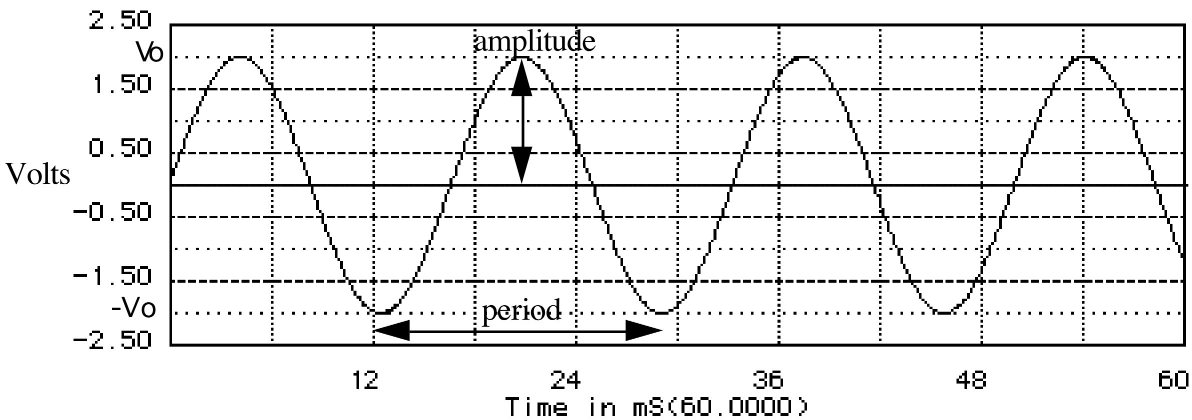
R-L-C AC Circuits

<u>What does AC stand for?</u>

• AC stands for "Alternating Current". Most of the time however, we are interested in the voltage at a point in the circuit, so we'll concentrate on voltages here rather than currents. We encounter AC circuits whenever a periodic voltage is applied to a circuit. The most common periodic voltage is in the form of a sine (or cosine) wave:

$V(t) = V_0 \cos \omega t$ or $V(t) = V_0 \sin \omega t$



In this notation V_0 is the *amplitude:*

V_0 = Peak Voltage (V_P)

V_0 = 1/2 Peak-to-Peak Voltage (V_{PP})

$V_0 = \sqrt{2}\, V_{RMS} = 1.41\, V_{RMS}$

Note: V_{PP} is easiest to read off scope.
V_{RMS} is usually what multimeters read.

In this notation ω is the *angular frequency:*

$\omega = 2\pi f$, with f = frequency of the waveform.

The frequency (f) and period (T) are related by:

T (sec) = $1/f$ (sec^{-1})

Household line voltage is usually 110-120 V_{RMS} (156-170 V_P), f = 60 Hz.

It is extremely important to be able to analyze circuits (systems) with sine or cosine inputs since (almost) any waveform can be constructed from a sum of sines and cosines. This is the "heart" of *Fourier analysis* (Simpson, Chapter 3). The response of a circuit to a complicated waveform (e.g. a square wave) can be understood by analyzing each of the individual sine or cosine components that make up the complicated waveform. Usually only a few of these components are important in determining the circuit's response to the input waveform.

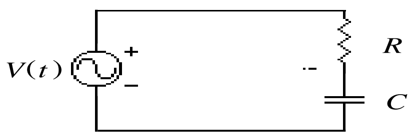
R-C Circuits and AC waveforms

• There are many different techniques for solving AC circuits, all of them are based on Kirchhoff's laws. When we solve for the voltage and/or current in an AC circuit we are really solving a differential equation. The different circuit techniques are really just different ways of solving the same differential equation.

- brute force solution to differential equation
- complex numbers (algebra)
- Laplace transforms (integrals)

• We will solve the following RC circuit using the brute force method and complex numbers method.

Let the input (driving) voltage be $V(t) = V_0 \cos\omega t$ and we want to find $V_R(t)$ and $V_C(t)$.



