Conservation Laws

Read
M&S Chapters 2, 3, and 5

- When something doesn't happen there is usually a reason!
  - The following decays have never been observed despite years of searching:

$$n \to pe^-$$

$$p \to ne^+\nu_e$$

$$\mu^- \to e^-\gamma$$

    - ☞ That something is a conservation law!
- A conserved quantity is related to a symmetry in the Lagrangian of the interaction. (Noether's Theorem)
  - A symmetry is associated with a transformation that leaves the Lagrangian invariant.
    - time invariance leads to energy conservation
    - translation invariance leads to linear momentum conservation
    - rotational invariance leads to angular momentum conservation
  - Familiar conserved quantities that are sacred:

Quantity	**Strong**	**EM**	**Weak**
Energy	Y	Y	Y
Linear momentum	Y	Y	Y
Angular momentum	Y	Y	Y
Electric charge	Y	Y	Y

Other Conserved Quantities

Quantity	Strong	EM	Weak	Comments
Baryon number	Y	Y	Y	no $p \to \pi^+\pi^0$
Lepton number	Y	Y	Y	no $\mu^- \to e^-\gamma$
Top	Y	Y	N	discovered 1995
Bottom	Y	Y	N	discovered 1977
Charm	Y	Y	N	discovered 1974, Nobel 1976
Strangeness	Y	Y	N	discovered 1947
Isospin	Y	N	N	proton = neutron ($m_u \approx m_d$)
Charge conjugation (C)	Y	Y	N	particle ⇔ anti-particle
Parity (P)	Y	Y	N	Nobel 1957
CP or Time (T)	Y	Y	N	small violation, Nobel 1980
CPT	Y	Y	Y	sacred
G parity	Y	N	N	works for pions only

- Neutrino oscillations give first evidence of lepton number violation!
  - These experiments were designed to look for baryon number violation!!
- Classic example of <u>strangeness violation</u>: $\Lambda \to p\pi$ ($S = -1 \to S = 0$)
- Very subtle example of <u>*CP* violation</u>:
  - expect: $K^0_{long} \to \pi^+\pi^0\pi$
  - observe: $K^0_{long} \to \pi^+\pi \approx 2 \times 10^{-3}$

Some Sample Reactions

- $\nu_\mu + p \to \mu^+ + n$:
  - What force is involved here?
    - Must be weak interaction since neutrino is involved.
  - Is this reaction allowed or forbidden?
    - Consider quantities conserved by weak interaction: lepton #, baryon #, *q, E, p, L* etc.

      muon lepton number : $1+0 \neq -1+0$

      baryon number : $0+1=0+1$

      ☞ Reaction not allowed!
- $\nu_e + p \to e^- + \pi^+ + p$:
  - Must be weak interaction since neutrino is involved.
    - Conserves all weak interaction quantities

      electron lepton number : $1+0=1+0+0$

      baryon number : $0+1=0+0+1$

      ☞ Reaction is allowed!
- $\Lambda \to e^- + \pi^+ + \bar{\nu}_e$:
  - Must be weak interaction since neutrino is involved.

    electron lepton number : $0=1+0-1$

    baryon number : $1 \neq 0+0+0$

    ☞ Reaction is not allowed!
- $K^+ \to \mu^+ + \pi^0 + \nu_\mu$:
  - Must be weak interaction since neutrino is involved.

    muon lepton number : $0=-1+0+1$

    ☞ Reaction is allowed!

More Reaction Examples

- Let's consider the following reactions to see if they are allowed:
  a) $K^+p \to \Lambda\pi^+\pi^+$ b) $K^-p \to \Lambda n$ c) $K^0 \to \pi^+\pi^-$

$K^+ = \bar{s}u$

$K^- = s\bar{u}$

$K^0 = s\bar{d}$

$\Lambda = usd$

  - First we should figure out which forces are involved in the reaction.
  - All three reactions involve only strongly interacting particles (no leptons)
    - ☞ it is natural to consider the strong interaction first.
      a) Not possible via strong interaction since strangeness is violated (1 → -1).
      b) OK via strong interaction (e.g. strangeness –1 → -1).
      c) Not possible via strong interaction since strangeness is violated (1 → 0).
      - ★ If a reaction is possible through the strong force then it will happen that way!
      - ★ Next, consider if reactions a) and c) could occur through the electromagnetic interaction.
        - ☞ Since there are no photons involved in this reaction (initial or final state) we can neglect EM.
          - ❑ EM conserves strangeness.
      - ★ Next, consider if reactions a) and c) could occur through the weak interaction.
        - ◆ Here we must distinguish between interactions (collisions) as in a) and decays as in c).

