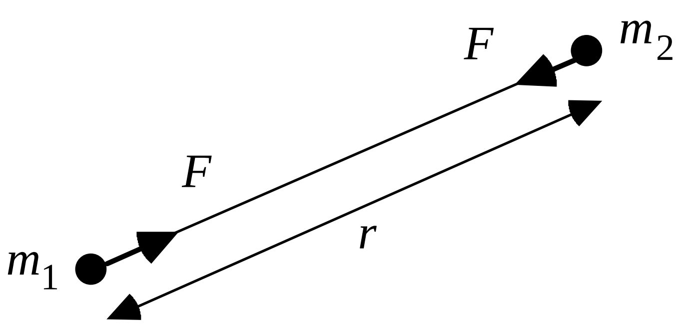
14. Gravitation

Universal Law of Gravitation (Newton):

The attractive force between two particles:

$$F = G\frac{m_1 m_2}{r^2}$$

where $G = 6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}$ is the universal gravitational constant.



- Particle #1 feels a pull toward particle #2 and particle #2 feels a pull towards particle #1 -- action-reaction forces.
- The law is for pairs of "point-like" particles.
- Every particle in the universe pulls on every other particle in the universe, e.g. the moon is pulling on you now.
- The force does not depend on what is between two objects, i.e. it cannot be shielded by a material (e.g. wall) between them.

- This is one of the four fundamental forces.

Principle of Superposition:

The total force on a point particle is equal to the sum of all the forces on the particle.

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots + \vec{F}_{1n}$$

$$= \sum_{i=2}^{n} \vec{F}_{1i}$$

- $\vec{F}_{1i}$: force of the ith particle on particle #1.

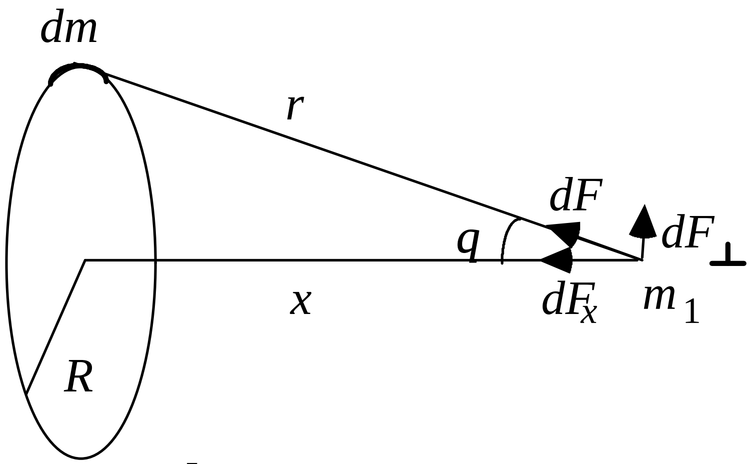
For a real object with a continuous distribution of particles:

$$dF_1 = G\frac{m_1 dm}{r^2}$$

$$F_1 = \int dF_1$$

$$= \int G\frac{m_1}{r^2}dm$$

Gravitational force from a thin ring:



$$F_x = \int dF_x$$

$$= \int G \frac{m_1 dm}{r^2} \cos q$$

$$= \int G \frac{m_1 dm}{r^2} \frac{x}{r}$$

$$= \int G \frac{m_1 x dm}{r^3}$$

$$= \int G \frac{m_1 x dm}{(x^2 + R^2)^{\frac{3}{2}}}$$

$$= G \frac{m_1 m x}{(x^2 + R^2)^{\frac{3}{2}}}$$

- The force points toward the ring.
- $F_x = 0$ if $x = 0$
- The force perpendicular to x ($dF_\perp$) cancels by symmetry. Use symmetry to simplify your problem.

Other Newton's results:

- A uniform spherical shell of matter attracts a particle outside as if all the shell mass was concentrated at the center.
- Similarly for a sphere of matter.
- Like the case of a ring, a particle inside a spherical shell of matter feels zero gravitational force from the shell.

Gravity:

Force on a mass *m* on Earth's surface:

$$F_g = \frac{GmM_{Earth}}{R_{Earth}^2}$$

$$= m\frac{GM_{Earth}}{R_{Earth}^2}$$

$$\frac{GM_{Earth}}{R_{Earth}^2} = \frac{\left(6.67\times10^{-11}\ \mathrm{m^3/kg/s^2}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{\left(6.37\times10^{6}\ \mathrm{m}\right)^2}$$

$$= 9.83\ \mathrm{m/s^2}$$

$$= g$$

For any planet, the acceleration of gravity at its surface is:

$$g_{planet} = \frac{GM_{planet}}{R_{planet}^2}$$

Example:

What is the acceleration due to gravity for (a) an airplane flying at an altitude of 10 km, (b) a shuttle at 300 km, (c) a geosynchronous satellite at 36000 km?