Brute Force Method: Start with Kirchhoff's loop law:

$$V(t) = V_R(t) + V_C(t)$$

$$V_0 \cos\omega t = IR + Q/C$$

$$= R\,dQ(t)/dt + Q(t)/C$$

We have to solve an inhomogeneous D.E. The usual way to solve such a D.E. is to assume the solution has the same form as the input:

$$Q(t) = \alpha \sin\omega t + \beta \cos\omega t$$

Plug our trial solution $Q(t)$ back into the differential equation:

$$V_0 \cos\omega t = \alpha R\omega \cos\omega t - \beta R\omega \sin\omega t + (\alpha/C)\sin\omega t + (\beta/C)\cos\omega t$$

$$= (\alpha R\omega + \beta/C)\cos\omega t + (\alpha/C - \beta R\omega)\sin\omega t$$

$$V_0 = \alpha R\omega + \beta/C$$

$$\alpha/C = \beta R\omega$$

$$\alpha = \frac{RC^2\omega V_0}{1+(RC\omega)^2}$$

$$\beta = \frac{CV_0}{1+(RC\omega)^2}$$

We can now write the solution for $V_C(t)$:

$$V_C(t) = Q/C$$

$$= (\alpha \sin\omega t + \beta \cos\omega t)/C$$

$$= \frac{RC\omega V_0}{1+(RC\omega)^2}\sin\omega t + \frac{V_0}{1+(RC\omega)^2}\cos\omega t$$

We would like to rewrite the above solution in such a way that only a cosine term appears. In this form we can compare it to the input voltage. From the previous page we have:

$$V_C(t) = \frac{RC\omega V_0}{1+(RC\omega)^2}\sin\omega t + \frac{V_0}{1+(RC\omega)^2}\cos\omega t$$

$$= \frac{V_0}{\sqrt{1+(RC\omega)^2}}\left[\frac{RC\omega}{\sqrt{1+(RC\omega)^2}}\sin\omega t + \frac{1}{\sqrt{1+(RC\omega)^2}}\cos\omega t\right]$$

We get the above equation in terms of cosine only using the following dirty trick from basic trig:

$$\cos(\theta_1 - \theta_2) = \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2$$

We can now define an angle such that:

$$\cos\phi = \frac{1}{\sqrt{1+(RC\omega)^2}},\quad \sin\phi = \frac{RC\omega}{\sqrt{1+(RC\omega)^2}},\quad \tan\phi = RC\omega$$

Finally (!) we can write the desired expression for $V_C(t)$:

$$V_C(t) = \frac{V_0}{\sqrt{1+(RC\omega)^2}}\cos(\omega t - \phi)$$

From the above expression we see that $V_C(t)$ and $V_0(t)$ are out of phase.

Using the above expression for $V_C(t)$, we obtain:

$$V_R(t) = IR$$

$$= R\frac{dQ}{dt}$$

$$= RC\frac{dV_C}{dt}$$

$$= \frac{-RC\omega V_o}{\sqrt{1+(RC\omega)^2}}\sin(\omega t - \phi)$$

Again, we would like to have cosines instead of sines. We do this using:

$$-\sin\theta = \cos(\theta + \pi/2)$$

Finally (!!) we have:

$$V_R(t) = \frac{RC\omega V_o}{\sqrt{1+(RC\omega)^2}}\cos(\omega t - \phi + \tfrac{\pi}{2})$$

- Some things to note:

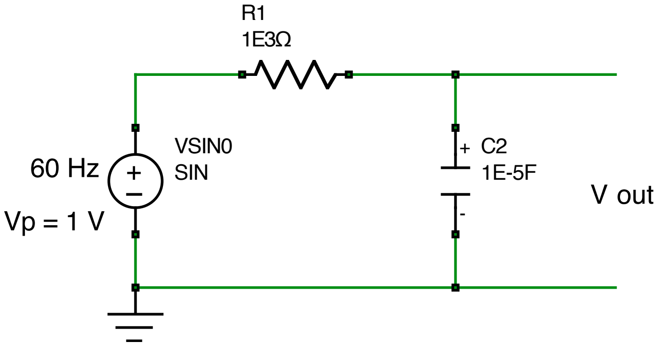
  $V_C(t)$, $V_R(t)$, and $I(t)$ are all out of phase with the applied voltage.

