4. Two and Three Dimensional Motion

• Describe motion using position, displacement, velocity, and acceleration vectors

Position vector:

vector from origin to location of the object. $\stackrel{\mathrm{r}}{r} = x \,\hat{i} + y \,\hat{j} + z \,\hat{k}$

Displacement:

1D:
$$\Delta x = x_2 - x_1$$

3D: $\Delta \hat{r} = \hat{r}_2 - \hat{r}_1$
 $= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$
 $\equiv r_x\hat{i} + r_y\hat{j} + r_z\hat{k}$

Velocity:

1D:
$$v = \frac{dx}{dt}$$

Acceleration:

1D:
$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

3D:
$$\begin{aligned} \overset{\mathbf{r}}{a} &= a_{x}\hat{i} + a_{y}\hat{j} + a_{z}\hat{k} \\ &= \frac{d^{\mathbf{r}}}{dt} \\ &= \frac{d^{2}\overset{\mathbf{r}}{r}}{dt^{2}} \\ a_{x} &= \frac{dv_{x}}{dt^{2}} = \frac{d^{2}x}{dt^{2}} \\ a_{y} &= \frac{dv_{y}}{dt} = \frac{d^{2}y}{dt^{2}} \\ a_{z} &= \frac{dv_{z}}{dt} = \frac{d^{2}z}{dt^{2}} \end{aligned}$$

- The definition of individual components of a 3D vector is the same as 1D.
- Writing the equation in vector format (i.e. ^r_v = ^d/_{dt}) is a compact way of writing three equations, one for each component.
 On most cases, work with each
 - component separately;
 - no need to solve everything at once.

Problem 12 (p. 73)

The position \hat{r} of a particle is given by $\hat{r} = (2.00t^3 - 5.00t)\hat{i} + (6.00 - 7.00t^4)\hat{j}$, where r is in meters and t in seconds. Calculate (a) \dot{r} , (b) \dot{v} , (c) \dot{a} at t = 2.00 s, (d) What is the orientation of a line tangential to the particle's path at t = 2.00 s? (a) $\dot{r}(t = 2.00 \text{ s}) = [2.00(2.00)^3 - 5.00(2.00)]\hat{i}$ $+[6.00-7.00(2.00)^4]\hat{j}$ $= 6.00 \ \hat{i} - 106 \ \hat{j}$ (b) $\stackrel{\mathrm{r}}{v}(t) = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j}$ $=\frac{d}{dt}\left[2.00t^{3}-5.00t\right]\hat{i}+\frac{d}{dt}\left[6.00-7.00t^{4}\right]\hat{j}$ $= (6.00t^2 - 5.00)\hat{i} + (-28.0t^3)\hat{j}$

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$$v(t = 2.00 \text{ s}) = [6.00(2.00)^2 - 5.00]\hat{i}$$

-28.0(2.00)³ \hat{j}
= 19.0 \hat{i} - 224 \hat{j}
(c) $\overset{r}{a}(t) = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j}$
= $\frac{d}{dt} [6.00t^2 - 5.00]\hat{i} + \frac{d}{dt} [-28.0t^3]\hat{j}$
= 12.0t \hat{i} - 84.0t² \hat{j}
 $\overset{r}{a}(t = 2.00 \text{ s}) = 24.0 \hat{i} - 336 \hat{j}$

(d) The orientation is along the velocity, which is tangent to the path of the particle.

Constant Acceleration:

If acceleration is constant then each component of acceleration is also constant,

i.e. $\stackrel{\mathrm{r}}{a} = \mathrm{constant} \Rightarrow \begin{cases} a_x = \mathrm{constant} \\ a_y = \mathrm{constant} \\ a_z = \mathrm{constant} \end{cases}$

Equation of Motion:

Vector Form:

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_{0} + \mathbf{v}_{0}t + \frac{1}{2}\mathbf{r}_{0}^{\mathrm{r}}t^{2} \\ \mathbf{r}_{0} &= \mathbf{v} + \mathbf{r}_{0}^{\mathrm{r}}t \\ |\mathbf{v}|^{2} &= |\mathbf{v}_{0}|^{2} + 2\mathbf{r}_{0}^{\mathrm{r}} \cdot (\mathbf{r} - \mathbf{r}_{0}^{\mathrm{r}}) \end{aligned}$$