(a) $$g_{plane} = \frac{GM_{Earth}}{\left(R_{Earth} + h_{plane}\right)^2}$$

$$= \frac{\left(6.67\times10^{-11}\ \mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{\left(6.37\times10^{6}\ \mathrm{m} + 1\times10^{4}\ \mathrm{m}\right)^2}$$

$$= 9.79\ \mathrm{m}/\mathrm{s}^2$$

(b) $$g_{shuttle} = \frac{GM_{Earth}}{\left(R_{Earth} + h_{shuttle}\right)^2}$$

$$= \frac{\left(6.67\times10^{-11}\ \mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{\left(6.37\times10^{6}\ \mathrm{m} + 3\times10^{5}\ \mathrm{m}\right)^2}$$

$$= 9.00\ \mathrm{m}/\mathrm{s}^2$$

Little different from g on Earth. Astronauts floating in a shuttle is not due to zero gravity.

(c) $$g_{satelite} = \frac{GM_{Earth}}{(R_{Earth} + h_{shuttle})^2}$$

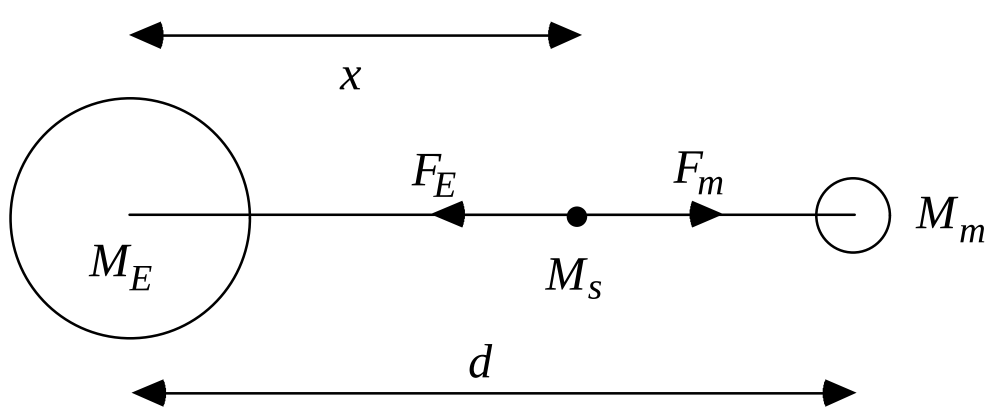
$$= \frac{(6.67\times10^{-11}\ \text{m}^3/\text{kg}/\text{s}^2)(5.98\times10^{24}\ \text{kg})}{(6.37\times10^{6}\ \text{m} + 3.6\times10^{7}\ \text{m})^2}$$

$$= 0.22\ \text{m}/\text{s}^2$$

Much lower than g on Earth but gravity still pulls on objects in space.

Example:

For a spaceship between the Earth and moon, at what distance from the Earth will the net gravitational force be zero?