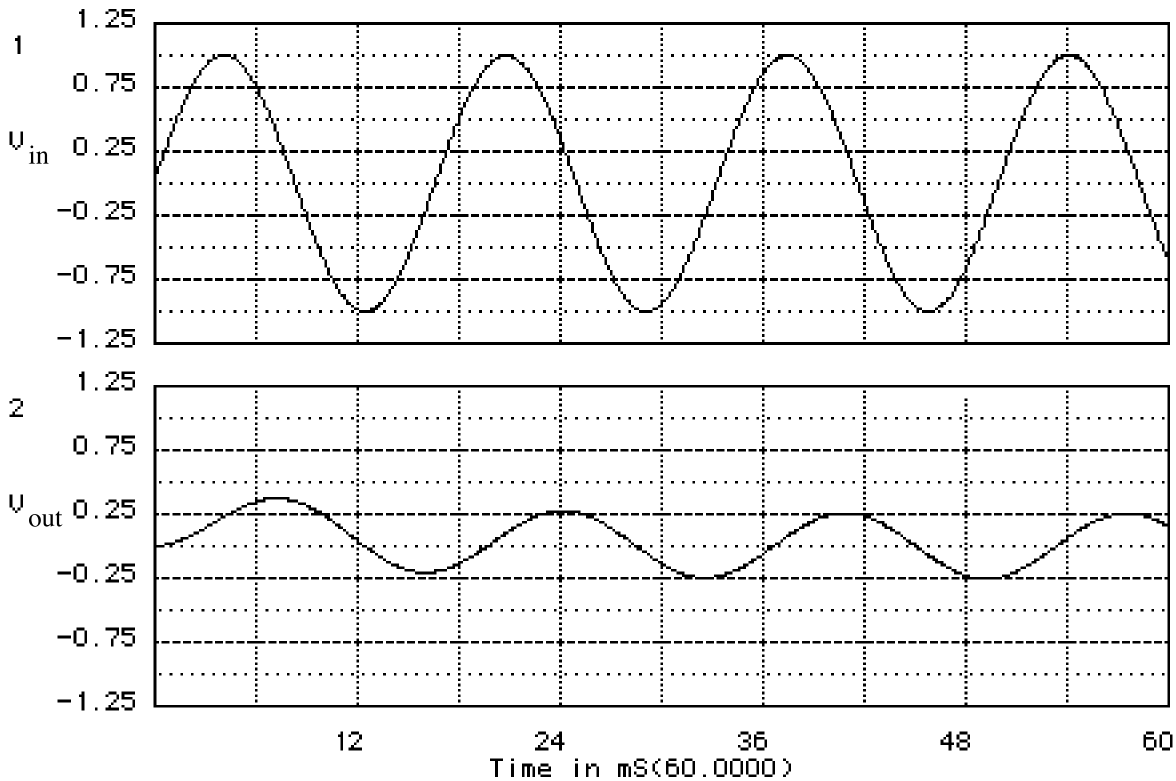
  $I(t)$ and $V_R(t)$ are in phase with each other.

  $V_C(t)$ and $V_R(t)$ are out of phase by 90^0.

  The amplitude of $V_C(t)$ and $V_R(t)$ depend on ω.

Example: RC Circuit





Solving circuits with complex numbers:

PROS: don't explicitly solve differential equations (lots of algebra).
can find magnitude and phase of voltage separately.

CONS: have to use complex numbers! No "physics" in complex numbers.

- What's a complex number? (see Simpson, Appendix E, P835)

Start with $j \equiv \sqrt{-1}$ (solution to $x^2 + 1 = 0$).

