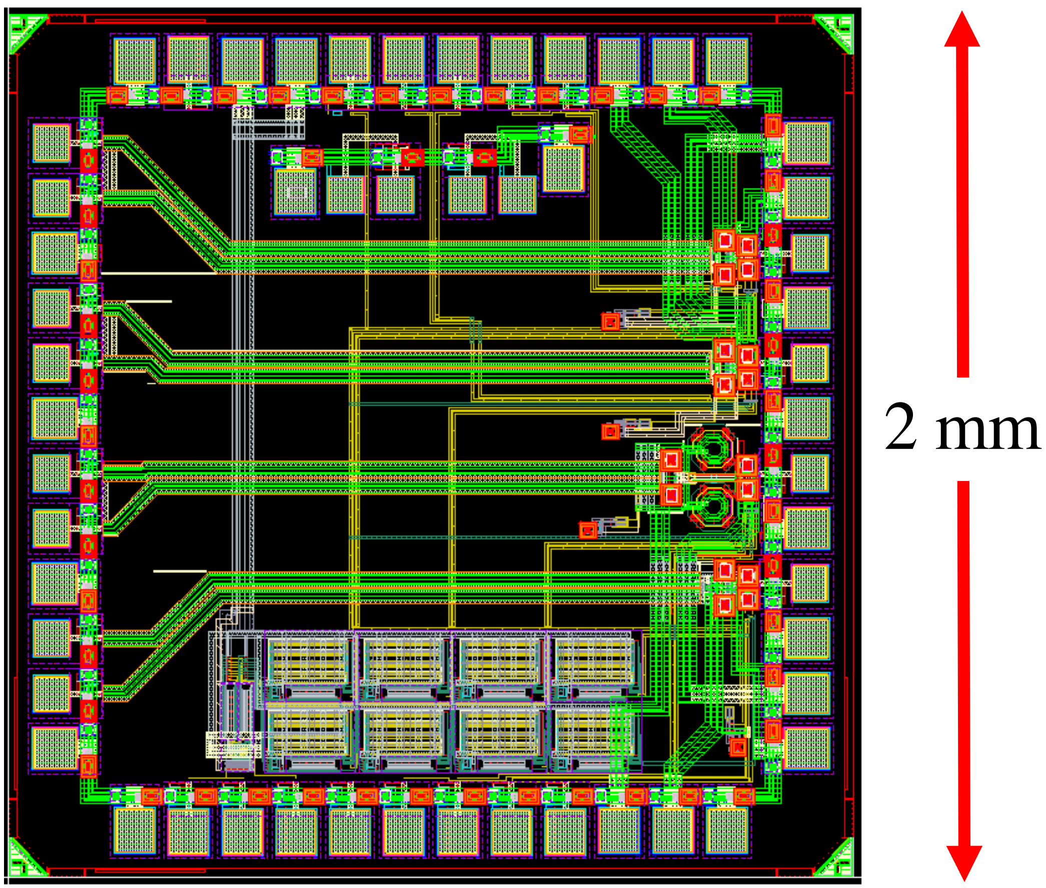
- Define inductance by: $V = LdI/dt$
  - Unit: Henry
  - Symbol:

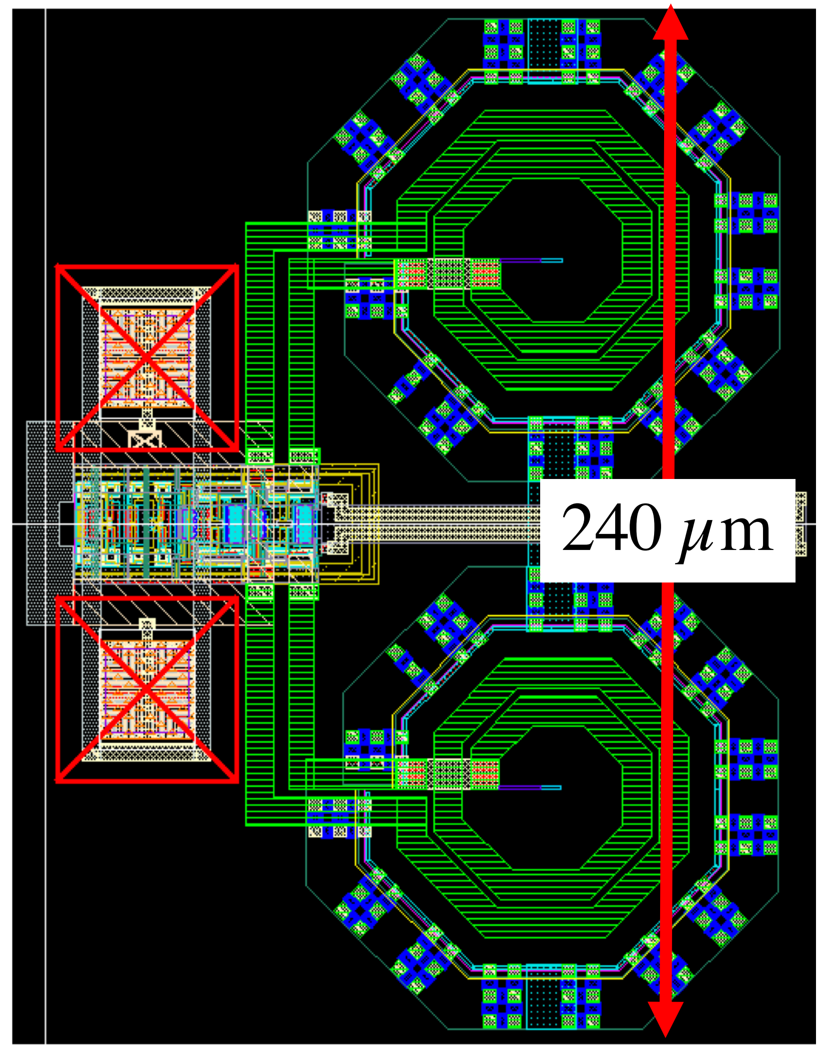
    

  - Inductors are usually made from a coil of wire
    - tend to be bulky and are hard to fabricate in small sizes (μm), seldom used in chips.
  - Two inductors next to each other (transformer) can step up or down a voltage
    - no change in the frequency of the voltage
    - provide isolation from the rest of the circuit

10 Gb/s VCSEL (Laser) Array Driver

- 4-channel test chip (65 nm CMOS)





- Capacitance and inductance on long transmission lines degrade signal

- How much energy is stored in an inductor?

$$dE = VdQ$$

$$I = \frac{dQ}{dt}$$

$$dE = VIdt$$

$$V = L\frac{dI}{dt}$$

$$dE = LIdI$$

$$E = L\int_0^I IdI$$

⇨ $E = \frac{1}{2}LI^2$

- How much power is dissipated in an inductor?

$$Power = \frac{dE}{dt}$$

$$= \frac{d}{dt}\left(\frac{LI^2}{2}\right)$$

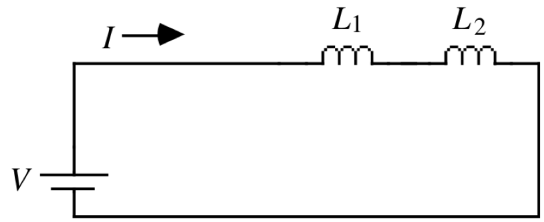
⇨ $P = LI\frac{dI}{dt}$

❑ dI/dt must be finite as we can't source (or sink) an infinite amount of power in an inductor!

THIS WOULD BE UNPHYSICAL.

⇨ the *current* across an inductor *cannot* change *instantaneously.*

- Two inductors in series:



- Apply Kirchhoff's Laws,

$$V = V_1 + V_2$$

$$= L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt}$$

$$\equiv L_{tot} \frac{dI}{dt}$$

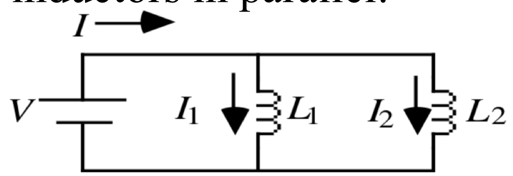
$$L_{tot} = L_1 + L_2$$

$$= \sum L_i$$

⇨ Inductors in series add like resistors in series.

The *total* inductance is *greater* than the individual inductances.