$$\sum F_x = F_{Earth} - F_{moon} = 0$$

$$\frac{GM_E M_s}{x^2} - \frac{GM_m M_s}{(x-d)^2} = 0$$

$$\frac{M_E}{x^2} - \frac{M_m}{(x-d)^2} = 0$$

$$(x-d)^2 - \frac{M_m}{M_E} x^2 = 0$$

$$x^2 - 2xd + d^2 - rx^2 = 0$$

$$(1-r)x^2 - 2xd + d^2 = 0$$

$$x = \frac{2d \pm \sqrt{4d^2 - 4(1-r)d^2}}{2(1-r)}$$

$$= \frac{2d \pm \sqrt{4rd^2}}{2(1-r)}$$

$$= \frac{2d \pm 2\sqrt{r}d}{2(1-r)}$$

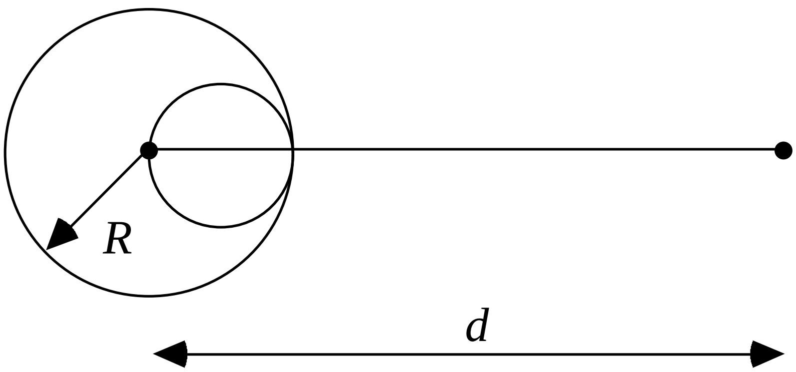
$$= \frac{1 \pm \sqrt{r}}{1-r} d$$

- $x = \frac{1+\sqrt{r}}{1-r} d > d \Rightarrow$ unphysical

- $x = \frac{1-\sqrt{r}}{1-r} d < d \Rightarrow \text{OK}$

Problem 15:

Calculate the gravitational force due to a hollowed sphere, assuming that the mass of the sphere was *M* before hollowing.



- Calculating the force by integrating over the solid part of the sphere is difficult
  Use Principle of Superposition

$$F = F(solid\,sphere) - F(hollowed\,material)$$

$$= \frac{GMm}{d^2} - \frac{GM_{hm}m}{\left(d - \frac{R}{2}\right)^2}$$

$$= \frac{GMm}{d^2} - \frac{Gm}{\left(d - \frac{R}{2}\right)^2} \times M \times \frac{\frac{4}{3}p\left(\frac{R}{2}\right)^3}{\frac{4}{3}pR^3}$$

$$= \frac{GMm}{d^2} - \frac{GMm}{\left(d - \frac{R}{2}\right)^2} \times \frac{1}{8}$$

$$= \frac{GMm}{d^2}\left[1 - \frac{1}{2\left(2 - \frac{R}{d}\right)^2}\right]$$

Satellite:

Gravitational Force is the centripetal force that keeps a satellite moving in a circular orbit.

$$F_c = F_g$$

$$m\frac{v^2}{r} = \frac{GM_E m}{r^2}$$

$$v^2 = \frac{GM_E}{r}$$

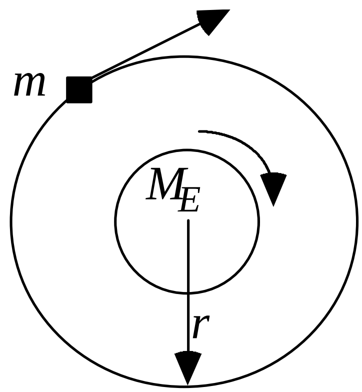
$$v = \sqrt{\frac{GM_E}{r}}$$

- The speed is independent of the mass of the satellite   a penny and a semi-truck would orbit at the same speed for the same radius.
- A satellite orbiting further away from the Earth has lower speed.
- The period of the orbit:

$$T_c = \frac{2pr}{v} = \frac{2pr}{\sqrt{GM_E / r}} = \sqrt{\frac{4p^2 r^3}{GM_E}}$$

- Communication satellites are positioned in orbit so that they "appear" stationary in the sky by placing them in a circular orbit with a period of 1 day.

What is the radius of the orbit of a geo-synchronous satellite?



$$r^3 = \frac{GM_E T_c^{\ 2}}{4p^2}$$

$$= \frac{(6.67\times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2})(5.98\times 10^{24}\ \text{kg})(86400\ \text{s})^2}{4p^2}$$

$$r = 4.2\times 10^7\ \text{m}$$

$$= 42{,}000\ \text{km}$$

Apparent Weightlessness:

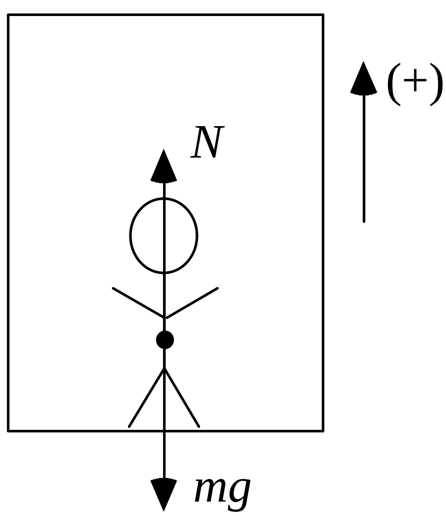
The weight of an object is defined as the force of gravity that the object feels. For an astronaut in a spacecraft, the weight is just:

$$F_g = \frac{GMm}{r^2}$$

where M is the mass of the earth and m is the mass of the astronaut. However, we see pictures of astronauts floating around in the space shuttle in an apparently weightless environment. Are they actually weightless?