A complex number can be written in two forms:

$X = A + jB$, with A and B *real* numbers

or

$X = R\, e^{j\theta}$, with $R = (A^2 + B^2)^{1/2}$, and $\tan\theta = B/A$ (remember $e^{j\theta} = \cos\theta + j\sin\theta$)

Define the complex conjugate of X as:

$$X^* = A - jB \quad \text{or} \quad X^* = R\,e^{-j\theta}$$

The magnitude of X can be found from:

$$|X| = (XX^*)^{1/2} = (X^*X)^{1/2} = (A^2 + B^2)^{1/2}$$

Suppose we have 2 complex numbers, X and Y with phases θ and ϕ respectively,

$$Z = \frac{X}{Y} = \frac{|X|e^{j\theta}}{|Y|e^{j\phi}} = \frac{|X|}{|Y|}e^{j(\theta-\phi)}$$

The magnitude of Z is just $|X|/|Y|$, while the phase of Z is θ - ϕ.

- So why is this useful?

Consider the case of the capacitor and AC voltage:

$$\begin{aligned} V(t) &= V_0 \cos\omega t \\ &= \mathrm{Real}\left(V_0 e^{j\omega t}\right) \end{aligned}$$

$$\begin{aligned} I(t) &= C\frac{dV}{dt} \\ &= -C\omega V_0 \sin\omega t \\ &= \mathrm{Real}\left(j\omega C V_0 e^{j\omega t}\right) \\ &= \mathrm{Real}\left[\frac{V_0 e^{j\omega t}}{1/j\omega C}\right] \\ &= \mathrm{Real}\left[\frac{V}{X_C}\right] \qquad \text{with } V \text{ and } X_C \text{ complex} \end{aligned}$$

We now have Ohm's law for capacitors using the capacitive reactance X_C.

$$X_C = \frac{1}{j\omega C}$$

We can make a similar case for the inductor ($V = LdI/dt$):

$$\begin{aligned} I(t) &= \frac{1}{L}\int V_0 \cos\omega t\; dt \\ &= \frac{V_0 \sin\omega t}{L\omega} \\ &= \mathrm{Real}\left[\frac{V_0 e^{j\omega t}}{j\omega L}\right] \\ &= \mathrm{Real}\left[\frac{V}{X_L}\right] \qquad \text{with } V \text{ and } X_L \text{ complex} \end{aligned}$$

We now have Ohm’s law for inductors using the inductive reactance X_L:

$$X_L = j\omega L$$

• X_C and X_L act like frequency dependent resistors. However they also have a *phase* associated with them due to their complex nature.

$X_L \to 0$ as $\omega \to 0$ (short circuit, DC)
$X_L \to \infty$ as $\omega \to \infty$ (open circuit)
$X_C \to 0$ as $\omega \to \infty$ (short circuit)
$X_C \to \infty$ as $\omega \to 0$ (open circuit, DC)

• Back to the RC circuit. Allow voltages, currents, and charge to be complex

$$
\begin{aligned}
V_{in} &= V_0 \cos \omega t \\
&= \mathrm{Real}\left(V_0 e^{j\omega t}\right) \\
&= \mathrm{Real}\left(V_R + V_C\right)
\end{aligned}
$$

We can write an expression for the charge (Q) taking into account the phase difference (ϕ) between applied voltage and the voltage across the capacitor (V_C).

$$
\begin{aligned}
Q(t) &= CV_C(t) \\
&= Ae^{j(\omega t - \phi)}
\end{aligned}
$$

where Q and V_C are complex, A and C real.

We can find the complex current by differentiating the above:

$$
\begin{aligned}
I(t) &= dQ(t)/dt \\
&= j\omega Ae^{j(\omega t - \phi)} \\
&= j\omega Q(t) \\
&= j\omega CV_C(t) \\
V_{in} &= V_C + V_R \\
&= V_C + IR \\
&= V_C + j\omega CV_C R \\
V_C &= \frac{V_{in}}{1 + j\omega RC} \\
&= V_{in} \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \\
&= V_{in} \frac{X_C}{R + X_C}
\end{aligned}
$$

The above looks like a voltage divider equation!!!!!

We can easily find the magnitude of V_C

$$|V_C| = \frac{|V_{in}||X_C|}{|R + X_C|}$$

$$= \frac{V_0 \dfrac{1}{\omega C}}{\sqrt{R^2 + (1/\omega C)^2}}$$

$$= \frac{V_0}{\sqrt{1 + (RC\omega)^2}}$$

which is the same as the result on page 3.

