Capacitors and Inductors

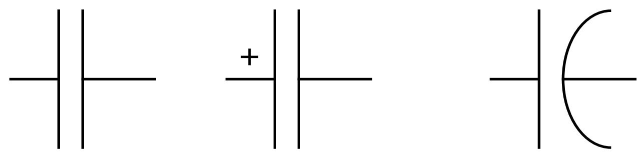
1) Capacitance:

•Capacitance (C) is defined as the ratio of charge (Q) to voltage (V) on an object.

•Define capacitance by: $C = Q/V$ = Coulombs/Volt = Farad.

•Capacitance of an object depends on geometry and its dielectric constant.

•Symbol(s) for capacitors:



polarized

•A capacitor is a device that stores electric charge (memory devices).

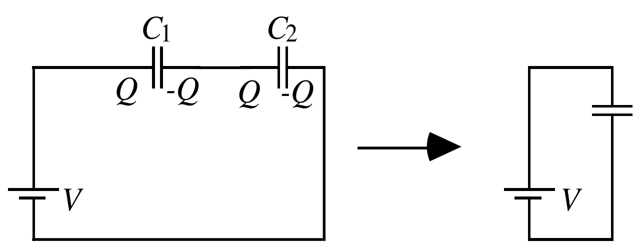
•A capacitor is a device that stores energy

$$E = \frac{Q^2}{2C}$$

•Capacitors are easy to fabricate in small sizes (μm), use in chips.

<u>Some Simple Capacitor circuits:</u>

•Two capacitors in series:



$$C = \frac{C_1 C_2}{C_1 + C_2}$$

Apply Kirchhoff's law:

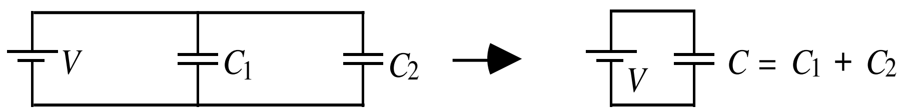
$$\begin{aligned} V &= V_1 + V_2 \\ &= \frac{Q}{C_1} + \frac{Q}{C_2} \\ &\equiv \frac{Q}{C_{tot}} \end{aligned}$$

$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} = \sum \frac{1}{C_i}$$

i.e. capacitors in series add like resistors in parallel.

Note the *total* capacitance is *less* than the individual capacitance.

•Two capacitors in parallel:



Again, use Kirchhoff's law:

$$V = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

The total charge in the circuit is:

$$\begin{aligned} Q &= Q_1 + Q_2 \\ &= V(C_1 + C_2) \\ &\equiv VC_{tot} \end{aligned}$$

$$C_{tot} = C_1 + C_2 = \sum C_i$$

i.e. capacitors in parallel add like resistors in series.

Note the *total* capacitance is *more* than the individual capacitance.

<u>Energy and Power in Capacitors</u>

•How much energy is stored in a capacitor?

If a charge (Q) moves through a potential difference (V) the amount of energy (E) the charge gains or loses is:

$$E = Q \cdot V$$

If we consider the case of the capacitor where we add charge and keep the voltage constant, the change in energy is:

$$dE = V \cdot dQ$$

$$V = Q / C$$

$$dE = \frac{Q}{C} dQ$$

$$E = \int_0^Q \frac{Q}{C} dQ$$

$$E = \frac{Q^2}{2C} \quad \text{or} \quad \frac{CV^2}{2}$$

Example: How much energy can a "typical" capacitor store?

Pick a 4 μF Cap (it would read 4 mF) rated at 3 kV.

Then $E = 0.5 \cdot (4\times10^{-6}) \cdot (3\times10^{3})^2 = 18$ Joules

This is the same as dropping a 2 kg weight (about 4 pounds) 1 meter.

