Relativistic Kinematics

Read:
Appendix A of M&S

- HEP: particles (e.g. protons, pions, electrons) are usually moving at speeds close to the speed of light.
  - ☞ classical relationship for the kinetic energy of the particle in terms of its mass and velocity is not valid:

    kinetic energy $= T \neq \frac{1}{2}mv^2$
  - ☞ must use special relativity to describe the energy and momentum of a particle.
  - ■ total energy of a particle moving with speed v:

$$\begin{aligned} E &= m_0c^2 + T \\ &= mc^2 \\ &= \frac{m_0c^2}{\sqrt{1-(v/c)^2}} \\ &= \gamma m_0 c^2 \end{aligned}$$

m_0: rest mass

m: relativistic mass

  - ■ total momentum of the particle:

$$\begin{aligned} \vec{p} &= m\vec{v} \\ &= \frac{m_0\vec{v}}{\sqrt{1-(v/c)^2}} \\ &= \gamma m_0 \vec{v} \end{aligned}$$

  - ■ energy and momentum are related:

$$E^2 = (\vec{p}c)^2 + (m_0c^2)^2$$

4-Vectors

- It is sometimes convenient to describe a particle (or a collection of particles) by a 4-vector:

  $$p = (E, \vec{p}) = (E, p_x, p_y, p_z)$$

  $c = 1$

  - length of a 4-vector:

    $$m_0^2 = E^2 - \vec{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

    - this is true in ALL reference frames (lab, center of mass…) because it is a Lorentz invariant.
    - A particle is said to be "on the mass shell" if m = rest mass, i.e. real
  - A 4-vector with length L^2 is classified as follows:

    Time like if $L^2 > 0$

    Space like if $L^2 < 0$

    Light like if $L^2 = 0$ — *Photon!*

- Can use 4-vector algebra to prove some physical phenomenon:
  - photon can convert into an electron-positron pair in the presence of a nucleus but not in free space:

    $$p_\gamma = p_{e^+} + p_{e^-}$$

    $$\begin{aligned} p_\gamma^2 &= (p_{e^+} + p_{e^-})^2 \\ &= p_{e^+}^2 + p_{e^-}^2 + 2p_{e^+} \cdot p_{e^-} \\ &= p_{e^+}^2 + p_{e^-}^2 + 2(E_{e^+}E_{e^-} - \vec{p}_{e^+} \cdot \vec{p}_{e^-}) \\ &= m_e^2 + m_e^2 + 2(E_{e^+}E_{e^-} + p^2) > 0 \text{ but } p_\gamma^2 = m_\gamma^2 = 0 \end{aligned}$$

    - this is mathematical: what is the actual reason?
      - a nucleus recoil is needed to conserve momentum

Lorentz Invariant vs. Conserved Quantity

- For a Lorentz invariant quantity, you get the same number in two different reference systems.
  - Lorentz invariant quantity is a scalar.
  - Let E_{lab} and $\boldsymbol{P}_{lab}$ be the energy and momentum measured in the lab frame.
  - Let E_{cm} and $\boldsymbol{P}_{cm}$ be the energy and momentum measured in the center of mass frame.

$$E_{cm}^2 - \boldsymbol{p}_{cm}^2 = E_{lab}^2 - \boldsymbol{p}_{lab}^2 = (\text{rest mass})^2$$

  - $(E, \boldsymbol{p})$ is a Lorentz invariant.
- For a conserved quantity, you get the same number in the same reference system but at a different time.
  - $\boldsymbol{p}_{i_{lab}}$: initial momentm in lab (before interaction)
  - $\boldsymbol{p}_{f_{lab}}$: final momentm in lab (after interaction)
  - $\boldsymbol{p}_{i_{cm}}$: initial momentm in CM (before interaction)
  - $\boldsymbol{p}_{f_{cm}}$: final momentm in CM (after interaction)
  - Momentum conservation;

$$\Rightarrow \boldsymbol{p}_{i_{lab}} = \boldsymbol{p}_{f_{lab}} \text{ and } \boldsymbol{p}_{i_{cm}} = \boldsymbol{p}_{f_{cm}}$$

$$\text{but } \boldsymbol{p}_{i_{lab}} \neq \boldsymbol{p}_{f_{cm}}$$

4-Vector Matrix

- Can also manipulate 4-vectors using contravariant/covariant (up/down) notation:

$$m_0^2 = g_{uv} p^u p^v$$

$$g_{uv} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

  - Scalar product of two 4-vectors:

$$ab = \sum_{u=0}^{3} \sum_{v=0}^{3} g_{uv} a^u b^v = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