Is this solution the same as what we had when we solved by brute force?

$$V_C = \mathrm{Real}\left\{\frac{V_{in}}{1 + j\omega RC}\right\}$$

$$= \mathrm{Real}\left\{\frac{V_0 e^{j\omega t}}{1 + j\omega RC}\right\}$$

$$= \mathrm{Real}\left\{\frac{V_0 e^{j\omega t}}{\sqrt{1 + (\omega RC)^2}\, e^{j\phi}}\right\}$$

where ϕ is given by $\tan \phi = \omega RC$.

$$V_C = \mathrm{Real}\left\{\frac{V_0 e^{j(\omega t - \phi)}}{\sqrt{1 + (\omega RC)^2}}\right\}$$

$$= \frac{V_0 \cos(\omega t - \phi)}{\sqrt{1 + (\omega RC)^2}}$$

<u>YES the solutions are identical</u>.

- We can now solve for the voltage across the resistor.

Start with the voltage divider equation in complex form:

$$V_R = \frac{V_{in} R}{R + X_C}$$

$$|V_R| = \frac{|V_{in}| R}{|R + X_C|}$$

$$= \frac{V_0 R}{\sqrt{R^2 + (1/\omega C)^2}}$$

$$= \frac{V_0 \omega RC}{\sqrt{1 + (\omega RC)^2}}$$

This amplitude is the same as the brute force differential equation case!

• Important note:

When we add complex voltages together we must take into account the phase difference between them. Thus, the sum of the voltages at a given time satisfy:

$V_0^2 = |V_R|^2 + |V_C|^2$ and **not** $V_0 = |V_R| + |V_C|$

R-C Filters

• Filters:

Allow us to select (reject) wanted (unwanted) signals on the basis of their frequency structure.

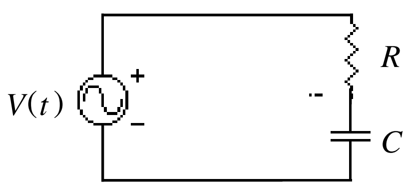
Allow us to change the phase of the voltage or current in a circuit.

Define the gain (G) or transfer (H) function of a circuit:

$G(j\omega) = H(j\omega) = V_{out} / V_{in}$ ($j\omega$ is often denoted by s).

G is independent of time, but can depend on ω, R, L, C.

For an RC circuit we can define G_R and G_C:

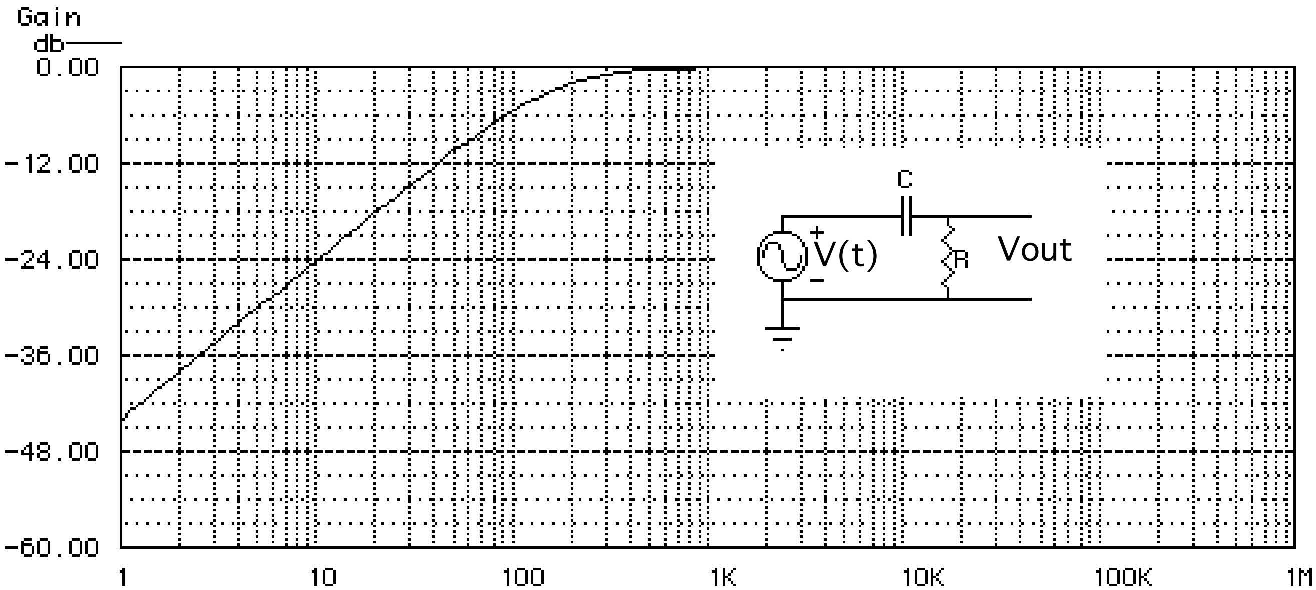


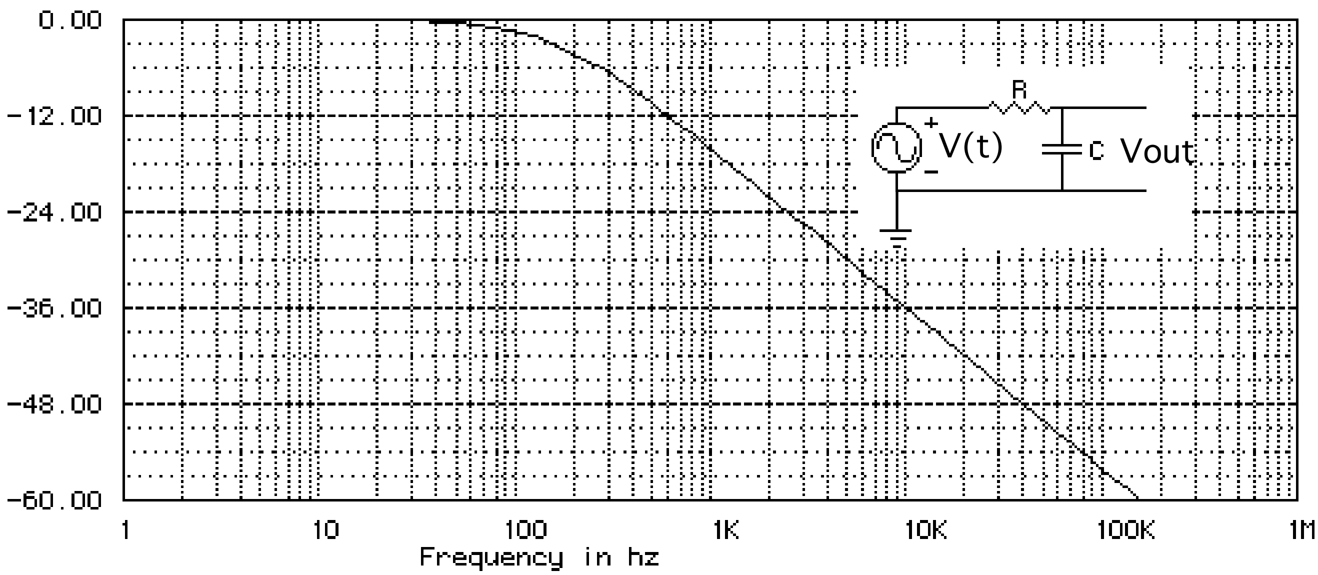
$$G_R \equiv \frac{V_R}{V_{in}} = \frac{R}{R + X_C} = \frac{R}{R + 1/j\omega C}$$

$$G_C \equiv \frac{V_C}{V_{in}} = \frac{X_C}{R + X_C} = \frac{1/j\omega C}{R + 1/j\omega C}$$

