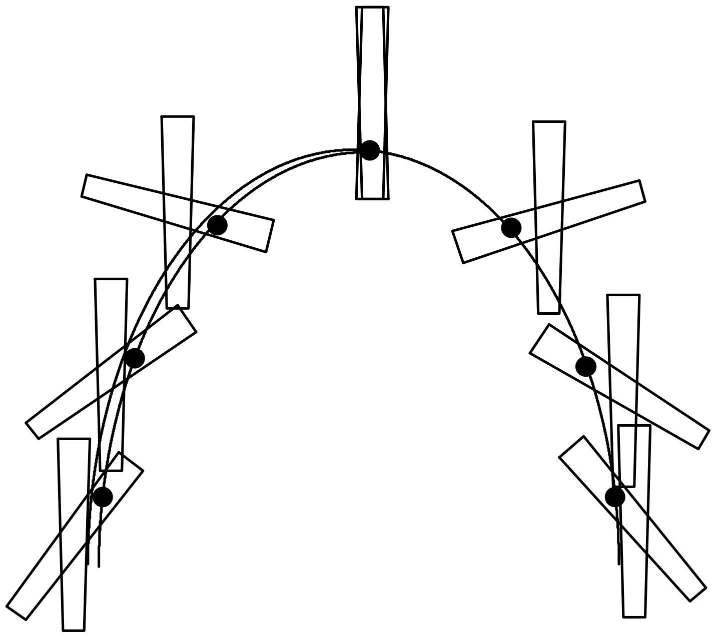
9. System of Particles

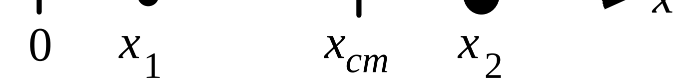


Consider a baseball bat being flopped into the air. Every part moves in a different way. However there is a special point on the bat that moves in a simple parabolic path. This point is called the center of mass.

Center of Mass:

The center of mass of a body or a system of bodies is the point that moves as though all the mass were concentrated there and all external forces were applied there.

Calculation of Center of Mass:



$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

- For this two-body system, the center of mass is closer to the heavier object.
- Center of mass does not need to be located on either particle.

Generalizing to more than two bodies along a line:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum m_i x_i}{M_{total}}$$

The *y* and *z* coordinates of the center of mass of a system of particles extended in three dimensions) :

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum m_i y_i}{M_{total}}$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_n z_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum m_i z_i}{M_{total}}$$

or vectorically:

$$\vec{r}_{cm} = \frac{1}{M_{total}} (m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots + m_n \vec{r}_n)$$

For a real extended object, we need to integrate over the distribution of mass:

$$x_{cm} = \frac{1}{M} \int x dm$$

$$y_{cm} = \frac{1}{M} \int y dm$$

$$z_{cm} = \frac{1}{M} \int z dm$$

For an object with constant density:

$$r = \frac{dm}{dv} = \frac{M}{V}$$

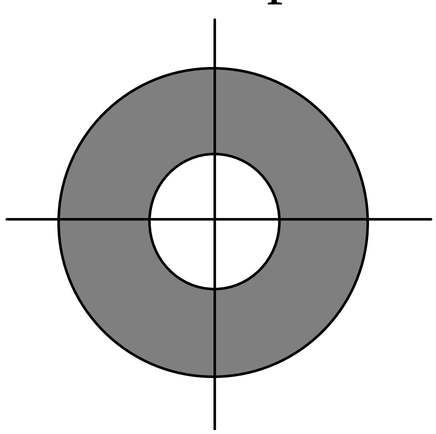
$$x_{cm} = \frac{1}{V}\int x dv$$

$$y_{cm} = \frac{1}{V}\int y dv$$

$$z_{cm} = \frac{1}{V}\int z dv$$

Use of Symmetry in Estimating Center of Mass:

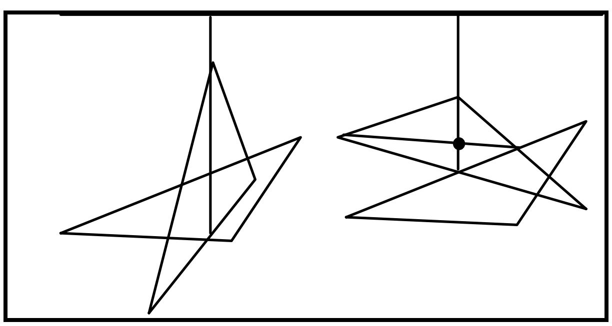
Sometimes it is easier to determine the center of mass using symmetry. For an object of uniform density, the center of mass is at the geometric center. For example, the center of mass of a doughnut is in the middle of the hole. Note that it is in free space where there is no material.



Similarly, by symmetry argument, the center of mass of an uniform square box is in the center of the box.

For irregularly shaped objects, the center of mass can be determined experimentally:

1. Take an object and hang it by a string from a point on the object
2. Draw a line on the object vertically along the string and straight down
3. Hand the object from a second point on the object and draw a second line
4. The intersection of the two lines is the center of mass



Newton's 2nd Law for a System of Particles:

$$m\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n$$

$$m\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n$$

$$m\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n$$

$$= \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$$

$$m\vec{a}_{cm} = \sum \vec{F}_{ext} + \sum \vec{F}_{int}$$

$$\sum \vec{F}_{ext} = m\vec{a}_{cm}$$

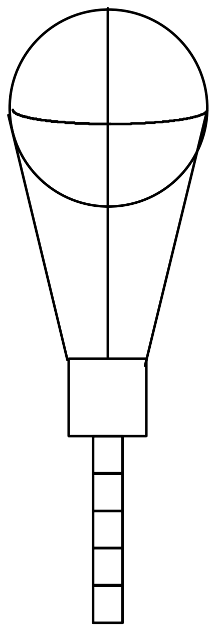
$$\sum F_{x_{ext}} = ma_{x_{cm}}$$

$$\sum F_{y_{ext}} = ma_{y_{cm}}$$

$$\sum F_{z_{ext}} = ma_{z_{cm}}$$

Example:

A man of mass *m* clings to a rope ladder suspended below a stationary balloon of mass *M*. (a) If he begins to climb the ladder at speed *v* with respect to the ladder, in what direction and with what speed with respect to the earth will the balloon move? (b) What is the state of the motion after the man stops climbing?



- The man and the balloon represent a closed system with no net force. So the center of mass of the system is initially at rest and will remain at rest even though there are internal forces.

(a) If the man is climbing up the ladder at speed v and the balloon and ladder are moving down with v_b, then the man's velocity with respect to the ground is:

$$v \qquad v - v_b$$

$$+ M)v \qquad mv_g - Mv \qquad 0$$

$$mv - mv \qquad Mv_b = 0$$

$$+ Mv_b =$$

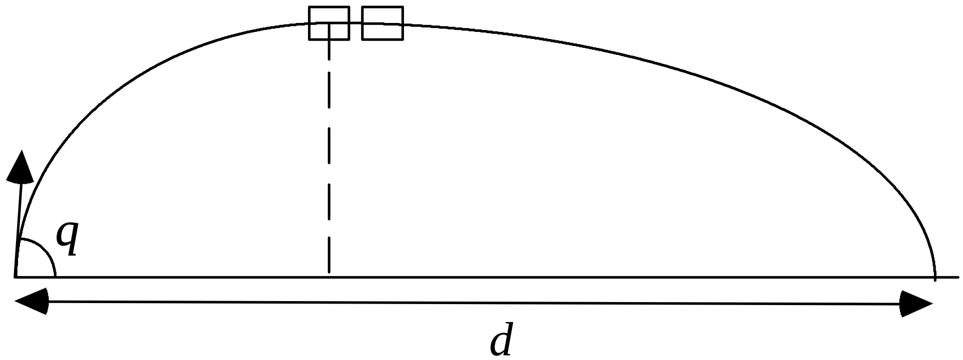
$$= \frac{mv}{m \qquad}$$

- If the balloon is very heavy, $M >> m$, then the velocity of the balloon is very small.

