Abstract

We use a two-level hierarchical Bayesian model to calibrate the parameters of the Rossiter model (1964) of cavity flow noise based on experimental data. The parameters of interest are the phase shift of the acoustic scattering process ($\alpha$) and the ratio of the convective speed of the disturbance to the free stream velocity ($\beta$). We implement a Metropolis within Gibbs algorithm to explore the joint posterior distribution. With scant data, the priors were made to reflect the expert's belief system. However, other possibilities associated with different amounts of prior information assumed, including the case of ignorance were also explored. Results indicate that the location of the joint posterior distribution is not invariant to the mode of the prior distribution. The 95% posterior intervals based on the expert’s prior are (0.10, 0.49) for $\alpha$ and (0.37, 0.80) for $\beta$. The posterior means obtained, $\alpha$ = 0.25 and $\beta$ = 0.60, are consistent with theoretical expectations. The uncertainty measures represented by the width of the posterior intervals, however, still leaves ample room for improvement.

1. Introduction

In the present study, we applied a Bayesian approach to calibrate the parameters of a model as an alternative to the current practice of “fitting by eye”. Even as the method relies on the expert’s knowledge and experience, it does not provide a systematic basis for measuring uncertainties in the outputs. Here, we try to optimize information from the data as well as the expert while accounting for uncertainties in the model inputs.

The object of interest is the Rossiter model for cavity flow noise. One of its application domains is the representation of a plane’s vibration caused by the opening of the wheels’ cavity as it prepares for a touchdown.

1.1 Self-Excited Resonant Flow Noise

We briefly illustrate the phenomenon described by the Rossiter model with a diagram shown in Figure 1.

When the wheels’ doors are opened, a cavity is exposed. The air passing over the cavity creates some eddies (in green) that move with the ambient air; these reach and impact the other end of the cavity (in red). The impact creates noise that propagates at the speed of sound until it reaches its point of origin. It then stimulates the production of new eddies. This self-excitation process explains the vibration in the aircraft when it approaches landing. The frequency of vibration depends on the speed of the surrounding air and on the number of waves in between the cavity.
1.2 The Rossiter Model

Rossiter (1964) developed a semi-empirical formula for resonant Strouhal number, a unitless function of the noise frequency. It is given as follows:

\[
St_v = \frac{f_n L}{U_\infty} = \frac{n - \alpha}{M_\infty \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{-1/2} + \frac{1}{\beta}}
\]

where \( f_n \) is the frequency at mode number \( n \), \( L \) is the length of the longitudinal section, \( U_\infty \) is the free stream velocity, \( \alpha \) is the phase shift (in fractions of wavelength) of the acoustic scattering process, \( \beta \) is the ratio of the convective speed of the disturbance to the free stream velocity, \( M_\infty \) is the surrounding wind velocity, and \( \gamma \) is a physical constant.

1.3 Data Collection

Our data source is Dr. Marco Debiasi, a researcher at the Gas Dynamics and Turbulence Laboratory of the Ohio State University. Dr. Debiasi is currently investigating the area of Closed Loop Active Flow Control. His experimental facility features the actuator, shown in Figure 2 which generated the data we analyzed.
The experiment was run at regularly spaced settings of the control variable called the Mach number. The levels explored are in the range [0.2, 0.7] with increments of 0.01. A digital device attached to the actuator read and recorded the observed frequencies in continuous time. Thus, at each Mach number setting, a “noise frequency curve” was plotted. Since only the peaks of the curve are considered informative about the values of the calibration parameters, these were determined in turn by the expert through visual inspection. The mode number \( n \) for each peak was resolved in like fashion. Though more aptly treated as a variable, we assume that its identification is made with negligible error, and therefore treat it as known. This is a point for further refinement in future investigations. Figure 3 on the left gives the totality of recorded noise intensities or frequencies against the Mach number, while Figure 4 with the dots shows the dominant (black) and subdominant (white) peaks identified.

In all there were 90 points, the set on which we based our statistical modeling. The lines correspond to the predicted frequencies according to the Rossiter formula. Our objective is to find the values of \( \alpha \) and \( \beta \) that provide the best fit to the data.

2. Statistical Formulation

In this section, we introduce our notation and elaborate on our statistical model. Unknown process quantities are denoted by letters of the Greek alphabet, and those quantities which we treat as known, by letters of the Roman alphabet.

2.1 Notation

Let \( Y \) be the \( 90 \times 1 \) vector containing the experimental data, \( X \) be the \( 51 \times 1 \) vector of Mach numbers, the control variable. The design matrix, \( B \) is of dimension \( 90 \times 51 \). Suppose \( \zeta(x) \) is the true frequency at \( X=x \). Then \( Y(x) = \zeta(x) + \varepsilon(x) \), where \( \varepsilon(x) \) accounts
for measurement error. We assume that the errors are independently and identically distributed as \( N(0, \sigma^2) \). Moreover, we postulate that \( \zeta(x) = B \eta(x, \alpha, \beta) + \delta(x) \), where the first term is the expected Rossiter frequency at \( X=x \). The second term is a stochastic discrepancy term representing the deviation of the Rossiter model from the true process at its best settings for the unknown calibration parameter \( (\alpha, \beta) \). We assume that \( \delta(x) \) is a Gaussian process with mean function \( \mu(x) = 0 \) and covariance function \( \gamma(x, x') = \lambda^{-1} \exp(-\phi \mid x - x' \mid^2) + \kappa d_{ij} \), where \( \lambda \) is the marginal precision, \( \phi \) is the dependence parameter, \( \kappa \) is a positive number related to the variance of the process (included to attain numerical stability), and \( d_{ij} = 1 \ (i=j) \).

