

7. Kinetic Energy and Work

Kinetic Energy:

The kinetic energy of a moving object:

$$k = \frac{1}{2} mv^2$$

- Kinetic energy is proportional to the square of the velocity. If the velocity of an object doubles, the kinetic energy increases by a factor of four.
- Kinetic energy is proportional to the mass. If a bowling ball and a ping pong ball have the same velocity, the bowling ball has much larger kinetic energy.
- Kinetic energy is always positive.
- unit : Joule (J) = $\text{kg} \cdot \text{m}^2 / \text{s}^2$

Example:

If we drop a 3-kg ball from a height of $h = 10$ m, the velocity when the ball hits the ground is given by:

$$v^2 = v_0^2 + 2a(y - y_0)$$

$$= 0 - 2g(0 - h)$$

$$v = \sqrt{2gh}$$

$$= \sqrt{2(9.8 \text{ m/s}^2)(10 \text{ m})}$$

$$= 14 \text{ m/s}$$

$$\text{Initial: } k = \frac{1}{2}mv^2 = 0$$

$$\text{Final: } k = \frac{1}{2}mv^2$$

$$= \frac{1}{2}(3 \text{ kg})(14 \text{ m/s})^2$$

$$= 294 \text{ J}$$

So as the ball falls, its kinetic energy increases. It is the gravitational force that accelerates the ball, causing the speed to increase. The increase in speed also increases the kinetic energy. The process of a force changing the kinetic energy of an object is called **work**.

Work:

Work is the energy transferred to or from an object by means of a force acting on the object.

- energy transferred to an object is positive work, e.g. gravity performs positive work on a

falling ball by transferring energy to the ball, causing the ball to speed up.

- energy transferred from an object is negative work, e.g. gravity performs negative work on a ball tossed up by transferring energy from the ball, causing the ball to slow down.
- both kinetic energy and work are scalars.
- unit: J

Work Energy Theorem:

The work done is equal to the change in the kinetic energy:

$$\Delta K = K_f - K_i = W$$

In the above example with the ball falling from a height of $h = 10$ m, the work done by gravity:

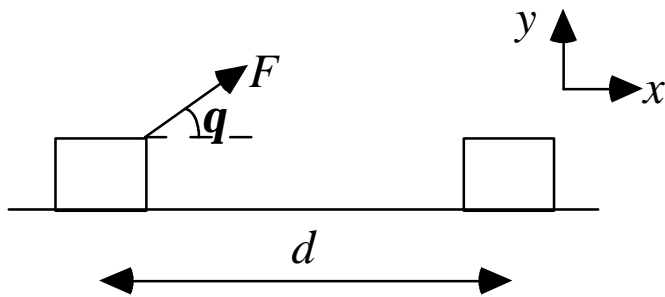
$$W = \Delta k = k_f - k_i = 294 \text{ J} - 0\text{J} = 294 \text{ J}.$$

If a ball rises to a height of $h = 10$ m, the work done by gravity:

$$W = \Delta k = k_f - k_i = 0\text{J} - 294 \text{ J} = -294 \text{ J}.$$

Work Done by a Force:

Consider a box being dragged a distance d across a frictionless floor:



$$v^2 = v_0^2 + 2a_x(x - x_0)$$

$$v^2 = v_0^2 + 2a_x d$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + ma_x d$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = ma_x d$$

$$k_f - k_i = (F \cos q)d$$

$$\therefore W = (F \cos q)d$$

- q is the angle between the force vector and the direction of motion.
- If the force is perpendicular to the direction of motion, then the work done:

$$W = (F \cos q)d = Fd \cos 90^\circ = Fd \times 0 = 0.$$

- The work energy theorem and the relationship between work and force are valid only if the force does not cause any other form of energy to change, e. g. we can not apply the theorem when friction is

involved because it causes a change in the thermal energy (temperature).

Work Done by Multiple Forces:

The total work done by many forces acting on an object:

$$W_{tot} = F_1 \cos \theta_1 d + F_2 \cos \theta_2 d + F_3 \cos \theta_3 d + \dots$$

where the angles are the angle between each force and the direction of motion. The total work is just the sum of individual work from each force:

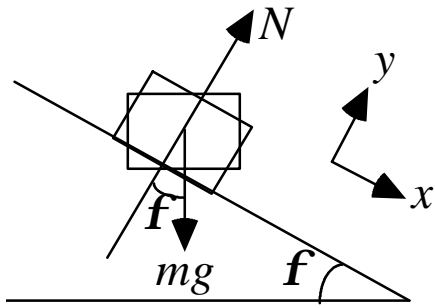
$$W_{tot} = W_1 + W_2 + W_3 + \dots$$

The work energy theorem relates the changes in the kinetic energy to the total work performed on the object:

$$\Delta K = W_{tot}$$

Example:

A 3-kg box initially at rest slides 3 m down a frictionless 30° incline. What is the work done on the object? What is the kinetic energy and speed at the bottom?



