11. Rotation

Translational Motion:

Motion of the center of mass of an object from one position to another.

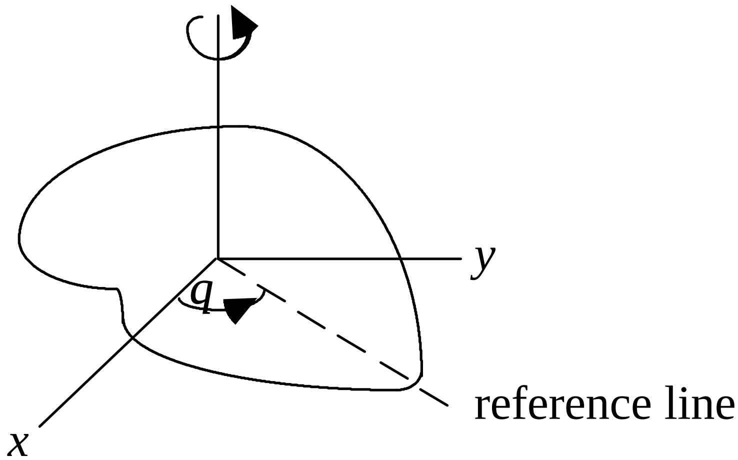
All the motion discussed so far belongs to this category, except uniform circular motion.

Rotational Motion:

Motion of an object about an axis: e.g. a basketball spinning on your finger, an ice skater spinning on his skates, the rotation of a bicycle wheel. Uniform circular motion is a special case of rotational motion.

- Will limit our discussion to rigid bodies, i.e. objects that don't deform.
- Also will limit ourselves to rotational motion about a fixed axis, i.e. the axis is not moving.

Define a coordinate system to describe the rotational motion:



- Every particle on the body moves in a circle whose center is on the axis of rotation.
- Each point rotates through the same angles over a fixed time period.

★ Will define a set of quantities to describe rotational motion similar to position, displacement, velocity, and acceleration used to describe translational motion.

Angular Position (*q*):

This is the angular location of the reference line which rotates with the object relative to a fixed axis.

- Unit: radian (not degree)
- If a body makes two complete revolutions, the angular position is 4*p*. (Don't "reset" the angular position to less that 2*p*).

- Define the counterclockwise direction as the direction of increasing q and the clockwise direction as the direction of decreasing q.
- The distance that a point on the object moves is the arc length defined by q and the distance from the axis of rotation:

$$s = rq$$

Angular Displacement:

The change in the angular position from one time to another:

$$\Delta q = q_2 - q_1$$

- Δq can be positive or negative.
- Every point on the rigid body has the same angular displacement even though they may have traveled a different distance.

Angular Velocity (w):

The rate of change in the angular position.

Average Angular Velocity:

$$< w > = \frac{q_2 - q_1}{t_2 - t_1} = \frac{\Delta q}{\Delta t}$$

Instantaneous Angular Velocity:

$$w = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$$

- All particles on the object have the same angular velocity even though they may have different linear velocities v.
- w can be positive or negative depending on whether the body rotates with increasing q (counterclockwise) or decreasing q (clock - wise).
- Unit: rad / s (preferred) or rev / s.

Angular Acceleration (*a*):

The rate of change of angular velocity.

Average Angular Acceleration:

$$< a >= \frac{w_2 - w_1}{t_2 - t_1} = \frac{\Delta w}{\Delta t}$$

Instantaneous Angular Acceleration:

$$a = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \frac{dw}{dt}$$

- All points on the object have the same angular acceleration even though they may have different linear accelerations.
- Unit: rad / s^2 (preferred) or rev / s^2.

Example:

What are the angular speeds and angular accelerations of the second, minute, and hour hands of an accurate watch?

$$w_{\text{second hand}} = \frac{2p}{60\ \text{s}} = 0.10\ \text{rad/s}$$

$$w_{\text{minute hand}} = \frac{2p}{1\ \text{hr}} = \frac{2p}{3600\ \text{s}} = 0.0017\ \text{rad/s}$$

$$w_{\text{hour hand}} = \frac{2p}{12\ \text{hr}} = \frac{2p}{43200\ \text{s}} = 0.00014\ \text{rad/s}$$

Since the watch is accurate, the angular velocity is not changing and hence the angular acceleration is zero for all the hands.