◆ The probability of an interaction (e.g. a)) involving only baryons and mesons occurring through the weak interactions is so small that we neglect it.

- ◆ Reaction c) is a decay.
  - ❑ Many particles decay via the weak interaction through strangeness changing decays.
    - ☞ this can (and does) occur via the weak interaction.

- To summarize:
  a) not strong, weak, and EM
  b) strong
  c) not strong and EM, OK via weak interaction

- Don't even bother to consider Gravity!

Conserved Quantities and Symmetries

- Every conservation law corresponds to an invariance of the Hamiltonian (or Lagrangian) of the system under some transformation.
  - We call these invariances symmetries.
- There are two types of transformations: continuous and discontinuous.
  - Continuous: give additive conservation laws

    $x \rightarrow x + dx$ or $\theta \rightarrow \theta + d\theta$
    - Examples of conserved quantities:
      - momentum
      - electric charge
      - baryon number
  - Discontinuous: give multiplicative conservation laws
    - Parity transformation: $(x, y, z) \rightarrow (-x, -y, -z)$
    - Charge conjugation (particle $\leftrightarrow$ antiparticle): $e^- \leftrightarrow e^+$
    - Examples of conserved quantities:
      - Parity (in strong and EM)
      - Charge conjugation (in strong and EM)
      - Parity and charge conjugation (in strong, EM, almost always in weak)

Momentum Conservation

- Example of classical mechanics and momentum conservation.
  - In general a system can be described by the following Hamiltonian:

    $H = H(p_i, q_i, t)$ $\quad p_i$ = momentum coordinate $\quad q_i$ = space coordinate $\quad t$ = time coordinate
- Consider the variation of H:

$$dH = \sum_{i=1}^{3} \frac{\partial H}{\partial q_i} dq_i + \sum_{i=1}^{3} \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

  - For a space translation (q_i) only, $dp_i = dt = 0$:

$$dH = \sum_{i=1}^{3} \frac{\partial H}{\partial q_i} dq_i$$

    - Using Hamilton's canonical equations:

$$\dot{q}_i = \frac{\partial q_i}{\partial t} = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \frac{\partial p_i}{\partial t} = \frac{\partial H}{\partial q_i}$$

$$dH = \sum_{i=1}^{3} \frac{\partial H}{\partial q_i} dq_i = \sum_{i=1}^{3} \dot{p}_i dq_i$$

    - If H is invariant under a translation (dq) then by definition we must have:

$$dH = \sum_{i=1}^{3} \dot{p}_i dq_i = 0$$

$$\dot{p}_1 = \dot{p}_2 = \dot{p}_3 = 0$$

$$\frac{dp_1}{dt} = \frac{dp_2}{dt} = \frac{dp_3}{dt} = 0$$

☞ Each p component is constant in time and momentum is conserved.

Conserved Quantities and Quantum Mechanics

- In quantum mechanics quantities whose operators commute with the Hamiltonian are conserved.
- The expectation value of an operator Q:

$$\langle Q\rangle = \int \Psi^* Q\Psi d\bar{x} \quad \text{with } \Psi = \Psi(\bar{x},t) \text{ and } Q = Q(\bar{x},\dot{\bar{x}},t)$$

- How does <Q> change with time?

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\int \Psi^* Q\Psi d\bar{x} = \int \frac{\partial\Psi^*}{\partial t} Q\Psi d\bar{x} + \int \Psi^* \frac{\partial Q}{\partial t}\Psi d\bar{x} + \int \Psi^* Q\frac{\partial\Psi}{\partial t} d\bar{x}$$

- Recall Schrodinger's equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi \text{ and } -i\hbar\frac{\partial\Psi^*}{\partial t} = \Psi^* H^+$$

$H^+ = H^{*T} =$ Hermitian conjugate of H

- Substituting the Schrodinger equation into the time derivative of Q:

$$\frac{d}{dt}\langle Q\rangle = -\frac{1}{i\hbar}\int \Psi^* H^+ Q\Psi d\bar{x} + \int \Psi^* \frac{\partial Q}{\partial t}\Psi d\bar{x} + \frac{1}{i\hbar}\int \Psi^* QH\Psi d\bar{x}$$

$$= \int \Psi^*\left(\frac{\partial Q}{\partial t} + \frac{1}{i\hbar}[-H^+Q + QH]\right)\Psi d\bar{x}$$

$$= \int \Psi^*\left(\frac{\partial Q}{\partial t} + \frac{1}{i\hbar}[Q,H]\right)\Psi d\bar{x}$$

H is Hermitian ($H^+ = H$)

☞ If $\partial Q/\partial t = 0$ and $[Q,H] = 0$ then <Q> is conserved.