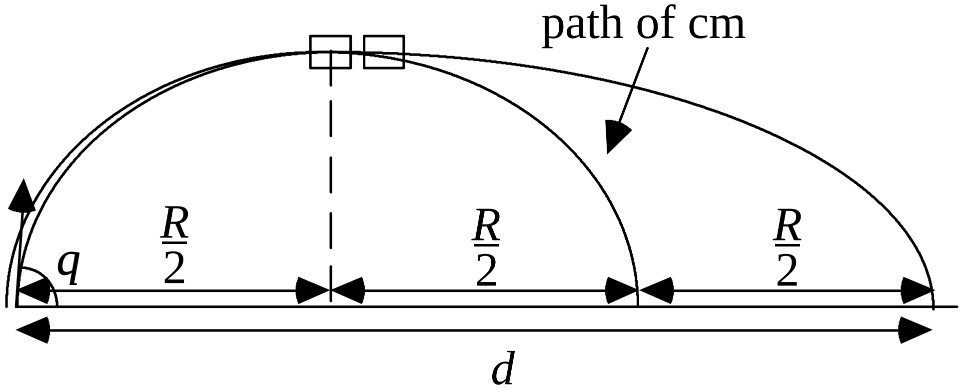
(b) When the man stops climbing (v = 0), the balloon will stop moving as well.

Problem 21 (p. 209):

A shell is fired from a gun with a muzzle velocity of 20 m/s at angle of 60° with the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass. One fragment, whose speed immediately after the explosion is zero, falls vertically. Neglecting air drag, how far from the gun does the other fragment land?



- Since the explosion represents an internal force, it will not alter the path of the center of mass. So when the two pieces hit the ground, the center of mass will be exactly where the shell would have been had it not exploded.



- The total distance of the part of the shell that flies furthest is just 3/2 the distance that the shell would have flown.

$$v_y = v_{0_y} + at$$

$$0 = v_0 \sin q - gt$$

$$t = \frac{v_0 \sin q}{g}$$

$$d = \tfrac{3}{2}(v_0 \cos q \times 2t)$$

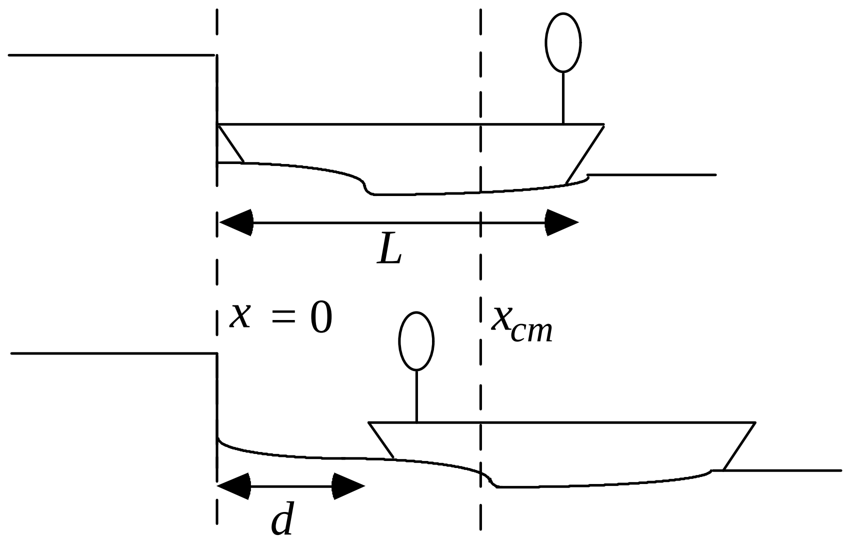
$$= \tfrac{3}{2}\left(v_0 \cos q \times \frac{2 v_0 \sin q}{g} \right)$$

$$= \frac{3 v_0^2 \sin 2q}{2g}$$

$$= 53 \text{ m}$$

Example:

A 400-kg boat has its stern just touching a dock. A 80-kg person is standing at the bow a distance of 10 m away. The person begins walking toward the stern to get off the boat. Assuming that there is no friction between the water and the boat, how far is the person from the dock when he reached the stern.



- Since there are no external forces on the person-boat system, the center of mass must remain at the same point.

Person at the bow:

$$x_{cm} = \frac{m_p L + m_b \frac{L}{2}}{m_p + m_b}$$

Person at the stern:

$$x_{cm} = \frac{m_p d + m_b\left(\frac{L}{2} + d\right)}{m_p + m_b}$$

$$m_p d + m_b\left(\frac{L}{2} + d\right) = m_p L + m_b\left(\frac{L}{2}\right)$$

$$m_p d + m_b d = m_p L$$

$$d = \frac{m_p L}{m_p + m_b}$$

$$= \frac{(80 \text{ kg})(10 \text{ m})}{80 \text{ kg} + 400 \text{ kg}}$$

$$= 1.66 \text{ m}$$

★ If the boat is very heavy, the distance the boat moves is small.

Linear Momentum:

Linear momentum is defined as the mass of the particle times the velocity of the particle:

$$\vec{p} = m\vec{v}$$

- The momentum vector points in the same direction as the velocity vector.
- We often drop the word "linear" and just say "momentum."

- Unit: kg·m / s.

Newton's Second Law:

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

- This is the original Newton's 2nd Law.

Conservation of Momentum:

For a system of particles:

$$\vec{p} = m\vec{v}_{cm}$$

$$\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt} \qquad (1)$$

If there is no net external force:

$$\frac{d\vec{p}}{dt} = \sum \vec{F}_{ext} = 0$$

$$\vec{p}_i = \vec{p}_f$$

- Momentum of the system is not changing with time, i.e. momentum is conserved.

In equation (1):

$$\text{If } \sum F_{x_{ext}} = 0 \Rightarrow \frac{dp_x}{dt} = 0 \Rightarrow p_{x_i} = p_{x_f}$$

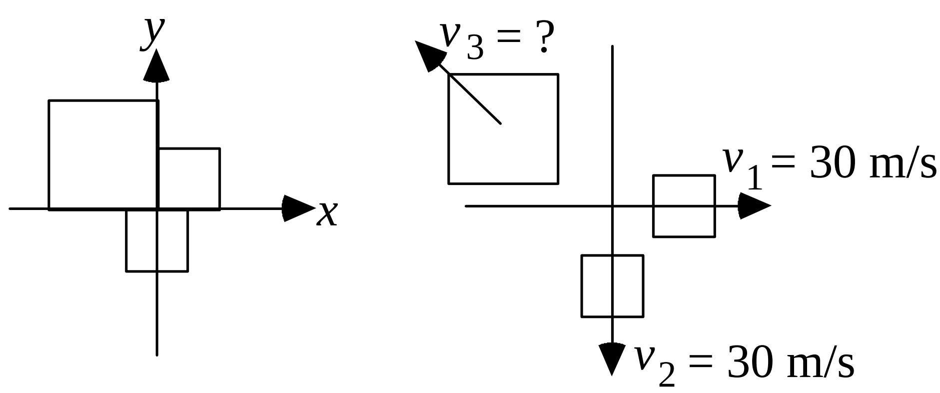
$$\text{If } \sum F_{y_{ext}} = 0 \Rightarrow \frac{dp_y}{dt} = 0 \Rightarrow p_{y_i} = p_{y_f}$$

$$\text{If } \sum F_{z_{ext}} = 0 \Rightarrow \frac{dp_z}{dt} = 0 \Rightarrow p_{z_i} = p_{z_f}$$

- If the sum of the external forces is zero in the *x*-direction but not zero in the *y*-direction, then the *x*-component of the momentum is conserved but the *y*-component is not conserved.