Constant Angular Acceleration:

The equations of motion for rotational motion look exactly like the equations of motion for translational motion with the replacements of the translational variables by angular variables:

Translational	Rotational
$x = x_0 + v_0 t + \frac{1}{2} a t^2$	$q = q_0 + w_0 t + \frac{1}{2} a t^2$
$v = v_0 + at$	$w = w_0 + at$
$v^2 = v_0^2 + 2a(x - x_0)$	$w^2 = w_0^2 + 2a(q - q_0)$

Example:

A flywheel completes 40 revolutions as it slows from an angular speed of 1.5 rad/s to a complete stop. (a) What is the angular acceleration? (b) What is the time required for it to come to rest? (c) How much time is required for it to complete the first 20 revolutions?

$$w^2 = w_0^2 + 2a(q - q_0)$$

$$0 = w_0^2 + 2a(q - q_0)$$

$$a = \frac{-w_0^2}{2(q - q_0)}$$

$$= \frac{-(1.5 \text{ rad / s})^2}{2 \times 40 \times 2p}$$

$$= -4.5 \times 10^{-3} \text{ rad / s}^2$$

$$w = w_0 + at$$

$$0 = w_0 + at$$

$$t = -\frac{w_0}{a}$$

$$= -\frac{1.5 \text{ rad / s}}{-4.5 \times 10^{-3} \text{ rad / s}^2}$$

$$= 335 \text{ s}$$

$$q = q_0 + w_0 t + \tfrac{1}{2} a t^2$$

$$\tfrac{1}{2} a t^2 + w_0 t - q = 0$$

$$\tfrac{1}{2}(-4.5 \times 10^{-3}) t^2 + 1.5t - 20 \times 2p = 0$$

$$t = 88 \text{ s}$$

Example:

A wheel turns through 90 revolutions in 15 s, its angular speed at the end of the period being 10 rev/s. (a) What was its angular speed at the beginning of the 15 s interval, assuming constant acceleration? (b) How much time has elapsed between the time when the wheel was at rest and the beginning of the 15 s interval?

(a) $$w = w_0 + at \qquad (1)$$

- We are interested in w_0, but a is also unknown. There are a total of two unknowns (w_0 and a). Therefore, another equation with two unknowns is needed (set $q_0 = 0$).

$$q = w_0 t + \tfrac{1}{2} a t^2$$

(1): $$at = w - w_0$$

$$q = w_0 t + \tfrac{1}{2}(w - w_0)t$$

$$= \tfrac{1}{2} w_0 t + \tfrac{1}{2} wt$$

$$w_0 = \frac{2q}{t} - w$$

$$w_0 = \frac{2 \times 90 \text{ rev}}{15 \text{ s}} - 10 \text{ rev/s}$$
$$= 2 \text{ rev/s}$$