•How much power is dissipated in a capacitor?

$$Power = \frac{dE}{dt}$$

$$= \frac{d}{dt}\left(\frac{CV^2}{2}\right)$$

$$P = CV\frac{dV}{dt}$$

Note: *dV/dt* must be finite otherwise we source (or sink) an infinite amount of power! THIS WOULD BE UNPHYSICAL!

Thus, the voltage across a capacitor cannot change instantaneously. This is a useful fact when trying to guess the transient (short term) behavior of a circuit.

However, the voltage across a resistor can change instantaneously as the power dissipated in a resistor does not depend on *dV/dt* ($P = I^2 \bullet R$ or V^2/R for a resistor).

•Why do capacitors come in such small values?

Example: Calculate the size of a 1 Farad parallel capacitor with air between the plates.
For a parallel plate capacitor:

$$C = \frac{k\varepsilon_o A}{d}$$

k = dielectric constant (= 1 for air)

$\varepsilon_o = 8.85 \times 10^{-12}\ \mathrm{N^{-1} m^{-2}}$

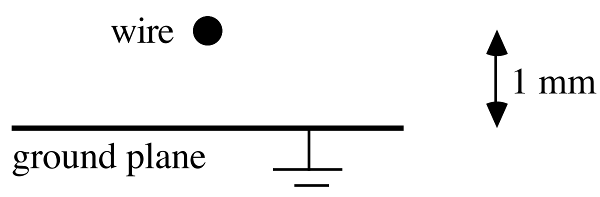
d = distance between plates (assumed 1 mm)

A = area of plates $= 1.1 \times 10^8\ \mathrm{m}^2$!!!!

This corresponds to square plate *6.5 miles* per side! Thus 1 Farad capacitor is gigantic in size. However, breakthroughs in capacitor technologies (driven by the computer industries) allow the production of 0.5-5 F capacitors of small size (1-2 cm high) and low cost (< $5).

•How small can we make capacitors?

A wire near a ground plane has $C \approx 0.1$ pf = 10^{-13} F.



•Some words to the wise on capacitors and their labeling.

Typical capacitors are multiples of microFarads (10^{-6} F) or picoFarads (10^{-12} F).

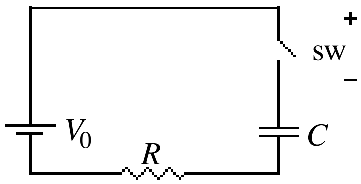
Caution: Whenever you see mF it almost always is micro, *not* milli F and *never* mega F.

picoFarad (10^{-12} F) is sometimes written as pf and pronounced *puff*.

There is no *single* convention for labeling capacitors. Many manufacturers have their own labeling scheme. See Horowitz and Hill lab manual for a discussion on this topic.

Resistors and Capacitors

•Examine voltage and current vs. time for a circuit with one R and one C.



Assume that at $t < 0$ all voltages are zero, $V_R = V_C = 0$.
At $t \geq 0$ the switch is closed and the battery (V_0) is connected.
Apply Kirchhoff's voltage rule:

$$\begin{aligned} V_0 &= V_R + V_C \\ &= IR + \frac{Q}{C} \\ &= R\frac{dQ}{dt} + \frac{Q}{C} \end{aligned}$$

Thus we have to solve a differential equation. For the case where we have a DC voltage (our example) it's easier to solve for the current (I) by differentiating both sides of above equation.

$$\begin{aligned} \frac{dV_0}{dt} &= \frac{1}{C}\frac{dQ}{dt} + R\frac{d^2Q}{dt^2} \\ 0 &= \frac{I}{C} + R\frac{dI}{dt} \\ \frac{dI}{dt} &= -\frac{I}{RC} \end{aligned}$$

This is just an exponential decay equation with time constant RC (sec). The current as a function of time through the resistor and capacitor is:

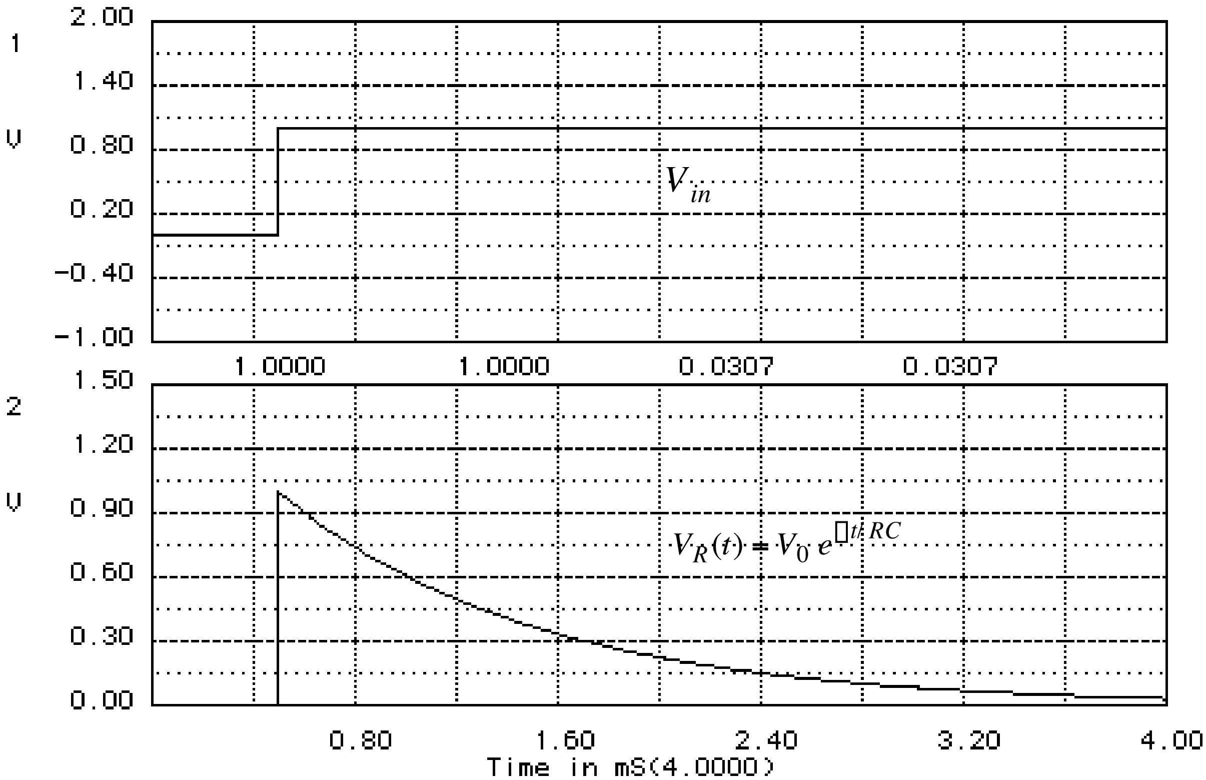
$$I(t) = I_0 e^{-t/RC}$$

•What's $V_R(t)$?

By Ohm's law:

$$\begin{aligned} V_R(t) &= I_R \cdot R \\ &= I_0 R\, e^{-t/RC} \\ &= V_0\, e^{-t/RC} \end{aligned}$$

At $t = 0$ all the voltage appears across the resistor, $V_R(0) = V_0$.
At $t =$, $V_R(\ ) = 0$.



•What's $V_C(t)$?

Easiest way to answer is to use the fact that $V_0 = V_R + V_C$ is valid for all t.

$$V_C = V_0 - V_R$$

$$V_C = V_0\left(1 - e^{-t/RC}\right)$$

At $t = 0$ all the voltage appears across the resistor so $V_C(0) = 0$.

At $t = \infty$, $V_C(\infty) = V_0$.

•Suppose we wait until $I = 0$ and then short out the battery.