Problem 43 (p. 211):

A vessel at rest explodes into three pieces. Two pieces of equal mass fly off perpendicular to one another with the same speed of 30 m/s. What are the direction and magnitude of the velocity of the third piece if it has a mass three times that of the other individual piece?



$$p_{x_i} = p_{x_f}$$

$$0 = p_{x_3} + p_{x_1}$$

$$= 3mv_{x_3} + mv_{x_1}$$

$$v_{x_3} = -\tfrac{1}{3}v_{x_1}$$

$$= -10 \text{ m / s}$$

$$p_{y_i} = p_{y_f}$$

$$0 = p_{y_3} + p_{y_2}$$

$$= 3mv_{y_3} + mv_{y_2}$$

$$v_{y_3} = -\tfrac{1}{3}v_{y_2}$$

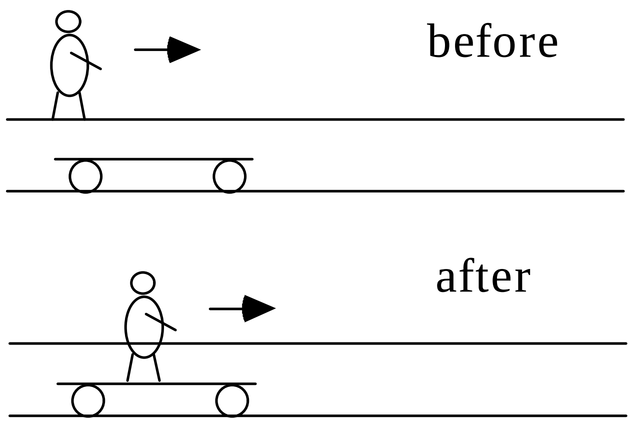
$$= -10 \text{ m / s}$$

$$v_3 = \sqrt{v_{x_3}{}^2 + v_{y_3}{}^2}$$

$$= 14 \text{ m / s (at } 45°\text{)}$$

Problem 44 (p.211):

A 242-kg sumo wrestler running at 5.3 m/s jumps onto a 2140-kg railroad flatcar initially at rest. What is the speed of the flatcar if he then (a) stands on it, (b) runs at 5.3 m/s relative to it, and (c) turns and runs at 5.3 m/s relative to the flatcar opposite his original direction?



- The momentum of the system before and after the sumo wrestler jumps on the flatcar must be the same.

(a)
$$p_i = p_f$$
$$m_w v_{i_w} + 0 = (m_w + m_c) v_f$$
$$v_f = \frac{m_w v_{i_w}}{m_w + m_c}$$
$$= 0.54 \text{ m / s}$$

(b)
$$p_i = p_f$$
$$m_w v_{i_w} + 0 = m_w (v_f + v_{i_w}) + m_c v_f$$
$$0 = m_w v_f + m_c v_f$$
$$v_f = 0 \text{ m / s}$$

(c)
$$p_i = p_f$$
$$m_w v_{i_w} + 0 = m_w (v_f - v_{i_w}) + m_c v_f$$
$$2 m_w v_{i_w} = m_w v_f + m_c v_f$$
$$v_f = \frac{2 m_w v_{i_w}}{m_w + m_c}$$
$$= 1.08 \text{ m / s}$$