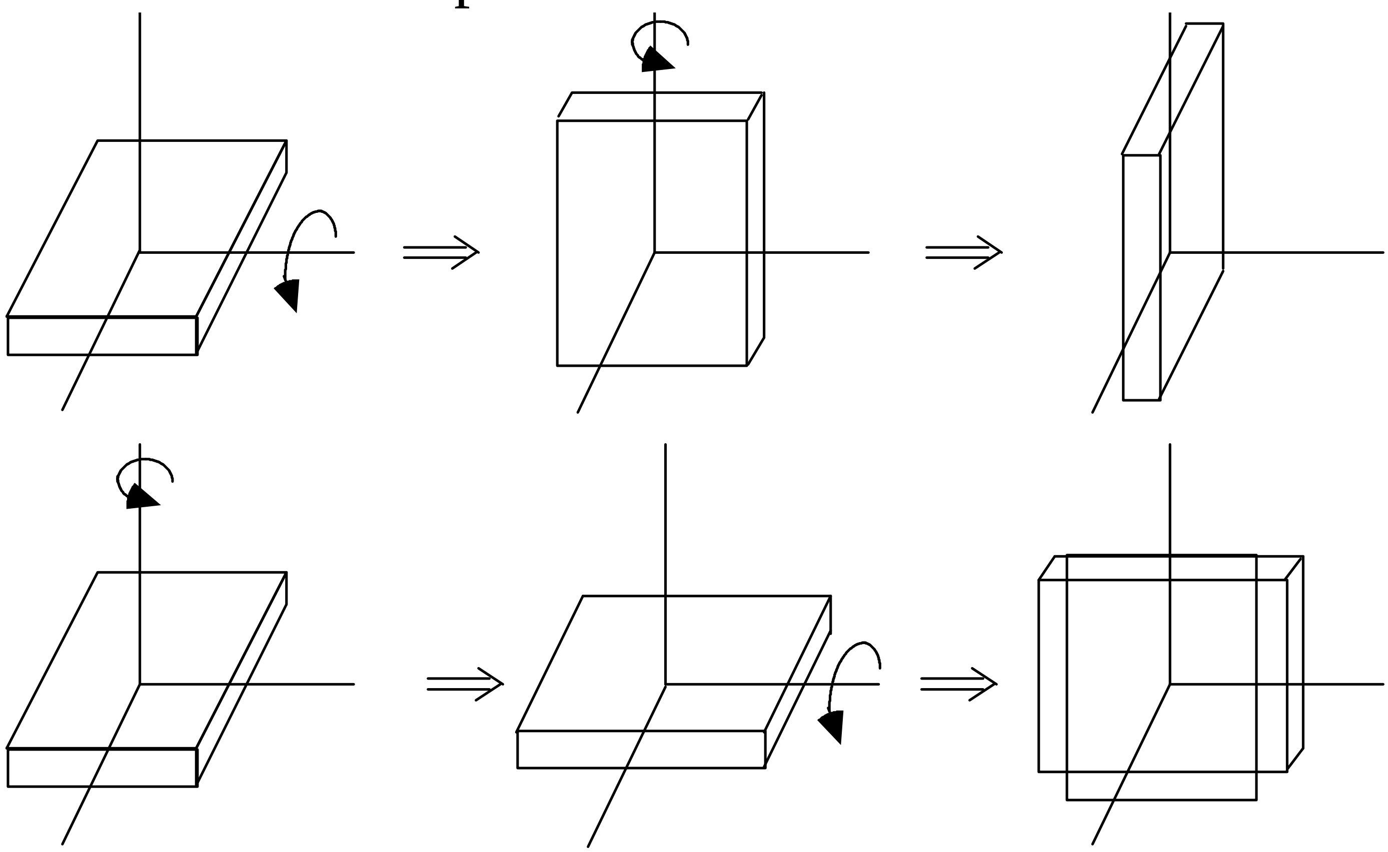
(b) Since we want to know how long it takes to go from rest to 2 rev/s, we need to know the angular acceleration:

(1): $$a = \frac{w - w_0}{t}$$
$$= \frac{10 \text{ rev/s - 2 rev/s}}{15 \text{ s}}$$
$$= 0.53 \text{ rev/s}^2$$

(1): $$\text{t} = \frac{w - w_0}{a}$$
$$= \frac{2 \text{ rev/s - 0}}{0.53 \text{ rev/s}^2}$$
$$= 3.8 \text{ s}$$

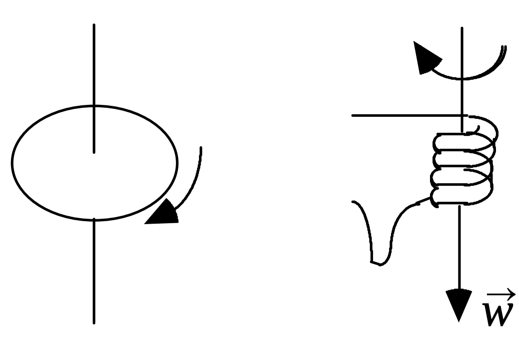
Are Angular Quantities Vectors?

We describe translational motion using vector quantities: position, displacement, velocity, and acceleration. We can also describe rotational motion using angular velocity and acceleration vectors. However, angular displacement does not behave like a vector; adding two vectors in different orders produces different vectors:



Angular Velocity Vector:

A vector with magnitude given by the angular speed and direction given by the axis of rotation according to the right hand rule:



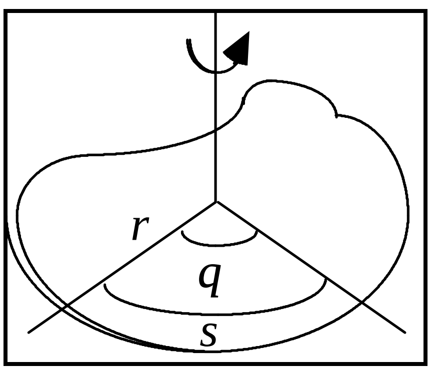
Angular Acceleration Vector:

The angular acceleration can also be represented by a vector:

$$\vec{a} = \frac{d\vec{w}}{dt}$$

For a rotation along a fixed axis, if $\vec{a}$ and $\vec{w}$ are parallel, then the object will spin faster and if $\vec{a}$ and $\vec{w}$ are anti-parallel, then the object will spin slower.