- The work done is performed by the force in the x direction since there is no motion in the y direction:

$$\begin{aligned}
 W &= F_x d \\
 &= (mg \sin f) d \\
 &= (3 \text{ kg})(9.8 \text{ m / s}^2)(\sin 30^\circ)(3 \text{ m}) \\
 &= 44 \text{ J}
 \end{aligned}$$

Alternatively,

$$\begin{aligned}
 W &= (F \cos \mathbf{q}) d \\
 &= F \cos(90^\circ - f) d \\
 &= F \sin f d
 \end{aligned}$$

★ The first method of using the component of the force in the direction of motion for the calculation is easier.

$$W = k_f - k_i$$

$$= k_f - 0$$

$$k_f = W = 44 \text{ J}$$

$$W = \frac{1}{2}mv^2$$

$$v^2 = \frac{2W}{m}$$

$$v = \sqrt{\frac{2W}{m}}$$

$$= \sqrt{\frac{2 \times 44 \text{ J}}{3 \text{ kg}}}$$

$$= 5.4 \text{ m / s}$$

★ calculate the speed without using the equation of motion.

Conceptual Question:

A 1-kg ball on a string is moving in uniform circular motion. The tension is providing the centripetal force on the ball. In one revolution, the work done by the tension on the ball is:

a) $2\pi rT$

b) $-2\pi rT$

c) $2prT \cos q$

Problem 15 (p150):

A 12-m long fire hose is uncoiled by pulling the nozzle end horizontally along a frictionless surface at the steady speed of 2.3 m/s. The mass of 1.0 m of hose is 0.25 kg. How much work has been done on the hose by the applied force when the entire hose is moving?

★ difficult to solve the problem by calculating the applied force that causes the front end of the hose to move a large distance (12 m) while the other end has not moved very far

use work energy theorem

$$m = \text{length} \times (\text{mass per unit length})$$

$$= 12 \text{ m} \times \frac{0.25 \text{ kg}}{1.0 \text{ m}}$$

$$= 3 \text{ kg}$$

$$W = k_f - k_i$$

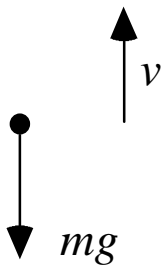
$$= \frac{1}{2}mv^2 - 0$$

$$= \frac{1}{2}(3 \text{ kg})(2.3 \text{ m / s})^2$$

$$= 7.9 \text{ J}$$

Work done by Gravity:

Ball traveling up:

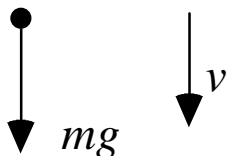


- Work done by gravity is negative because gravity and the direction of motion are in opposite direction.

consistent with the decrease in kinetic energy because the ball is slowing down:

$$W_g = \Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 < 0$$

Ball traveling down:



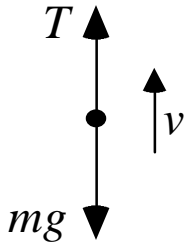
- Work done by gravity is positive because gravity and the direction of motion are in same direction.

consistent with the increase in kinetic energy because the ball is speeding up:

$$W_g = \Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 > 0$$

Question:

What is the total work done on a ball that is being pulled up a distance d with constant speed using a string?



$$\Sigma F = T - mg = ma = 0$$

$$T = mg$$

$$W_g = -mgd$$

$$W_T = Td = mgd$$

$$W_{tot} = W_g + W_T$$

$$= -mgd + mgd$$

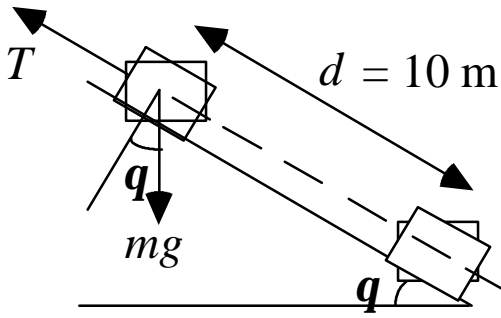
$$= 0$$

$$\Delta k = W_{tot} = 0$$

consistent with the fact that the ball is being pulled up with constant speed.