We now have

$$0 = V_R + V_C$$

$$V_R = -V_C$$

$$R\frac{dQ}{dt} = -\frac{Q}{C}$$

$$\frac{dQ}{dt} = -\frac{Q}{RC}$$

Solving the exponential equation yields,

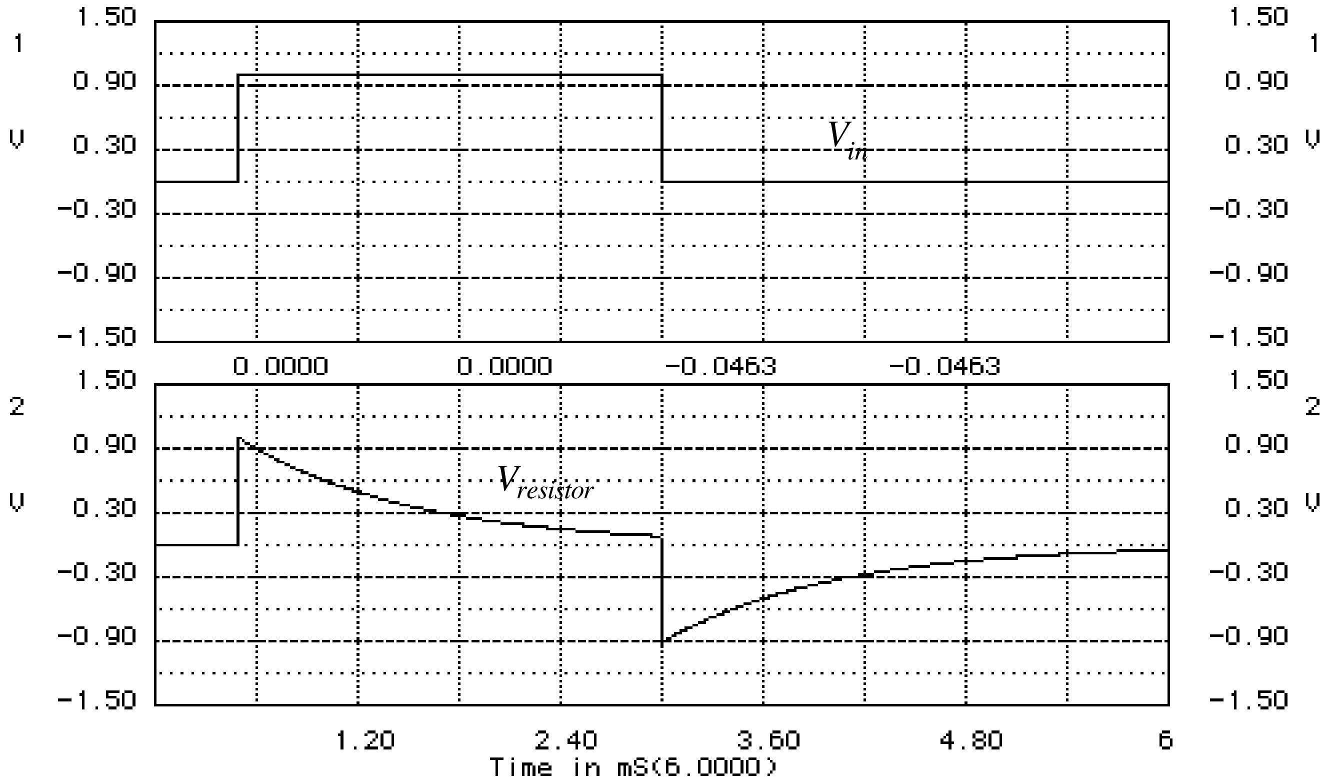
$$Q(t) = Q_0 e^{-t/RC}$$

We can find V_C using $V = Q/C$,

$$V_C(t) = V_0 e^{-t/RC}$$

Finally we can the voltage across the resistor using $V_R = -V_C$,

$$V_R(t) = -V_0 e^{-t/RC}$$



•Suppose $V(t) = V_0 \sin \omega t$ *instead of DC*, what happens to V_C and I_C?

$$Q(t) = CV(t)$$
$$= CV_0 \sin \omega t$$
$$I_C = dQ/dt$$
$$= \omega CV_0 \cos \omega t$$
$$= \omega CV_0 \sin(\omega t + \pi/2)$$

The current in the capacitor varies like a sine wave too, but it is 90^0 out of phase with the voltage.