Relationship Between Linear and Angular Variables:



Distance:

A point on the object swept out an arch when the object rotates with respect to a fixed axis. The distance that the point has moved is given by the length of the arc which is related to the angle of rotation:

$$s = rq \qquad (1)$$

- Different points move different distances. A point further from the center moves a larger distance.

Speed:

Differentiate Eq. (1):

$$\frac{ds}{dt} = r\frac{dq}{dt}$$

$$v = rw \qquad (2)$$

Acceleration:

Differentiate Eq. (2):

$$\frac{dv}{dt} = r\frac{dw}{dt} = ra$$

$$a_t = ra$$

- This is the acceleration that changes the speed of the object, but not its direction. This acceleration is therefore along the path of the object, i.e. tangent to the path

  tangential acceleration

- There is also an acceleration for the change in direction. This is the centripetal acceleration we derived for uniform circular motion:

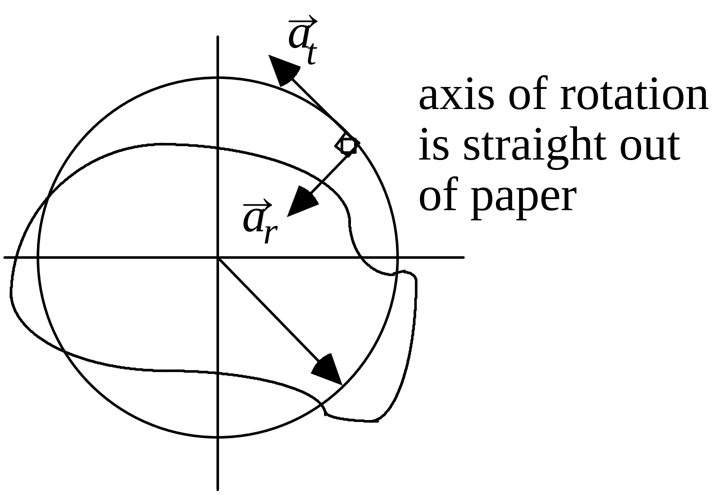
$$a_c = \frac{v^2}{r}$$

$$= \frac{(wr)^2}{r}$$

$$= w^2 r$$

$$= a_r$$

- This is along the radial direction toward the center and is known as the radial acceleration.



Example:

An astronaut is being tested in a centrifuge. The centrifuge has a radius of 10 m and, in starting, rotates according to $q = 0.30t^2$, where t is given in seconds and q in radians. When $t = 5.0$ s, what are the astronaut's (a) angular velocity,

(b) linear speed, and (c) tangential and radial accelerations?

$$\begin{aligned} w &= \frac{dq}{dt} \\ &= \frac{d}{dt}(0.30t^2) \\ &= 0.6t \\ &= 0.6 \times 5.0 \\ &= 3.0 \text{ rad / s} \end{aligned}$$

$$\begin{aligned} v &= wr \\ &= (3.0 \text{ rad / s})(10 \text{ m}) \\ &= 30 \text{ m / s} \end{aligned}$$

- "rad" disappeared because it is a unitless quantity: $q = s / r$. We keep "rad" as a reminder.

$$
\begin{aligned}
a_r &= w^2 r \\
&= (3.0 \text{ rad / s})^2 (10 \text{ m}) \\
&= 90 \text{ m / s}^2 \\
a_t &= r a \\
&= r \frac{dw}{dt} \\
&= r \frac{d}{dt}(0.6t) \\
&= r \times 0.6 \\
&= (10 \text{ m})(0.6 \text{ rad / s}^2) \\
&= 6.0 \text{ m / s}^2
\end{aligned}
$$

Rotational Kinetic Energy:

For an object in a translational motion, there is a kinetic energy associated with the motion:

$$K = \tfrac{1}{2} m v^2$$

For an object in rotational motion, there must be kinetic energy associated with each point on the object:

$$K = \sum \tfrac{1}{2} m_i v_i^2$$
$$= \sum \tfrac{1}{2} m_i (w r_i)^2$$
$$= \tfrac{1}{2} (\sum m_i r_i^2) w^2$$
$$= \tfrac{1}{2} I w^2$$

Moment of Inertia:

$$I = \sum m_i r_i^2$$

- Moment of inertia depends on the distribution of the mass of the object relative to the axis of rotation.
- r_i is the perpendicular distance of the particle from the axis of rotation.