- Two inductors in parallel:



  - Since the inductors are in parallel,

    $V_1 = V_2 = V$

  - The total current in the circuit is

$$I = I_1 + I_2$$

$$\frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

$$= \frac{V}{L_1} + \frac{V}{L_2}$$

$$\equiv \frac{V}{L_{tot}}$$

$$\frac{1}{L_{tot}} = \frac{1}{L_1} + \frac{1}{L_2}$$

$$L_{tot} = \frac{L_1 L_2}{L_1 + L_2}$$

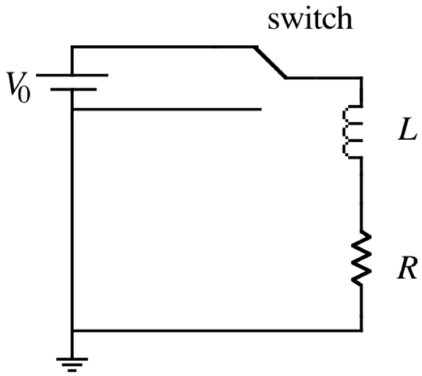
⇨ If we have more than 2 inductors in parallel, they combine like:

$$\frac{1}{L_{tot}} = \sum \frac{1}{L_i}$$

❑ Inductors in parallel add like resistors in parallel.

The *total* inductance is *less* than the individual inductances.

- Resistors and Inductors
  - Examine voltage and current versus time for a circuit with one R and one L.
    - Assume that at $t < 0$ all voltages are zero, $V_R = V_L = 0$.
    - At $t \geq 0$ the switch is closed and the battery (V_0) is connected.



    - Like the capacitor case, apply Kirchhoff's voltage rule:

$$V_0 = V_R + V_L$$
$$= IR + L\frac{dI}{dt}$$

    - Solving the differential equation, assuming at $t = 0$, $I = 0$:

$$I(t) = \frac{V_0}{R}\left(1 - e^{-tR/L}\right)$$

      ⇨ This is just an exponential decay equation with time constant L/R (seconds).
  - What's $V_R(t)$?
    - By Ohm's law $V_R = I_R R$ at any time:

$$V_R = I(t)R = V_0\left(1 - e^{-tR/L}\right)$$

    - At $t = 0$, none of the voltage appears across the resistor, $V_R(0) = 0$.
    - At $t = \infty$, $V_R(\infty) = V_0$.

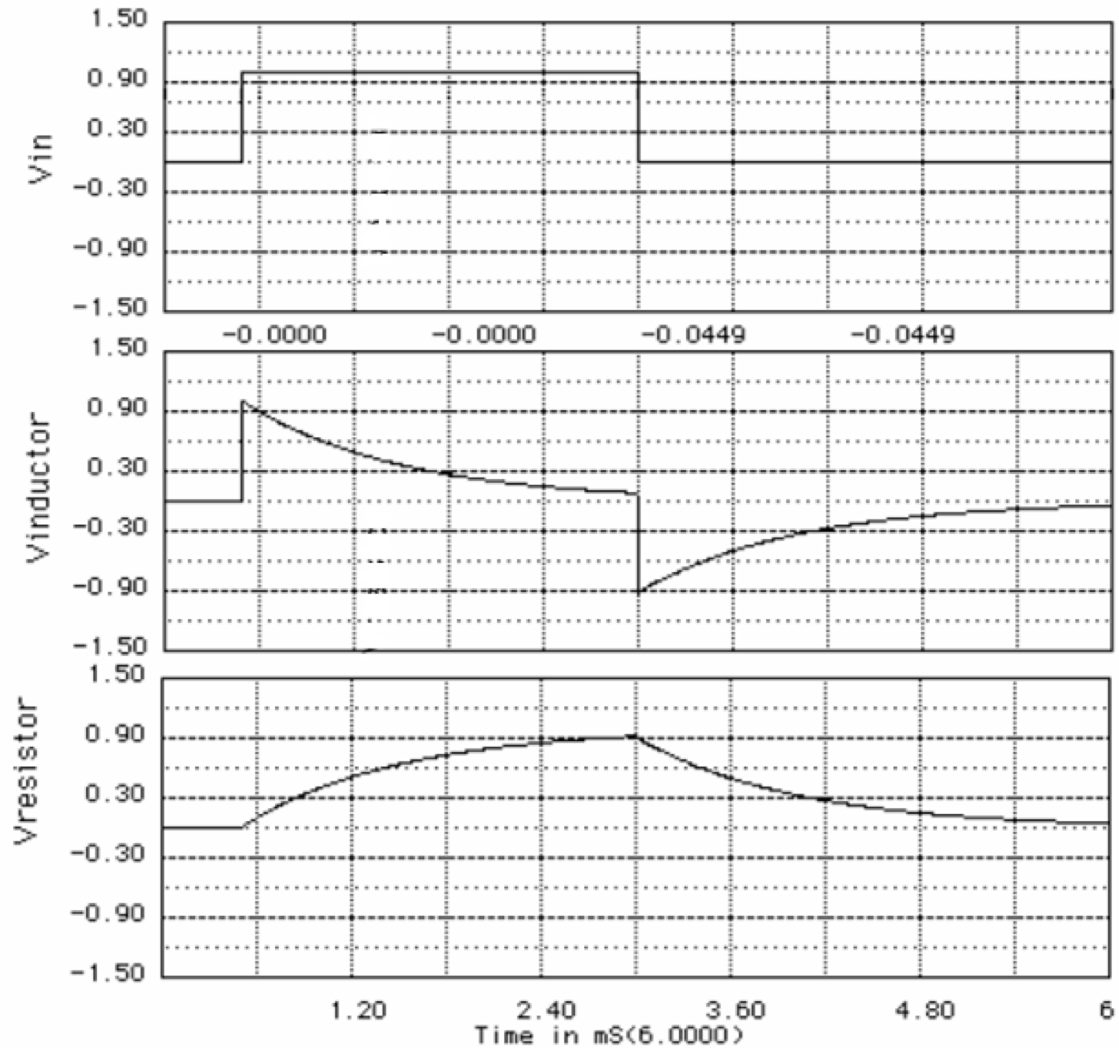
- What's $V_L(t)$?
  - Easiest way to answer is to use the fact that $V_0 = V_R + V_L$ is valid for all t.