Conservation of Electric Charge and Gauge Invariance

- Conservation of electric charge: $\Sigma Q_i = \Sigma Q_f$
- Evidence for conservation of electric charge:
  - Consider reaction $e^- \to \gamma \nu_e$:
    - Violates charge conservation but not lepton number or any other quantum number.
    - If the above transition occurs in nature then we should see x-rays from atomic transitions.
    - The absence of such x-rays leads to the limit:

      $\tau_e > 2\text{x}10^{22}$ years
- There is a connection between charge conservation, gauge invariance, and quantum field theory.
- A Lagrangian that is invariant under a transformation $U = e^{i\theta}$ is said to be gauge invariant.
- There are two types of gauge transformations:

  local: $\theta = \theta(\boldsymbol{x},t)$ θ is an operator

  global: θ = constant, independent of $(\boldsymbol{x},t)$
  - Conservation of electric charge is the result of global gauge invariance.
  - Photon is massless due to local gauge invariance.
- Recall Maxwell's Equations are invariant under a gauge transformation:

  $$\text{vector potential}: \overline{A}' \to \overline{A} + \nabla\Lambda$$

  $$\text{scalar potential}: \phi' \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

  $\Lambda(\overline{x},t)$: arbitrary scalar function
  - New set of potential also yields the same fields that particles interact with:

  $$\overline{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\overline{A}}{\partial t}$$

  $$\overline{B} = \nabla \times \overline{A}$$

Maxwell's EQs are locally gauge invariant.

Gauge invariance and Group Theory

- Consider a transformation (U) that acts on a wavefunction (ψ):

  $\psi' = U\psi$

  - Let U be a continuous transformation then U is of the form:

    $U = e^{i\theta}$ θ is an operator.

    - If θ is a Hermitian operator ($\theta = \theta^+ = \theta^{*T}$) then U is a unitary transformation:

      $U = e^{i\theta} \quad U^+ = (e^{i\theta})^{*T} = e^{-i\theta^{*T}} e^{-i\theta} \Rightarrow UU^+ = e^{i\theta}e^{-i\theta} = 1$

      - U is not a Hermitian operator since $U \neq U^+$

- In the language of group theory θ is said to be the generator of U
  - There are four properties that define a group:
    - Closure: if A and B are members of the group then so is AB
    - Identity: for all members of the set I exists such that $IA = A$
    - Inverse: the set must contain an inverse for every element in the set $AA^{-1} = I$
    - Associativity: if A, B, C are members of the group then $A(BC) = (AB)C$
  - If $\theta = (\theta_1, \theta_2, \theta_3\ldots)$ then the transformation is "Abelian" if:

    $U(\theta_1)U(\theta_2) = U(\theta_2)U(\theta_1)$

    ☞ The operators commute.

    - If the operators do not commute then the group is non-Abelian.
  - The transformation with only one θ forms the unitary Abelian group U(1).
  - The Pauli (spin) matrices generate the non-Abelian group SU(2):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

S = "special" = unit determinant
U = unitary
N = dimension (e.g. 2)

Global Gauge Invariance and Charge Conservation

- The relativistic Lagrangian for a free electron is:

  $L = i\overline{\psi}\gamma_u \partial^u \psi - m\overline{\psi}\psi \qquad \hbar = c = 1$

  ψ: the electron field (a 4-component spinor)

  m: the electron mass

  γ_u: "gamma" matrices, four ($u = 1,2,3,4$) 4×4 matrices that satisfy $\gamma^u\gamma^v + \gamma^v\gamma^u = 2g^{uv}$

  $\partial^u = (\partial^0, \partial^1, \partial^2, \partial^3) = (\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)$

This Lagrangian gives the Dirac equation:
$i\hbar\gamma^u\partial_u\psi - mc\psi = 0$

- Let's apply a global gauge transformation to L:

  $\psi' = e^{i\lambda}\psi \quad \overline{\psi}' = \overline{\psi}e^{-i\lambda}$

  $L' = i\overline{\psi}'\gamma_u\partial^u\psi' - m\overline{\psi}'\psi'$

  $L' = i\overline{\psi}e^{-i\lambda}\gamma_u\partial^u e^{i\lambda}\psi - m\overline{\psi}e^{-i\lambda}e^{i\lambda}\psi$

  $L' = i\overline{\psi}e^{-i\lambda}e^{i\lambda}\gamma_u\partial^u\psi - m\overline{\psi}e^{-i\lambda}e^{i\lambda}\psi$ since λ is a constant

  $L' = i\overline{\psi}\gamma_u\partial^u\psi - m\overline{\psi}\psi = L$

  ☞ By Noether's Theorem there must be a conserved quantity associated with this symmetry!