For a continuous object:

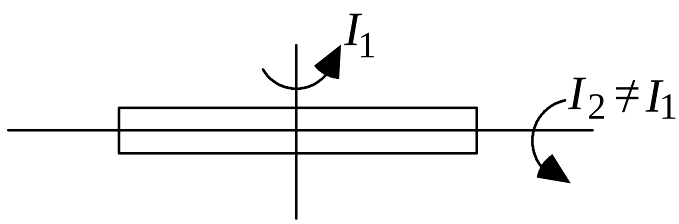
$$I = \int r^2 dm = \int r^2 r(r) dV$$

- Rotational kinetic energy is very similar to the translational kinetic energy with the following replacement:

mass   moment of inertia

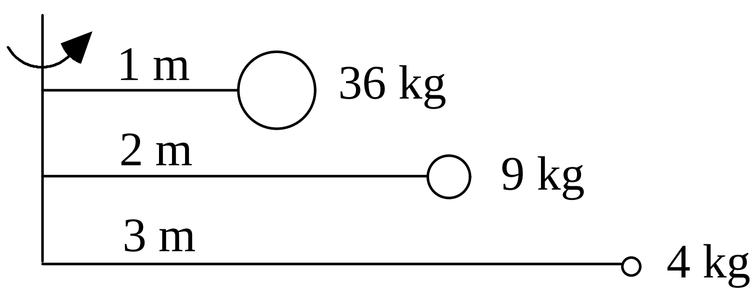
translational speed   rotational speed

- Moment of inertia depends on the axis of rotation, unlike mass which is a fixed quantity independent of the orientation of the object. The same object rotating about two different axes has two different moment of inertia:

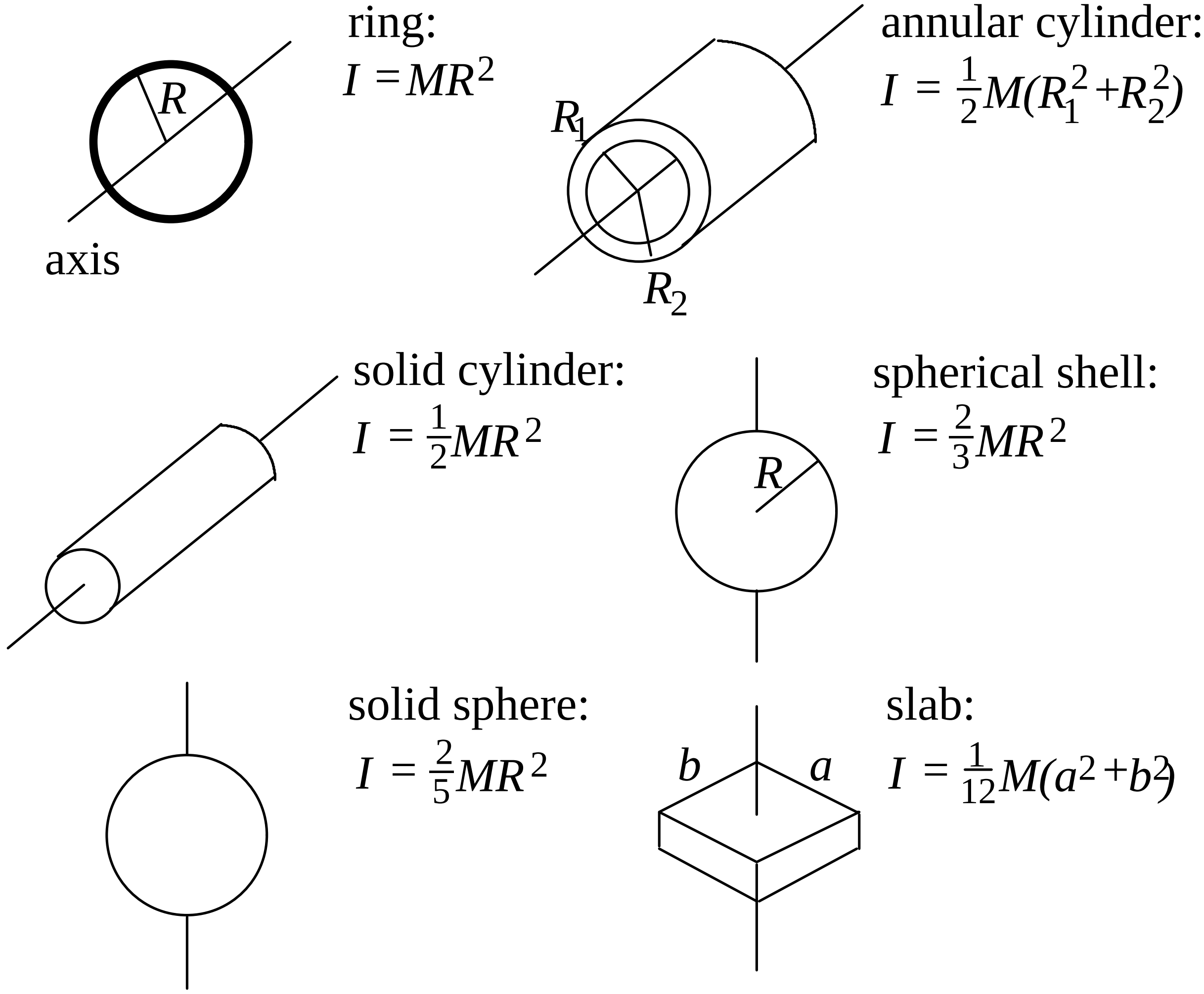


Conceptual Question:

In the figure below are three masses that rotate about a vertical axis. If these masses rotate with the same angular speed, rank them in order of increasing kinetic energy.



Example of Moment of Inertia:



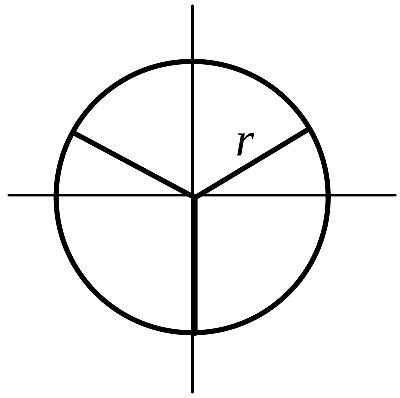
Addition of Moment of Inertia:

If an object is composed of two pieces, then the moment of inertia of one object about an axis of rotation is just the sum of the individual moments of inertia about that axis:

$$I_{tot} = I_1 + I_2$$

Example:

A wheel with axis of rotation out of paper:



$$I_{tot} = I_{hoop} + I_{spoke}$$

$$= M_h r^2 + \tfrac{1}{3} M_s r^2$$

Parallel Axis Theorem:

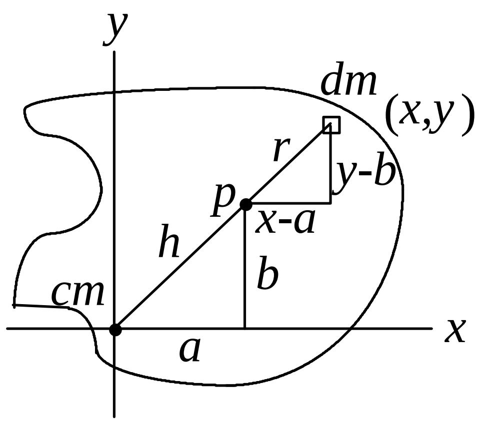
If we know the moment of inertia for an axis of rotation that passes through the center of mass of an object, then the moment of inertia about any new axis that is parallel to that axis is:

$$I = I_{cm} + mh^2$$

where h is the distance between the two axis.

- The added piece (mh^2) is just as if all the mass were concentrated at the center of mass.