$$V_L = V_0 - V_R$$

$$V_L(t) = V_0 e^{-tR/L}$$

  - At $t = 0$, all the voltage appears across the inductor so $V_L(0) = V_0$.
  - At $t = \infty$, $V_L(\infty) = 0$.

Pick $L/R = 1$ ms



- Suppose $V_L(t) = V_0 \sin\omega t$ instead of DC, what happens to V_L and I_L?

$$V = L\frac{dI_L}{dt}$$

$$I_L = \frac{1}{L}\int_0^t V dt$$

$$= -\frac{V_0}{\omega L}\cos\omega t$$

$$I_L(t) = \frac{V_0}{\omega L}\sin(\omega t - \pi/2)$$

⇨ The current in an inductor varies like a sine wave too, but 90^0 out of phase with the voltage.

- We can write an equation that looks like Ohm's law by defining V^*:

  $V^* = V_0\sin(\omega t - \pi/2)$

  ⇨ $V^* = I_L\omega L = I_L R^*$

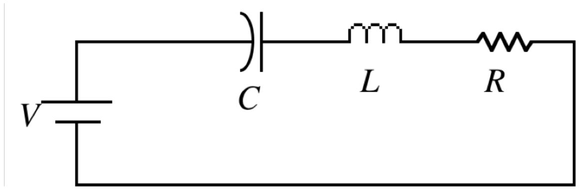
  - ωL can be identified as a kind of resistance, <u>inductive reactance</u>:

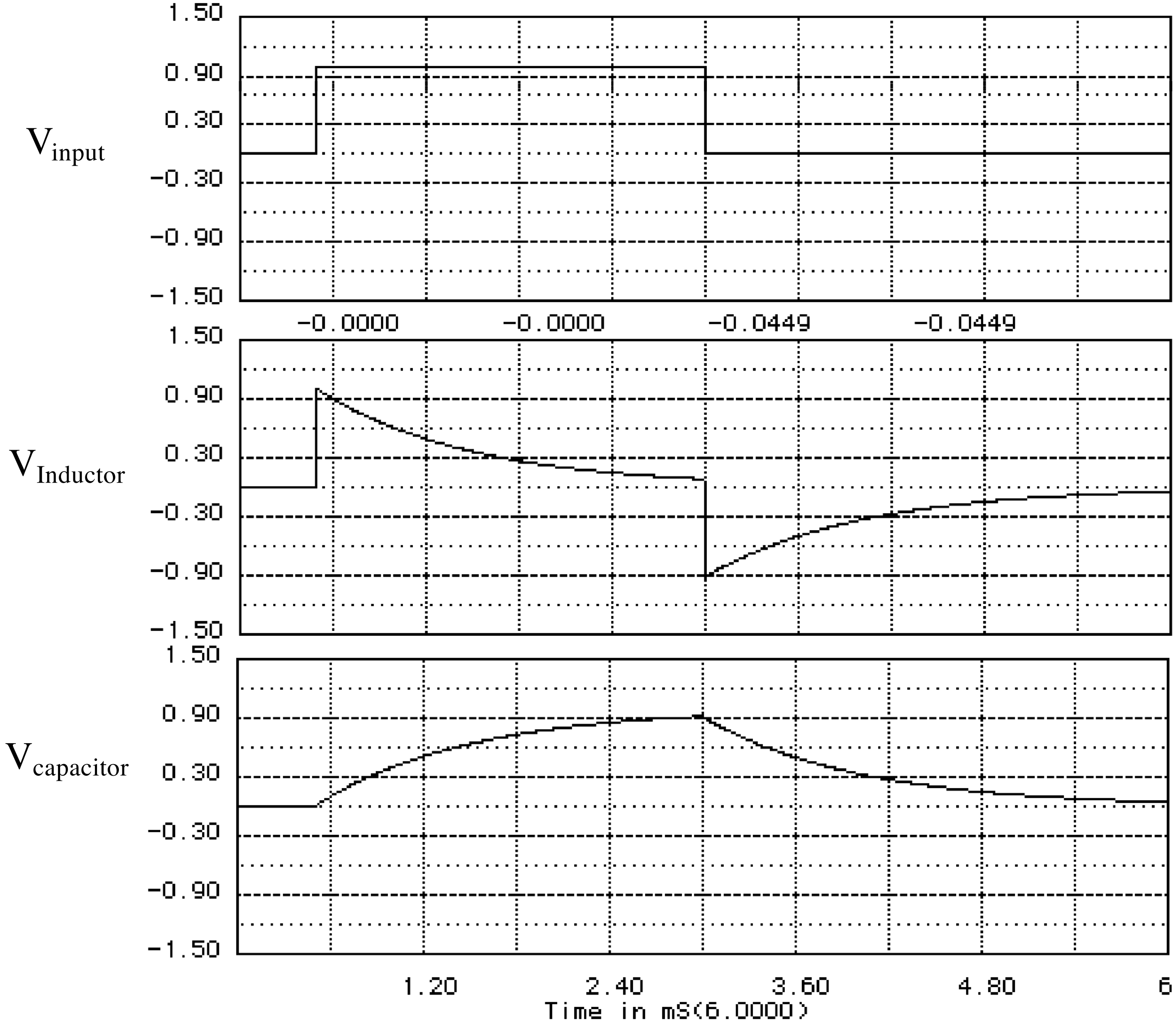
    $X_L \equiv \omega L$ (Ohms)

    - $X_L = 0$ if $\omega = 0$
      - ⇨ low frequencies: an inductor looks like a short circuit (low resistance).
    - $X_L = \infty$ if $\omega = \infty$
      - ⇨ high frequencies: an inductor looks like an open circuit.
      - ⇨ connect a circuit to power supply via an inductor to filter out ripple (noise).

Choke on power cord

- Some things to remember about R, L, and C's.
  - For DC circuits, after many time constants (L/R or RC):
    - ⇨ Inductor acts like a wire (0 Ω).
    - ⇨ Capacitor acts like an open circuit (∞ Ω).
  - For circuits where the voltage changes very rapidly or transient behavior:
    - ⇨ Capacitor acts like a wire (0 Ω).
    - ⇨ Inductor acts like an open circuit (∞ Ω).
  - Example, RLC circuit with DC supply:
    - At $t = 0$, voltages on R, C are zero and $V_L = V_0$.
    - At $t = \infty$, voltages on R, L are zero and $V_C = V_0$.





1.50
0.90
0.30
-0.30
-0.90
-1.50
V_{input}
-0.0000
-0.0000
-0.0449
-0.0449
$V_{Inductor}$
$V_{capacitor}$
1.20
2.40
3.60
4.80
6
Time in mS(6.0000)