The Dirac Equation on One Page

- The Dirac equation: $i\hbar\gamma^{u}\partial_{u}\Psi - mc\Psi = 0$

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix}$$

$$i\hbar[\gamma^{0}\frac{\partial\Psi}{\partial t} - \gamma^{1}\frac{\partial\Psi}{\partial x} - \gamma^{2}\frac{\partial\Psi}{\partial y} - \gamma^{3}\frac{\partial\Psi}{\partial z}] - mc\Psi = 0$$

$$i\hbar[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\frac{\partial\Psi}{\partial t} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\frac{\partial\Psi}{\partial x} - \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\frac{\partial\Psi}{\partial y} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\frac{\partial\Psi}{\partial z}] - mc\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A solution (one of 4, two with $+E$, two with $-E$) to the Dirac equation is:

$$\Psi = \sqrt{(|E|+mc^{2})/c}\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_{z}}{E+Mc^{2}} \\ \dfrac{c(p_{x}+ip_{y})}{E+Mc^{2}} \end{pmatrix}\exp[-(i/\hbar)(Et - \overline{p}\bullet\overline{r}] = U\exp[-(i/\hbar)(Et - \overline{p}\bullet\overline{r}]$$

- The function U is a (two-component) spinor and satisfies the following equation:

$$(\gamma^{u}p_{u} - mc)U = 0$$

- Spinors are most commonly used in physics to describe spin 1/2 objects:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ represents spin up } (+\hbar/2) \text{ while } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ represents spin down } (-\hbar/2)$$

  - Spinors also have the property that they change sign under a 360^{0} rotation!

Global Gauge Invariance and Charge Conservation

- We need to find the quantity that is conserved by our symmetry.
- In general if a Lagrangian density, $L = L(\phi, \partial\phi/\partial x_u)$ with a field ϕ, is invariant under a transformation:

$$\delta L = 0 = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \frac{\partial \phi}{\partial x_u}}\delta\frac{\partial \phi}{\partial x_u}$$

Result from field theory

- For our global gauge transformation we have:

$$\phi' = e^{i\lambda}\phi \approx (1+i\lambda)\phi$$

$$\delta\phi = \phi' - \phi = (1+i\lambda)\phi - \phi = i\lambda\phi \text{ and } \delta\frac{\partial \phi}{\partial x_u} = \frac{\partial \delta\phi}{\partial x_u} = i\lambda\frac{\partial \phi}{\partial x_u}$$

  - Plugging this result into the equation above we get (after some algebra…)

$$\delta L = 0 = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \frac{\partial \phi}{\partial x_u}}\delta\frac{\partial \phi}{\partial x_u} = \left[\frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x_u}\left(\frac{\partial L}{\partial \frac{\partial \phi}{\partial x_u}}\right)\right]\delta\phi + \frac{\partial}{\partial x_u}\left[\frac{\partial L}{\partial \frac{\partial \phi}{\partial x_u}}\delta\phi\right]$$

    - The first term is zero by the Euler-Lagrange equation.
    - The second term gives us a continuity equation.

E-L equation in 1D:
$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

Global Gauge Invariance and Charge Conservation

- The continuity equation is:

$$\frac{\partial}{\partial x_u}\left[\frac{\partial L}{\partial \frac{\partial \phi}{\partial x_u}}\delta\phi\right]=\frac{\partial}{\partial x_u}\left[\frac{\partial L}{\partial \frac{\partial \phi}{\partial x_u}}i\lambda\phi\right]=\frac{\partial J^u}{\partial x_u}=0 \quad \text{with} \quad J^u=i\lambda\frac{\partial L}{\partial \frac{\partial \phi}{\partial x_u}}\phi$$

Result from quantum field theory

- Recall that in classical E&M the (charge/current) continuity equation has the form:

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\vec{J}=0$$

$$(J^0, J^1, J^2, J^3)=(\rho, J_x, J_y, J_z)$$

- Also, recall that the Schrodinger equation gives a conserved (probability) current:

$$i\hbar\frac{\partial \psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi$$

$$\Rightarrow\begin{cases}\rho=c\psi^*\psi\\ \vec{J}=-i\hbar c[\psi^*\nabla\psi-(\nabla\psi^*)\psi]\end{cases}$$

- If we use the Dirac Lagrangian in the above equation for L we find:

$$J^u=-\lambda\overline{\psi}\gamma^u\psi$$

Conserved quantity

  - This is just the relativistic electromagnetic current density for an electron.
  - The electric charge is just the zeroth component of the 4-vector:

$$Q=\int J^0 d\vec{x}$$

  - ☞ If there are no current sources or sinks ($\nabla\bullet J=0$) charge is conserved as:

$$\frac{\partial J^u}{\partial x_u}=0 \quad \Rightarrow\frac{\partial J^0}{\partial t}=0$$

Local Gauge Invariance and Physics

- Some consequences of local gauge invariance:
  - For QED local gauge invariance implies that the photon is massless.
  - In theories with local gauge invariance a conserved quantum number implies a long range field.
    - e.g. electric and magnetic field
- There are other quantum numbers that are similar to electric charge (e.g. lepton number, baryon number) that don't seem to have a long range force associated with them!
  - Perhaps these are not exact symmetries!
    - Evidence for neutrino oscillation implies lepton number violation.
  - Theories with local gauge invariance can be renormalizable
    - can use perturbation theory to calculate decay rates, cross sections etc.
- Strong, weak and EM theories are described by local gauge theories.
  - U(1) local gauge invariance first discussed by Weyl in 1919
  - SU(2) local gauge invariance discussed by Yang & Mills in 1954 (electro-weak)

    $$\psi' = e^{i\tau\cdot\lambda(x,t)}\psi$$

    - τ is represented by the 2x2 Pauli matrices (non-Abelian)
  - SU(3) local gauge invariance used to describe strong interaction (QCD) in 1970's

    $$\psi' = e^{i\tau\cdot\lambda(x,t)}\psi$$

    - τ is represented by the 3x3 matrices of SU(3) (non-Abelian)

Local Gauge Invariance and QED

- Consider the case of local gauge invariance, $\lambda = \lambda(x,t)$ with transformation:

  $$\psi' = e^{i\lambda(\vec{x},t)}\psi \quad \overline{\psi}' = \overline{\psi}e^{-i\lambda(\vec{x},t)}$$

- The relativistic Lagrangian for a free electron is not invariant under this transformation.

  $$L = i\overline{\psi}\gamma_u\partial^u\psi - m\overline{\psi}\psi \qquad \hbar = c = 1$$

- The derivative in the Lagrangian introduces an extra term:

  $$\partial\psi' = e^{i\lambda(\vec{x},t)}\partial^u\psi + e^{i\lambda(\vec{x},t)}\psi\partial^u[i\lambda(\vec{x},t)]$$

- We can make a Lagrangian that is locally gauge invariant by adding an extra piece to the free electron Lagrangian that will cancel the derivative term.
- We need to add a vector field A_u which transforms under a gauge transformation as:

  $$A_u \to A_u + \partial_u\Lambda(\vec{x},t) \text{ with } \lambda(\vec{x},t) = -q\Lambda(\vec{x},t) \text{ (for electron } q = -|e|)$$

- The new, locally gauge invariant Lagrangian is:

  $$L = i\overline{\psi}\gamma_u\partial^u\psi - m\overline{\psi}\psi - \frac{1}{16}F^{uv}F_{uv} - q\overline{\psi}\gamma_u\psi A^u$$

Locally Gauge Invariance QED Lagrangian

- Several important things to note about the above Lagrangian:
  - A_u is the field associated with the photon.
  - The mass of the photon must be zero or else there would be a term in the Lagrangian of the form:

    $m_\gamma A_u A^u$
    - However, $A_u A^u$ is not gauge invariant!
  - $F_{uv} = \partial_u A_v - \partial_v A_u$ and represents the kinetic energy term of the photon.
  - The photon and electron interact via the last term in the Lagrangian.
    - This is sometimes called a current interaction since:

      $$q\overline{\Psi}\gamma_u \psi A^u = J_u A^u$$
    - In order to do QED calculations we apply perturbation theory (via Feynman diagrams) to $J_u A_u$ term.
  - The symmetry group involved here is unitary and has one parameter $\Rightarrow U(1)$