Component Form:

x - direction	y - direction	z - direction
$x = x_0 + v_{o_x}t + \frac{1}{2}a_xt^2$	$y = y_0 + v_{0_y}t + \frac{1}{2}a_yt^2$	$z = z_0 + v_{0_z}t + \frac{1}{2}a_zt^2$
$v_x = v_{o_x} + a_x t$	$v_y = v_{0_y} + a_y t$	$v_z = v_{0_z} + a_z t$
$v_x^2 = v_{0_x}^2 + 2a_x(x - x_0)$	$v_y^2 = v_{0_y}^2 + 2a_y(y - y_0)$	$v_z^2 = v_{0_z}^2 + 2a_z(z - z_0)$

- Time is the only common variable between motion in different directions
- Equation of motion for each direction is independent of any other quantities from other directions
 Motion in each direction is independent of motion in other directions

Example:

Spacecraft P131 is flying initially in a straight line with a speed of 15 km/s. Suddenly, it fires thrusters which produce an acceleration of 0.1 km/s² in the direction perpendicular to the direction of motion for 100 s. After the thrusters are turned off, (a) what is the velocity of the spacecraft with respect to the original direction of motion? (b) What is the speed? (c) How far is it from the point it started firing the thrusters?



(a)
$$v_y = v_{0y} + a_y t$$

 $= 0 + (0.1 \text{ km / s}^2)(100 \text{ s})$
 $= 10 \text{ km / s}$
 $v_x = v_{0x}$
 $= 15 \text{ km / s}$
(b) $v = \sqrt{v_x^2 + v_y^2}$
 $= \sqrt{(15 \text{ km / s})^2 + (10 \text{ km / s})^2}$
 $= 18 \text{ km / s}$
(c) $x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$
 $= v_{0x} t$
 $= (15 \text{ km / s})(100 \text{ s})$
 $= 1500 \text{ km}$

$$y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2$$

= $\frac{1}{2}a_yt^2$
= $\frac{1}{2}(0.1 \text{ km} / \text{s}^2)(100 \text{ s})^2$
= 500 km
 $r = \sqrt{x^2 + y^2}$
= $\sqrt{(1500 \text{ km})^2 + (500 \text{ km})^2}$
= 1600 km

 ☆ Solved each component separately and combined then at the end when necessary.

Questions:

(a) Which path best represents the path of the spacecraft?



(b) If spacecraft P132 is flying next to P131 with same initial velocity, but does not fire thrusters,

which plot best represents the position of the two crafts at 0, 25, 50, 75, and 100 s?



Projectile Motion

- 2D motion: vertical and horizontal
- Horizontal motion is independent of vertical motion. Time is the only variable that connects the two motions.
- Acceleration due to gravity acts only in the vertical direction.
- Horizontal velocity remains constant (neglecting air resistance)
- At any given height, the magnitude of the vertical velocity is the same whether rising or falling.
- At the maximum height, vertical velocity

is zero.The trajectory is parabolic.



Horizontal Motion:

$$x = x_0 + v_{0_x} t$$
$$= x_0 + v_0 (\cos q_0) t$$
Vertical Motion:

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

= $y_0 + v_0(\sin q_0)t - \frac{1}{2}gt^2$
 $v_y = v_{0y} - gt$
= $v_0 \sin q_0 - gt$
 $v_y^2 = v_{0y}^2 - 2g(y - y_0)$
= $(v_0 \sin q_0)^2 - 2g(y - y_0)$

Question:

If a bullet is fired directly toward a ball that has just been released, will the bullet hit the ball?



Plan: Calculate the bullet flight time *t* and then compare the vertical locations of bullet and ball

$$x = x_{0} + v_{0} (\cos q_{0})t$$

$$R = 0 + v_{0} (\cos q_{0})t$$

$$\Rightarrow t = \frac{R}{v_{0} \cos q_{0}}$$

$$y_{bullet} = y_{0} + v_{0} (\sin q_{0})t - \frac{1}{2}gt^{2}$$

$$= 0 + v_{0} \sin q_{0} \frac{R}{v_{0} \cos q_{0}} - \frac{1}{2}g \frac{R^{2}}{v_{0}^{2} \cos^{2} q_{0}}$$

$$= R \tan q_{0} - \frac{gR^{2}}{2v_{0}^{2} \cos^{2} q_{0}}$$

$$= R \frac{H}{R} - \frac{gR^{2}}{2v_{0}^{2} \cos^{2} q_{0}}$$

$$= H - \frac{gR^{2}}{2v_{0}^{2} \cos^{2} q_{0}}$$

$$= H + 0 - \frac{1}{2}g \frac{R^{2}}{v_{0}^{2} \cos^{2} q_{0}}$$

$$= y_{bullet} !!!$$

• The bullet will hit the ball for any *H*, *R*, *v*₀, and *q*₀.