Example:

A 10-kg crate initially at rest is being lowered down a frictionless 10-m ramp using a rope. The speed of the crate is 3 m/s at the bottom of the 40° ramp. Determine (a) the work done by gravity, (b) the work done by the rope, and (c) the tension in the rope.



$$(a) \quad W_g = mg \sin q d$$

$$= 630 \text{ J}$$

$$(b) \quad W_{tot} = W_g + W_T = \Delta K$$

$$mg \sin q d + W_T = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$$W_T = \frac{1}{2}mv^2 - mg \sin q d$$

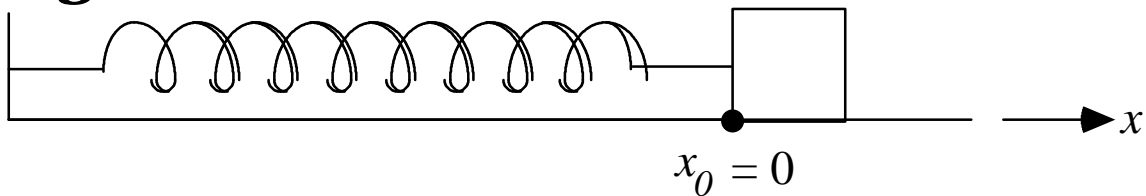
$$= -585 \text{ J}$$

$$(c) \quad W_T = -Td$$

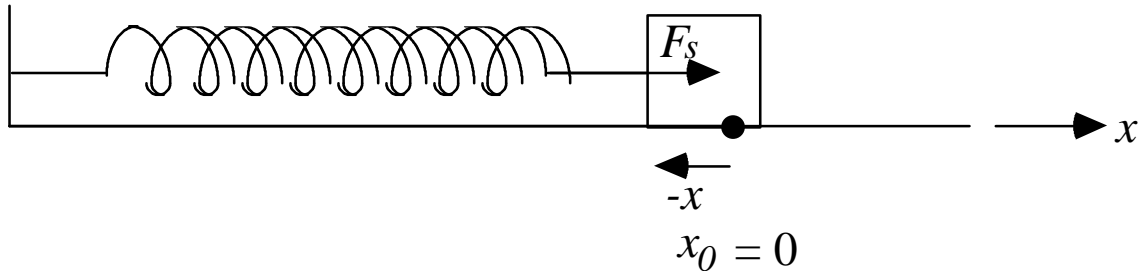
$$T = -\frac{W_T}{d}$$

$$= 58.5 \text{ N}$$

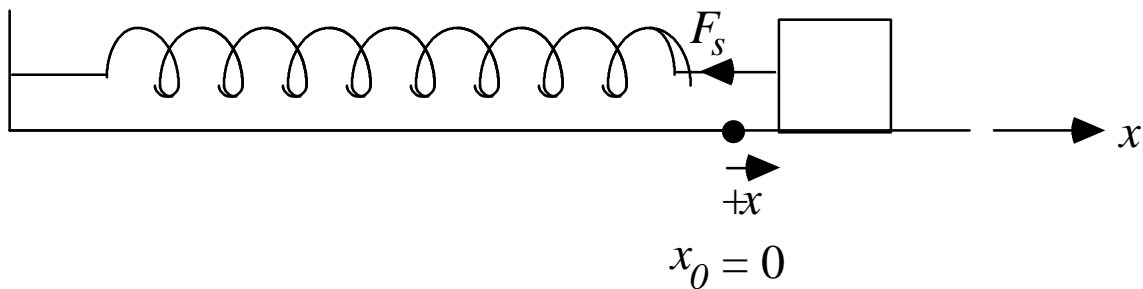
Spring Force and Work:



When a spring is in a relaxed state, i.e. neither compressed nor stretched, it applied no force to the attached object.



When a spring is compressed, the spring pushes the object back toward the relaxed position.



When a spring is stretched, the spring pulled the object back toward the relaxed position.

Hooke's Law:

The magnitude of the restoring force of a spring is proportional to the distance from the relaxed position:

$$F = -kx$$

where k is the spring constant.

- The direction of the spring force is always toward the "relaxed" or equilibrium position, i.e. this is a *restoring* force.
- The force is *not* constant: increases linearly with displacement.
- Hooke's law applies when the spring is not compressed or stretched too far, i.e. the spring should not be deformed.
- Unit of k : N/m.