We can write an equation that looks like Ohm's law by defining V^*:

$$V^* = V_0 \sin(\omega t + \pi/2)$$

Then the relationship between the voltage and current in C looks like:

$$V^* = I_C / \omega C$$
$$= I_C R^*$$

Indeed $1/\omega C$ can be identified as a kind of resistance. We call it capacitive reactance, X_C:

$$X_C \equiv 1/\omega C \text{ (Ohms)}, X_C = 0 \text{ if } \omega = \quad \text{and } X_C = \quad \text{if } \omega = 0.$$

Thus at high frequencies a capacitor looks like a short circuit, while at low frequencies a capacitor looks like an open circuit (high resistance).

2) Inductance:

•Define inductance by: $V = L dI/dt$, unit = Henry.
•Electric component commonly called inductors.
•Symbol(s) for inductor:

•Useful circuit element that provides a voltage proportional to dI/dt.
•An inductor is a device that stores energy

$$E = \tfrac{1}{2} L I^2$$

•Inductors are usually made from a coil of wire. They tend to be bulky and are hard to fabricate in small sizes (μm), not used in chips.
•Two inductors next to each other (transformer) can step up or down a voltage without changing the frequency of the voltage. Also provide isolation from the rest of the circuit.

<u>Energy and Power in Inductors</u>

•How much energy is stored in an inductor?

$$dE = V dQ$$

$$I = \frac{dQ}{dt}$$

$$dE = V I dt$$

$$V = L\frac{dI}{dt}$$

$$dE = L I dI$$

$$E = L\int_0^I I\, dI$$

$$E = \tfrac{1}{2} L I^2$$

•How much power is dissipated in an inductor?

$$Power = \frac{dE}{dt}$$

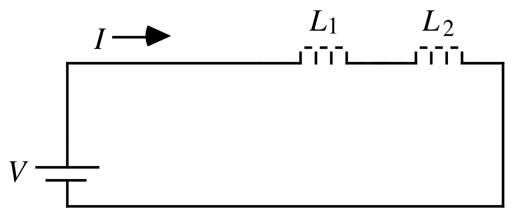
$$= \frac{d}{dt}\left(\frac{LI^2}{2}\right)$$

$$P = LI\frac{dI}{dt}$$

Note: dI/dt must be finite otherwise we source (or sink) an infinite amount of power in an inductor!
THIS WOULD BE UNPHYSICAL.

Thus the *current* across an inductor <u>*cannot*</u> change *instantaneously*.

•Two inductors in series:

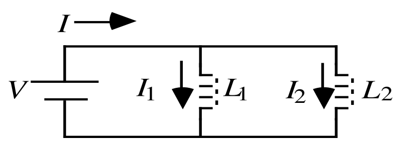


Apply Kirchhoff's Laws,

$$V = V_1 + V_2$$
$$= L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt}$$
$$\equiv L_{tot} \frac{dI}{dt}$$
$$L_{tot} = L_1 + L_2$$
$$= \sum L_i$$

Inductors in series add like resistors in series.
Note the *total* inductance is *greater* than the individual inductances.