  - Sum of two 4-vectors is also a 4-vector:

$$p_1 + p_2 = (E_1 + E_2, \vec{p}_1 + \vec{p}_2)$$

  - Length of the sum of two 4-vectors of two particles (1,2):

$$\begin{aligned}(p_1 + p_2)^2 &= (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \\ &= E_1^2 + E_1^2 + 2E_1^2 E_2^2 - \vec{p}_1^2 - \vec{p}_2^2 - 2\vec{p}_1 \cdot \vec{p}_2 \\ &= m_1^2 + m_2^2 + 2(E_1^2 E_2^2 - |\vec{p}_1||\vec{p}_2|\cos\theta) \\ &= m_{12}^2\end{aligned}$$

    - θ: angle between particles
    - m_{12}: effective (invariant) mass

Calculation of Speed

- Consider a proton at rest and an antiproton with 10 GeV/c of momentum in the lab frame.
  - What is the energy of the antiproton in the lab frame?

$$E = \sqrt{\vec{p}^2 + m_0^2} = \sqrt{10^2 + 0.938^2} = 10.044 \text{ GeV}$$

    - ☞ At high energy, $E >> m_0c^2$, $E \sim P$.
  - How fast is the antiproton moving in the lab frame?
    - use energy/momentum relationship between the rest and lab frames of the antiproton:

$$p_{lab} / E_{lab} = \gamma\beta m_0 c / \gamma m_0 c^2 = \beta / c = 10/10.044 = 0.966$$

$$v = 0.996c$$

Fast!

Colliding Beam vs Fixed Target Collisions

- As we will see later in the course, cross sections and the energy available for new particle production depend on the total energy in the center of mass (CM) frame.
  - We define the CM where the total momentum of the collision is zero:

    $(p_1 + p_2) = (E_1 + E_2, \boldsymbol{p}_1 + \boldsymbol{p}_2) = (E_1 + E_2, 0)$
    - If the masses of the two particles are equal as in the case of proton-antiproton collisions:

      $(p_1 + p_2) = (2E, 0)$
      - ☞ The CM energy is twice energy of either particle.
      - ★ The square of the energy in the CM is often called s $(= 4E^2)$
- How much energy is available in the CM from a 10 GeV/c antiproton colliding with a proton at rest?
  - Since $(p_1 + p_2)$ is a Lorentz invariant we evaluate in any frame we please!
    - We are given values in the lab frame:

      $(p_1 + p_2) = (E_1 + m_p, \boldsymbol{p}_1 + 0)$
      - ☞ The magnitude of this 4-vector:

        $$s = (E_1 + m_p)^2 - \boldsymbol{p}_1^2 = (10.044 + 0.938)^2 - 10^2 = 20.6 \text{ GeV}^2 = 4.54^2 \text{ GeV}^2$$
      - ☞ The total energy in the CM is 4.54 GeV
      - ★ We could have gotten the same CM energy with two colliding beams of 2.27 GeV!
- In general the energy available for new particle production increases as:
  - fixed target experiments: $(2m_{\text{target}}E_{\text{beam}})^{1/2}$
  - colliding beam experiments: $2E_{\text{beam}}$
    - ☞ colliding beam experiment is more efficient if practical