Since we **define** weight to be the force of gravity, the astronauts are not weightless. We can define an **apparent weight** as being the normal force between an object and the support (e.g. floor).

Consider a ride in elevator:



- If the elevator is accelerating upward,

$$N - mg = ma$$

$$N = mg + ma$$

$$> mg$$

the reading on a scale will give an apparent weight larger than the actual weight because the scale measures the normal force.

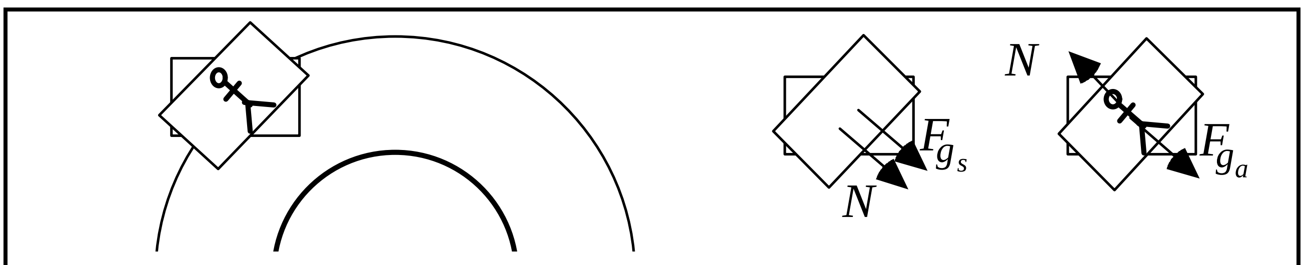
- If the elevator is accelerating downward, the apparent weight is smaller than the actual weight because normal force is less than *mg*.
- If the elevator is in a free fall,

$$N - mg = -mg$$

$$N = 0$$

the reading on a scale would be zero, giving a zero apparent weight.

Consider an astronaut in a shuttle:



Shuttle: $$G\frac{M_E m_s}{r^2} + N = m_s\frac{v^2}{r}$$

$$\frac{N}{m_s} = \frac{v^2}{r} - G\frac{M_E}{r^2} \qquad (1)$$

Astronaut: $$G\frac{M_E m_a}{r^2} - N = m_a\frac{v^2}{r}$$

$$\frac{N}{m_a} = G\frac{M_E}{r^2} - \frac{v^2}{r} \qquad (2)$$

$(1)+(2)$ $$\frac{N}{m_s} + \frac{N}{m_a} = 0$$

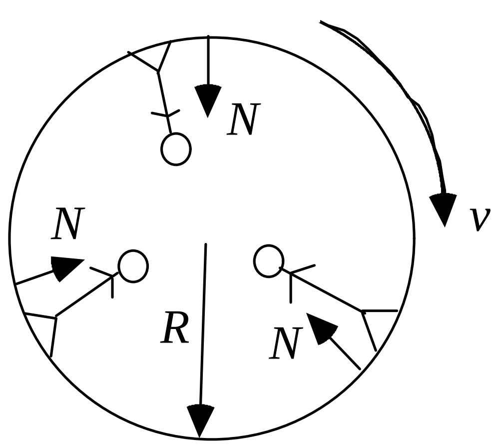
$$N = 0$$

astronauts floating in a shuttle. However, astronauts are not weightless:

$$F_g = G\frac{M_E m_a}{r^2} \neq 0$$

Artifical Gravity:

- can use cricular motion to create an apparent weight, similar to the idea of accelerating elevator.



Consider a spinning space station, the normal force exerted by the wall keeps the astronants in cricular motion. They feels the normal force like the gravity on Earth. Any value of *N* (artifical gravity) can be selected by adjusting *v* and *r*. For example, if we want a space station spinning at one revolution per minute to have an apparent gravity equal to that on Earth, what should be the radius of the station?

$$T = \frac{2pr}{v}$$

$$r = \frac{Tv}{2p}$$

$$mg = \frac{mv^2}{r}$$

$$v = \sqrt{rg}$$

$$r = \frac{T\sqrt{rg}}{2p}$$

$$\sqrt{r} = \frac{T\sqrt{g}}{2p}$$

$$r = \frac{T^2 g}{4p^2} = 890 \text{ m}$$