Proof:



The moment of inertia through p:

$$I = \int r^2 dm$$

$$= \int \left[(x-a)^2 + (y-b)^2 \right] dm$$

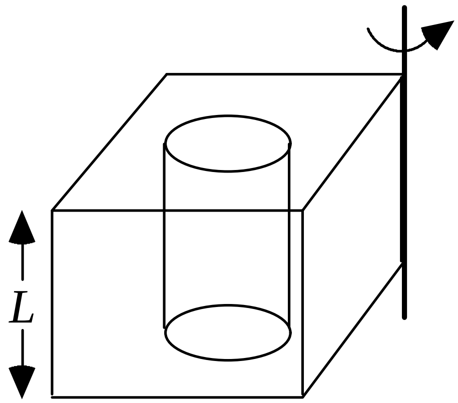
$$= \int (x^2 + y^2) dm - 2a \int x dm - 2a \int y dm + \int (a^2 + b^2) dm$$

$$= I_{cm} - 2a x_{cm} m - 2b y_{cm} m + h^2 m$$

$$= I_{cm} + mh^2$$

Example:

A uniform square block of mass m has a hole bored through it as shown. The radius of the hole is 1/4 of the length of one side of the cube. Determine the moment of inertia if the cube is rotating about an axis on the side as shown.



$$I_{tot} = I_{solid\,cube} - I_{cylinder}$$

$$I_{solid\,cube} = I_{cm} + mh^2$$

$$= \tfrac{1}{12} m(L^2 + L^2) + m\left[\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2\right]$$

$$= \tfrac{2}{3} mL^2$$

$$I_{cylinder} = \tfrac{1}{2} m_{cyl}\left(\frac{L}{4}\right)^2 + m_{cyl}\left[\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2\right]$$

$$= \tfrac{17}{32} m_{cyl} L^2$$

$$I_{cylinder} = \tfrac{17}{32} m \frac{p\left(\frac{L}{4}\right)^2 \times L}{L^3} \times L^2$$

$$= \tfrac{17}{512} pmL^2$$

$$I_{tot} = \tfrac{2}{3} mL^2 - \tfrac{17}{512} pmL^2$$

$$= 0.56 mL^2$$

Translational Motion:

$$x, v, a \xrightarrow{\text{What causes } a?} \text{Force } (F) \xrightarrow{\text{How are } F \text{ and } a \text{ related?}} \sum F = ma \longrightarrow \begin{array}{l} dW = F_{dir} dx \\ W = \Delta K \\ P = \frac{dW}{dt} = F_{dir} v \end{array}$$

Rotational Motion:

$$q, w, a \xrightarrow{\text{What causes } a\,?} \text{Torque } (t) \xrightarrow{\text{How are } t \text{ and } a \text{ related?}} \sum t = Ia \longrightarrow \begin{array}{l} dW = t dq \\ W = \Delta K \\ P = \frac{dW}{dt} = tw \end{array}$$

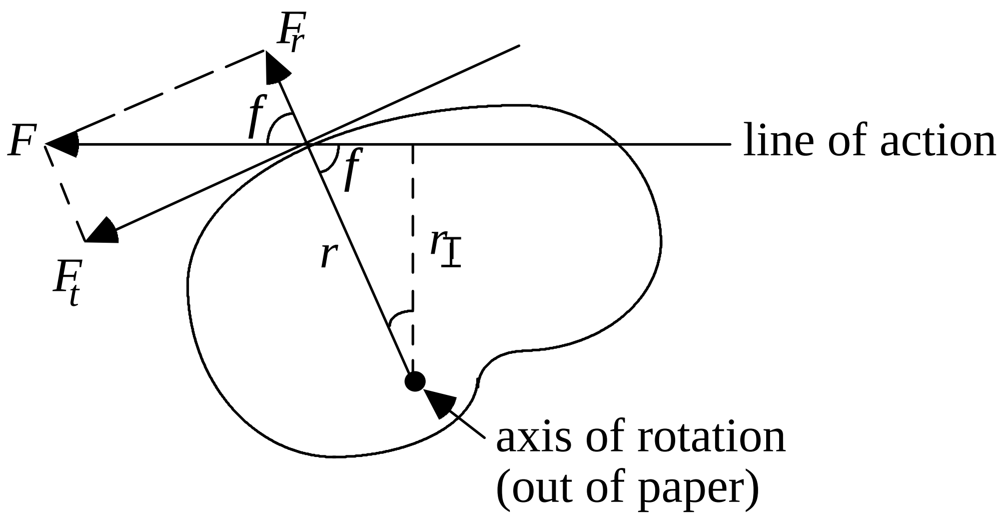
Torque:

- To open a door, where and in what direction you apply the force is important:
  - ★ the applied force should be as far away from the hinge as possible

door knob is located far away from the hinge

★ the applied force should be perpen-dicular to the door

torque is a quantity that takes the direction, magnitude, and location of the force into account in rotating an object.