Work Done By a Spring:

Since the force is not constant, we must perform an integral to determine the work: The work done by the spring when moving a small distance dx :

$$dW = Fdx = -kxdx$$

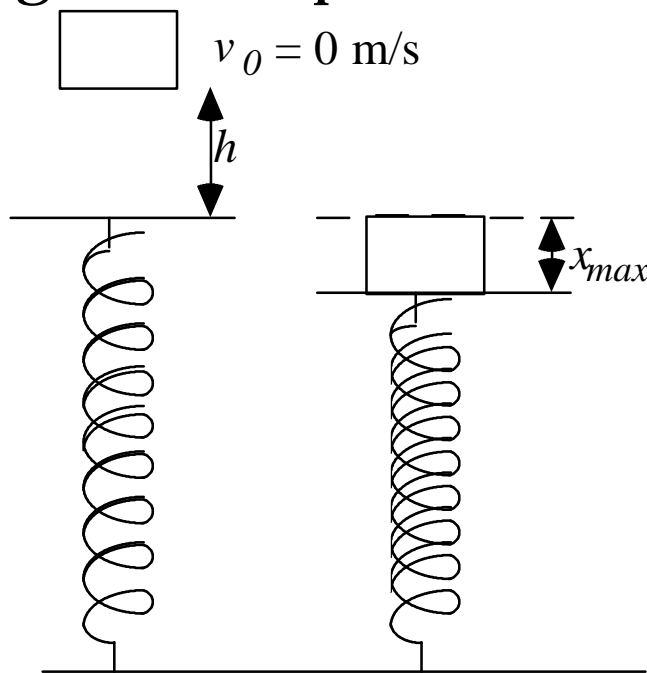
The total work done in moving from position x_i to x_f :

$$W = -\int_{x_i}^{x_f} kxdx = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

Example:

A 3-kg mass is dropped from a height $h = 5$ m above a platform with a spring of spring constant

$k = 500 \text{ N/m}$. What is the maximum distance the spring will compress?



- The total work done on the box must be zero because the box is at rest at the beginning and at the end when the spring is compressed:

$$\Delta W = W_g + W_s = \Delta K = K_f - K_i = 0$$

$$W_g = -W_s$$

$$mg(h + x_{\max}) = -(\frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2)$$

$$mg(h + x_{\max}) = \frac{1}{2}kx_{\max}^2$$

$$\frac{1}{2}kx_{\max}^2 - mgx_{\max} - mgh = 0$$

$$\frac{1}{2} \cdot 500 \cdot x_{\max}^2 - 3 \cdot 9.8 \cdot x_{\max} - 3 \cdot 9.8 \cdot 5 = 0$$

$$250x_{\max}^2 - 30x_{\max} - 150 = 0$$

$$x_{\max} = -0.7 \text{ m or } 0.83 \text{ m}$$

Power:

Consider a box being dragged up a ramp. The work done is:

$$W = F \cos \theta d$$

Whether the box is dragged quickly or slowly up the ramp, the work done is the same. However, doing the work quickly requires more effort. The quickness of doing the work is characterized using the concept of **Power**:

Power is the work done per unit time.

Average Power:

$$\langle P \rangle = \frac{W}{\Delta t}$$

Instantaneous Power:

$$P = \frac{dW}{dt}$$

Unit: Watts (W) = J/s

This unit is also being used in other science besides describing mechanical work, e.g. a 100-W light bulb, i.e. the light bulb uses 100 J of energy in 1 s.

1 horsepower = 746 W

For a constant force,

$$\begin{aligned} P &= \frac{dW}{dt} \\ &= \frac{d}{dt} [(F_{\text{dir. of motion}})x] \\ &= (F_{\text{dir. of motion}}) \frac{dx}{dt} \\ &= (F_{\text{dir. of motion}})v \end{aligned}$$

Example:

If a box is being pulled up a ramp at a speed of 0.5 m/s by a 100-N force, what is the power supplied by the rope? If the ramp is 10 m long, what is the work done by the rope?

$$P = FV = (100 \text{ N})(0.5 \text{ m / s}) = 50 \text{ W}$$

$$W = Fd = (100 \text{ N})(10 \text{ m}) = 1000 \text{ J}$$

$$= Pt = (50 \text{ W})\left(\frac{10 \text{ m}}{0.5 \text{ m / s}}\right) = (50 \text{ W})(20 \text{ s}) = 1000 \text{ J}$$