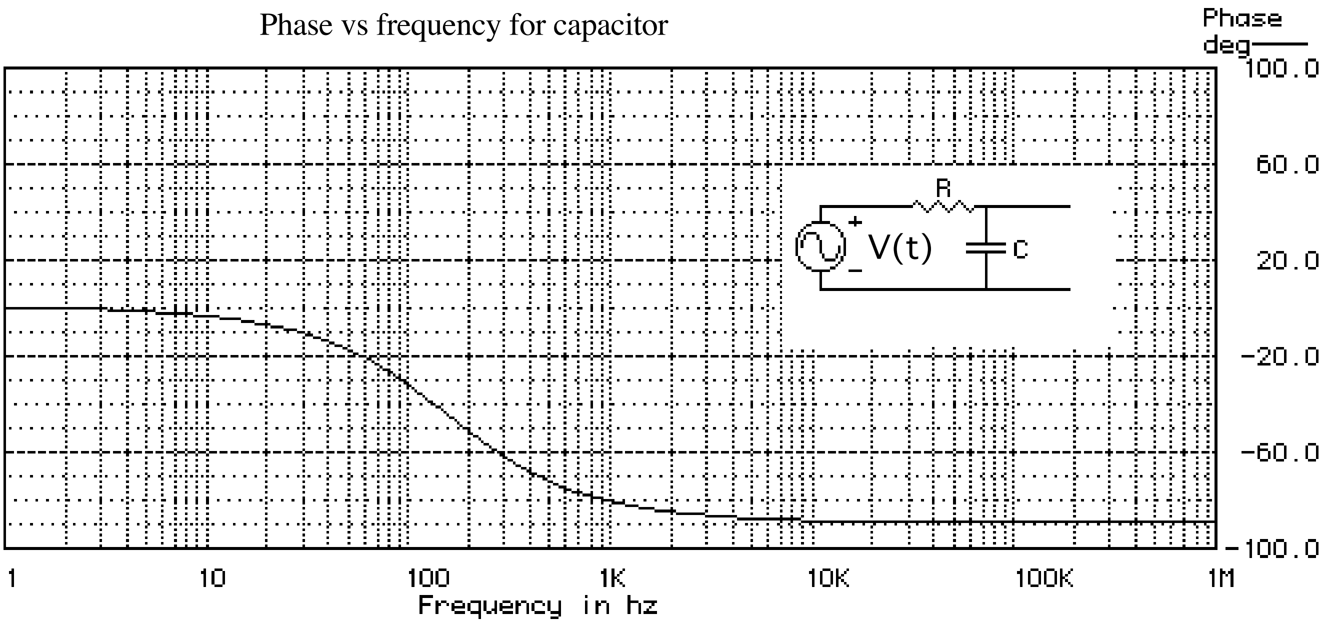
We can categorize the G's as follows:

	G_R	G_C
High Frequencies	$\to 1$, no phase shift **high pass filter**	$\to 1/j\omega CR \to 0$, phase shift
Low Frequencies	$\to j\omega CR \to 0$, phase shift	$\to 1$, no phase shift **low pass filter**


Gain
db
0.00
-12.00
-24.00
-36.00
-48.00
-60.00
1
10
100
1K
10K
100K
1M
C
+
V(t)
−
R
Vout


0.00
-12.00
-24.00
-36.00
-48.00
-60.00
1
10
100
1K
10K
100K
1M
Frequency in hz
R
+
V(t)
−
C
Vout

Phase vs frequency for capacitor



- Decibels and Bode Plots:

  Decibel (dB) describes voltage or power gain:

  $$\text{dB} = 20\log(V_{out}/V_{in})$$
  $$= 10\log(P_{out}/P_{in})$$

  Bode Plot is a log-log plot with dB on the Y axis and log(ω) or log(f) on the X axis.

- 3 dB point or 3 dB frequency:

  (also called break frequency, corner frequency, 1/2 power point)

  At the 3 dB point:

  $$\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{2}} \qquad \text{since } 3 = 20\log(V_{out}/V_{in})$$

  $$\frac{P_{out}}{P_{in}} = \frac{1}{2} \qquad \text{since } 3 = 10\log(P_{out}/P_{in})$$

  $\omega RC = 1$ *for high or low pass filter*