•Two inductors in parallel:



Since the inductors are in parallel,

$$V_1 = V_2 = V$$

The total current in the circuit is

$$I = I_1 + I_2$$
$$\frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$
$$= \frac{V}{L_1} + \frac{V}{L_2}$$
$$\equiv \frac{V}{L_{tot}}$$
$$\frac{1}{L_{tot}} = \frac{1}{L_1} + \frac{1}{L_2}$$
$$L_{tot} = \frac{L_1 L_2}{L_1 + L_2}$$

If we have more than 2 inductors in parallel, they combine like:

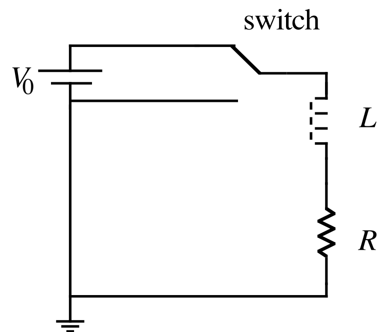
$$\frac{1}{L_{tot}} = \sum \frac{1}{L_i}$$

Inductors in parallel add like resistors in parallel.
Note: the *total* inductance is *less* than the individual inductances.

Resistors and Inductors

•Examine voltage and current versus time for a circuit with one R and one L.

Assume that at $t < 0$ all voltages are zero, $V_R = V_L = 0$.
At $t \geq 0$ the switch is closed and the battery (V_0) is connected.



Like the capacitor case, apply Kirchhoff's voltage rule:

$$V_0 = V_R + V_L$$

$$= IR + L\frac{dI}{dt}$$

Solving the differential equation, assuming at $t = 0$, $I = 0$:

$$I(t) = \frac{V_0}{R}\left(1 - e^{-tR/L}\right)$$

This is just an exponential decay equation with time constant L/R (seconds).

•What's $V_R(t)$?

By Ohm's law $V_R = I_R R$ at any time:

$$V_R = I(t)R = V_0\left(1 - e^{-tR/L}\right)$$

At $t = 0$, none of the voltage appears across the resistor, $V_R(0) = 0$.
At $t = \infty$, $V_R(\infty) = V_0$.

•What's $V_L(t)$?

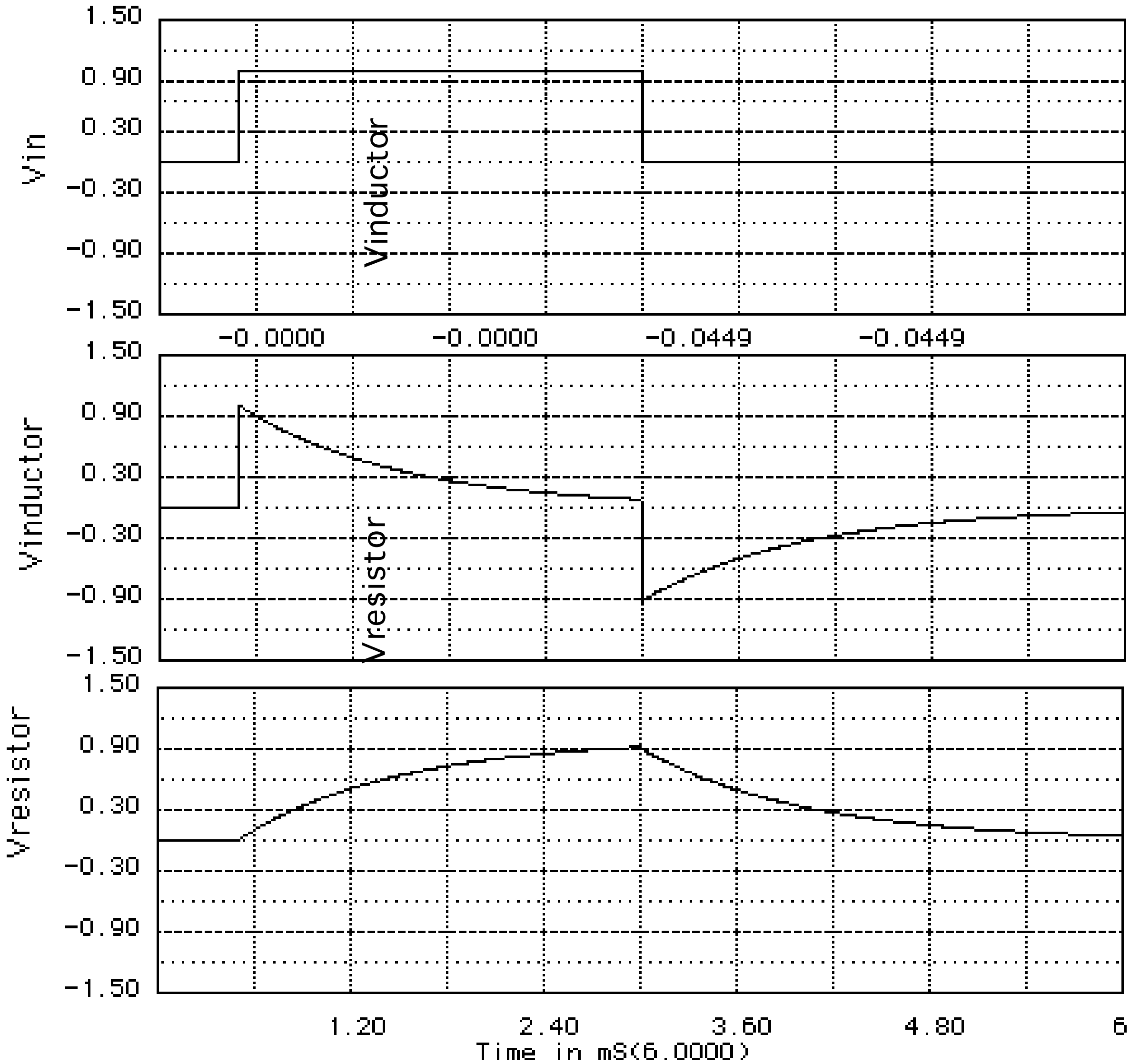
Easiest way to answer is to use the fact that $V_0 = V_R + V_L$ is valid for all t.