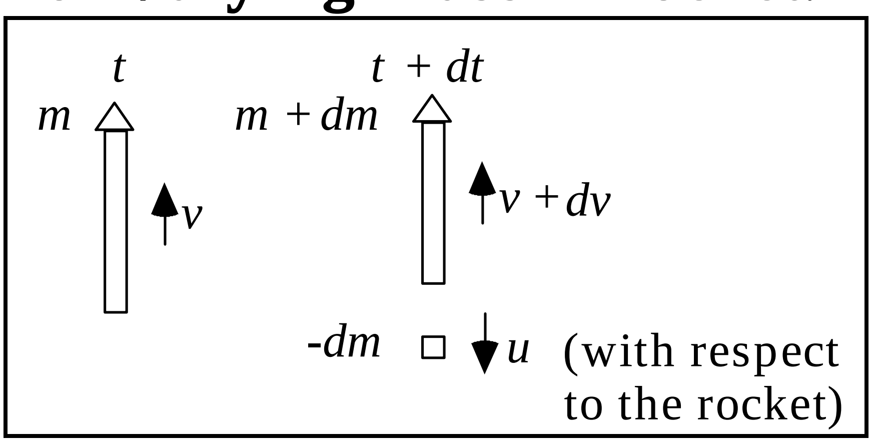
Conceptual Problems:

(a) A man is standing in the middle of a large frozen lake with a large bag of gold coins. Because the lake is frictionless, no net external force is acting on him and by conservation of momentum he can't walk anywhere. How can he get off the lake?

(b) An open freight car rolls friction free along a horizontal track in a pouring rain that falls vertically. As water accumulates in the car, its speed:

1. Increases
2. Decreases
3. Stays the same

System of Varying Mass -- Rocket:



In a rocket, the majority of its weight is in the fuel. The rocket speeds up due to conservation of momentum as the fuel is ejected backward.

$$
\begin{aligned}
p_i &= p_f \\
mv &= (m+dm)(v+dv)+(-dm)(v+dv-u) \\
mv &= mv+dmv+mdv+dmdv \\
&\quad -dmv-dmdv+dmu \\
0 &= mdv+dmu \\
-dmu &= mdv \qquad (1) \\
-\frac{dm}{dt}u &= m\frac{dv}{dt} \\
&= ma
\end{aligned}
$$

Let $R=-\frac{dm}{dt}$ be the rate at which fuel is being burned:

$$Ru = ma$$

- Ru is often referred to as the **thrust** of the rocket.

The instantaneous velocity is given by (1):

$$mdv = -udm$$

$$dv = -u\frac{dm}{m}$$

$$\int_{v_i}^{v_f} dv = -u\int_{m_i}^{m_f} \frac{dm}{m}$$

$$v_f - v_i = -u \ \mathrm{1n} \ m_f + u \ \mathrm{1n} \ m_i$$

$$v_f - v_i = u\mathrm{1n}\left(\frac{m_i}{m_f}\right)$$

$$m_f = m_i - Rt$$

$$v_f = v_i + u\mathrm{1n}\left(\frac{m_i}{m_i - Rt}\right)$$

Example:

The Saturn V rocket used for the Apollo mission to the moon had a first stage which supplied a thrust of 34 MN (mega Newtons) and burned fuel at a rate of 13.8 metric tons per second for a total of 150 s. The initial mass of the rocket was 2850 metric tons. Neglecting air resistance and gravity, determine the velocity of the rocket when the first stage burned out.

- First we need to determine the velocity u of the exhaust:

$$\text{Thrust} = Ru$$

$$u = \frac{\text{Thrust}}{R}$$

$$= \frac{3.4 \times 10^7 \text{ N}}{13.8 \times 10^3 \text{ kg/s}}$$

$$= 2464 \text{ m/s}$$

$$v_f = v_i + u \ln\left(\frac{m_i}{m_i - Rt}\right)$$

$$= 0 + 2464 \ln\left(\frac{2850}{2850 - 13.8 \times 150}\right)$$

$$= 3212 \text{ m/s } (7200 \text{ mph})$$

★ The actual final sped is 2400 m / s (5400 mph) due to air friction.