Antiproton Discovery

- In the early 1950's many labs were trying to find evidence of the antiproton.
  - At Berkeley a new proton accelerator (BEVATRON) was being designed for this purpose.
- What is the energy of proton beam (E_b) needed to create an antiproton in a fixed target experiment?
- The simplest reaction that conserves all the necessary quantities (energy, momentum, electric charge, baryon number):

$$pp \rightarrow p\bar{p}pp$$

  - The invariant mass square of the beam and target proton system is a sum of the two 4-vectors:

$$(p_b + p_t)^2 = (E_b + m_t, \vec{p}_b)^2 = m_b^2 + m_t^2 + 2E_b m_t = 2m_p^2 + 2m_p E_b$$

    - This must have a minimum value of $(4m_p)^2$ in the CM for the reaction to take place:

$$(p_b + p_t)^2 = 2m_p^2 + 2m_b E_b > (4m_p)^2$$

$$E_b > 7m_p = 6.6 \text{ GeV}$$

    - The antiproton was discovered at Berkeley in 1955 (Nobel Prize 1959).

Time Dilation

- Most of the particles we are concerned with in HEP are not stable.
  - They spontaneously decay into other particles after a certain amount of time:

Lepton	Mean Lifetime (s)
electron	stable
muon (μ)	$\sim 2\text{x}10^{-6}$
tau (τ)	$\sim 3\text{x}10^{-13}$

  - The above table gives the average lifetime of the leptons in their rest frame.
- We often need to know how long a particle will live (on average) in a frame where the particle is moving close to the speed of light (c).
  - ☞ use special relativity!
- Consider a particle moving in the lab frame with speed v along the x-axis.
  - The relationship between time and distance measured in the lab and CM frame is:

$$t_{lab} = \gamma(t_{cm} + \beta x_{cm})$$

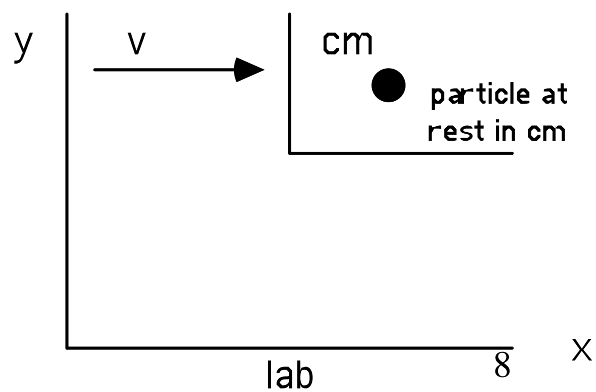
$$x_{lab} = \gamma(x_{cm} + \beta t_{cm})$$

    - In the lab frame the time between life and death of the particle:

$$\tau_{lab} = t_{2_{lab}} - t_{1_{lab}} = \gamma(t_{2_{cm}} + \beta x_{2_{cm}}) - \gamma(t_{1_{cm}} + \beta x_{1_{cm}}) = \gamma(t_{2_{cm}} - t_{1_{cm}}) = \gamma\tau$$

$$\text{CM}: x_{1_{cm}} = x_{2_{cm}}$$

    - ★ τ is called the proper lifetime: $\tau \leq \tau_{lab}$



More Time Dilation

- Consider a muon ($m_0 = 0.106$ GeV/c^2) with 1 GeV energy in the lab frame.
  - On average how long does this particle live in the LAB?
  - In the muon's rest frame it only lives (on average) $\tau = 2$ μsec.
  - But in the lab frame it lives (on average):

$$\tau_{lab} = \gamma\tau$$

$$\gamma = \frac{E}{m_0 c^2} = \frac{1}{0.106} \approx 10$$

$$\tau_{lab} = 10 \cdot 2\ \mu s = 20\ \mu s$$

  - On average how far does this particle travel in the LAB before decaying?

$$\Delta x_{lab} = \gamma\beta c\tau$$

$$\gamma\beta = \frac{p}{m_0 c} = \frac{1}{0.106} \approx 10$$

$$\Delta x_{lab} = 10 \cdot 3\times 10^8 \cdot 2\times 10^{-6} = 6\times 10^3\ m$$

☞ Big increase due to special relativity!