$$V_L = V_0 - V_R$$

$$V_L(t) = V_0 e^{-tR/L}$$

At $t = 0$, all the voltage appears across the inductor so $V_L(0) = V_0$.
At $t = \infty$, $V_L(\infty) = 0$.

Pick L/R = 1 millisecond:



•Suppose $V(t) = V_0 \sin \omega t$ instead of DC, what happens to V_L and I_L?

$$V = L\frac{dI_L}{dt}$$

$$I_L = \frac{1}{L}\int_0^t V dt$$

$$= -\frac{V_0}{\omega L}\cos \omega t$$

$$I_L(t) = \frac{V_0}{\omega L}\sin(\omega t - \pi / 2)$$

The current in an inductor varies like a sine wave too, but it is 90^0 out of phase with the voltage.

We can write an equation that looks like Ohm's law by defining V^*:

$V^* = V_0 \sin(\omega t - \pi / 2)$

Then the relationship between the voltage and current in L looks like:

$V^* = I_L \omega L = I_L R^*$

Indeed, ωL can be identified as a kind of resistance. We call it <u>inductive reactance</u>, X_L:

$X_L \equiv \omega L$ (Ohms), $X_L = 0$ if $\omega = 0$ and $X_L = \infty$ if $\omega = \infty$.

Thus at high frequencies an inductor looks like an open circuit, while at low frequencies an inductor looks like a short circuit (low resistance).

•Some things to remember about R, L, and C's.

For DC circuits, after many time constants (L/R or RC):

Inductor acts like a wire (0 Ω).

Capacitor acts like an open circuit (∞ Ω).

For circuits where the voltage changes very rapidly or transient behavior:

Capacitor acts like a wire (0 Ω).

Inductor acts like an open circuit (∞ Ω).

Example, RLC circuit with DC supply:

At $t = 0$, voltages on R, C are zero and $V_L = V_0$.

At $t = \infty$, voltages on R, L are zero and $V_C = V_0$.