Example:

A projectile is fired with an initial speed $v_0 = 30.0 \text{ m/s}$ at a target R = 20.0 m away. Find (a) the two possible firing angles and (b) the maximum height for the higher trajectory.

- One unknown (*q*), but two equations of motion (vertical and horizontal).
- Use time that connects the two equations as the second unknown.



$$y = y_0 + v_0(\sin q)t - \frac{1}{2}gt^2$$
$$0 = 0 + v_0(\sin q)t - \frac{1}{2}gt^2$$
$$v_0(\sin q)t = \frac{1}{2}gt^2$$
$$v_0\sin q = \frac{1}{2}gt$$
$$v_0\sin q = \frac{gR}{2v_0\cos q}$$
$$2\sin q\cos q = \frac{gR}{v_0^2}$$
$$\sin 2q = \frac{gR}{v_0^2}$$

$$2q = \sin^{-1} \left(\frac{gR}{v_0^2} \right)$$

$$q = \frac{1}{2} \sin^{-1} \left(\frac{gR}{v_0^2} \right)$$

$$= \frac{1}{2} \sin^{-1} \left[\frac{(9.8 \text{ m/s}^2)(20.0 \text{ m})}{(30 \text{ m/s})^2} \right]$$

$$= \frac{1}{2} \sin^{-1} (0.218)$$

$$= \frac{1}{2} (12.8^\circ \text{ or } 167^\circ)$$

$$= 6.4^\circ \text{ or } 83.7^\circ$$

(b) $v^2 = v_{0y}^2 - 2g(y - y_0)$
 $0 = (v_0 \sin q)^2 - 2gh_{\text{max}}$
 $h_{\text{max}} = \frac{(v_0 \sin q)^2}{2g}$

$$= \frac{\left[(30 \text{ m/s}) \sin 83.7^\circ \right]^2}{2 \times 9.8 \text{ m/s}^2}$$

$$= 45.4 \text{ m}$$

Conceptual Question:

1. A P131 student throws a ball straight up and the other straight down with same initial speed. Neglecting air resistance, the ball to hit the ground with greater speed is the one thrown:

(a) upward

(b) downward

(c) neither - they both hit at the same speed

2. The student throws one ball horizontally and at the same time releases another ball from rest. Which hits the ground first?

- (a) the ball thrown horizontally
- (b) the ball released at the same time
- (c) both hit at the same time

3. The figure below shows three paths for a kicked football. Ignoring air resistance, rank the paths according to (a) time of flight, (b) initial vertical velocity, (c) initial horizontal velocity, and (d) initial speed.



Uniform Circular Motion:

An object moves in a circle with a constant **speed** (not velocity!):



- Since velocity is tangent to the path, the velocity is always tangent to the circle.
- The speed is constant, i.e. the length of the velocity vector does not change but the direction does change a motion with a constant speed but changing velocity!
- Since the velocity vector is changing

there is an acceleration, i.e. the object is accelerating but the speed is constant!

Centripetal Acceleration:



$$\Delta t = \frac{\text{distance}}{v}$$
$$= \frac{r(2q)}{v}$$
$$v_{1_x} = v \cos q$$
$$v_{1_y} = v \sin q$$
$$v_{2_x} = v \cos q$$
$$v_{2_y} = -v \sin q$$
$$\langle a_x \rangle = \frac{\Delta v_x}{\Delta t}$$
$$= \frac{v \cos q - v \cos q}{\Delta t}$$
$$= 0$$
$$\langle a_y \rangle = \frac{\Delta v_y}{\Delta t}$$
$$= \frac{-v \sin q - v \sin q}{r2q}$$
$$= -\frac{v^2 \sin q}{rq}$$

$$a_{y} = \lim_{\Delta t \to 0} \langle a_{y} \rangle$$
$$\equiv \lim_{q \to 0} \langle a_{y} \rangle$$
$$= -\frac{v^{2} q}{r q}$$
$$= -\frac{v^{2}}{r}$$

• The acceleration always points directly toward the center

centripetal acceleration, i.e. "center seeking"



 Like the velocity vector, the direction of acceleration is always changing. Acceleration is not constant but the magnitude is a constant:

$$a_{c} = \frac{v^{2}}{r}$$
Period of Revolution:

$$T = \frac{circumference}{speed}$$

$$= \frac{2pr}{v}$$