The torque can be calculated using two methods:

- Use the tangential component of the force:

$$t = F_t r = (F \sin f) r$$

★ the radial component has no contri-bution in the rotational motion.

- Use the line of action:

$$t = F r_\perp = F r \sin f$$

★ $r_\perp$ is called the moment arm (level arm): it is the perpendicular distance between the axis of rotation and the **line of action**, a line that runs through the force vector.

- Unit: Nm--This is the same unit as energy but never quote torque in Joules!
- The further away from the axis of rotation a force is applied, the larger the torque.
- A force that tends to make an object rotate counter clockwise is defined as a positive torque, and a force that tends to make an object rotate clockwise is defined as a negative torque.
- The torque vector is perpendicular to the plane containing the force and the vector $\vec{r}$.

Newton's Second Law for Rotation:

$$F_t = ma_t$$

$$F_t r = ma_t r$$

$$t = mar^2$$

$$t = Ia$$

i.e. the angular acceleration is related to the applied torque with the moment of inertia as the proportionality constant.

If there are multiple forces:

$$\sum t = Ia$$

Work and Kinetic Energy:

The Work-Kinetic Energy theorem for translational motion:

$$W_{tot} = \Delta K = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2$$

Since the rotational kinetic energy is:

$$K = \tfrac{1}{2} I w^2$$

This yields a similar theorem for rotational motion:

$$W_{tot} = \Delta K = \tfrac{1}{2} I w_f^2 - \tfrac{1}{2} I w_i^2$$

Work:

$$dW = F_{dir\ of\ motion} dx$$
$$= F_t ds$$
$$= F_t r dq$$
$$dW = t dq$$
$$W = \int t dq$$

For a constant torque:

$$W = t \Delta q$$

Power:

$$P = \frac{dW}{dt}$$

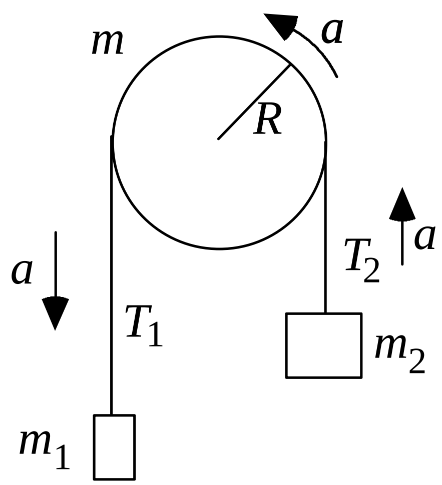
For a constant torque:

$$P = t \frac{dq}{dt}$$
$$P = t w$$

Example (A real pulley with mass):

Two blocks of mass m_1 = 400 g and m_2 = 600 g are connected by a massless cord around a uniform disk of mass m = 500 g and radius R = 12.0 cm. The system is released from rest.

Assuming that there is no slippage and ignoring the friction of the axis of rotation, find (a) the magnitude of acceleration of the blocks, (b) the tension T_1 in the cord on the left, and (c) the tension T_2 in the cord on the right.



Block 1: $\sum F = m_1 g - T_1 = m_1 a$ (1)

Block 2: $\sum F = T_2 - m_2 g = m_2 a$ (2)

Pulley: $$\sum t = T_1 R - T_2 R = I\alpha = I\frac{a}{R} \qquad (3)$$

(1) + (2): $$(m_1 - m_2)g - (T_1 - T_2) = (m_1 + m_2)a$$

(3): $$(m_1 - m_2)g - I\frac{a}{R^2} = (m_1 + m_2)a$$

$$(m_1 - m_2)g - \tfrac{1}{2}mR^2\frac{a}{R^2} = (m_1 + m_2)a$$

$$(m_1 - m_2)g - \tfrac{1}{2}ma = (m_1 + m_2)a$$

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + \tfrac{1}{2}m}$$

$$= -0.16g$$

$$= -1.57 \text{ m / s}^2$$

- answer is independent of radius

(1): $$T_1 = m_1 g - m_1 a = m_1(1 + 0.16)g$$

$$= 1.16 m_1 g$$

$$= 4.5 \text{ N}$$

(2): $$T_2 = m_2 g + m_2 a = m_2(1 - 0.16)g$$

$$= 0.84 m_2 g$$

$$= 4.9 \text{ N}$$

- The tensions are not the same or else there will be no net torque to rotate the pulley.