### 2.2 The Priors

We now present our representation of knowledge and uncertainties about the unknown quantities in our model. We succinctly write out the relationships as follows:

\[
Y(x) = \zeta(x) + \varepsilon(x), \quad \varepsilon(x) \sim \text{iid } N(0, \sigma^2) \tag{2.1}
\]

\[
\sigma^2 \sim \text{Inverse Gamma}(SA, SB) \tag{2.2}
\]

\[
\zeta(x) = B \eta(x, \alpha, \beta) + \delta(x), \quad \delta(x) \sim \text{MVN}(0, \Sigma) \tag{2.3}
\]

\[
\Sigma = \{ \gamma(x_i, x_j) \}_{ij} = \lambda^{-1} \exp(-\phi \mid x_i - x_j \mid^2) + \kappa d_{ij}
\]

\[
\lambda \sim \text{Gamma}(LA, LB)
\]

\[
\varphi \sim \text{Gamma}(PA, PB)
\]

\[
\kappa \sim \text{Gamma}(KA, KB)
\]

\[
\alpha \sim \text{Beta}(AA, AB)
\]

\[
\beta \sim \text{Beta}(BA, BB) \tag{2.5}
\]

The parameter vector is written as \( \theta = [\alpha, \beta, \sigma^2, \lambda, \varphi, \kappa] \), where we set independently distributed priors for the elements of \( \theta \). That is,

\[
[\theta] = [\alpha][\beta][\sigma^2][\lambda][\varphi][\kappa] \tag{2.6}
\]

Our sampling model is given by

\[
L[Y(x) | \theta] \propto \sigma^{-n/2} \exp(-1/2 \{Y(x) - \eta(x, \alpha, \beta)\}^T \sigma^{-2}(Y(x) - \eta(x, \alpha, \beta))) \tag{2.7}
\]

Thus the posterior distribution \( [\theta | Y] \) is

\[
[\theta | Y] \propto \sigma^{-n/2} \exp(-1/2 \{Y(x) - \eta(x, \alpha, \beta)\}^T \sigma^{-2}(Y(x) - \eta(x, \alpha, \beta))) \times
\]

\[
[\alpha][\beta][\sigma^2][\lambda][\varphi][\kappa] \tag{2.8}
\]
The table below gives the values we assigned to our second-level parameters.

<table>
<thead>
<tr>
<th>θ</th>
<th>Distribution</th>
<th>Initial Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
<td>Beta(AA,AB)</td>
<td>(AA + AB) = c (2, 10, 30); prior mode = c(0.25, 0.80)</td>
</tr>
<tr>
<td>β</td>
<td>Beta(BA,BB)</td>
<td>(BA + BB) = c (2, 10, 30); prior mode = c(0.10, 0.60)</td>
</tr>
<tr>
<td>σ²</td>
<td>IG(SA,SB)</td>
<td>SA = 4.0; SB = 0.5</td>
</tr>
<tr>
<td>λ</td>
<td>Gamma(LA,LB)</td>
<td>LA = 0.2; LB = 0.2</td>
</tr>
<tr>
<td>φ</td>
<td>Gamma(PA,PB)</td>
<td>PA = 20; PB = 2</td>
</tr>
<tr>
<td>κ</td>
<td>Gamma(KA,KB)</td>
<td>KA = 2; KB = 1</td>
</tr>
</tbody>
</table>

Table 1: Summary of Prior Specification

As was previously mentioned, we capitalized on expert knowledge to provide us with our prior distributions for α and β. However, we also explored other situations reflecting different degrees of lack of knowledge, including ignorance.

Current (extra-statistical) literature and the expert himself pegged the value of α at 0.25, and for β at 0.625. We took this as the prior modes of two independent Beta distributions. The Beta distribution was particularly chosen because the support of α and β both lie in the unit interval. Thus for the pair of hyper parameters (AA, AB) and (BA, BB), the actual values were determined supposing that the amount of prior information, ν=(AA+AB)=(BA+BB)=10. We also examined one scenario when ν=30. When ν=2, we have the case of ignorance, and the modes do not exist for either of the distributions. This was also investigated. Finally, we looked into what results would follow if the prior modes had been set to far away values (α=0.80, and β=0.10).

2.3 Markov Chain Monte Carlo Sampling

We implemented a Metropolis-Hastings within Gibbs sampling algorithm to approximate the posterior distribution. There were 7,000 iterations in all. The first 2,000 were discarded, and from the remaining 5,000, we sampled every 10th interval. Therefore, our posterior inferences are based on sample size of 500. On the average, each run took 12 minutes on a PC machine. The coding was done in Matlab version 7.1.

The steps are outlined as follows:

i.  \[ \eta (x) \mid \text{rest}, Y \] \sim \text{MVN (} \mu^*, \Sigma^* \text{)}

ii. \[ \sigma^2 \mid \text{rest}, Y \] \sim \text{Inverse Gamma}

iii. \[ (\alpha, \beta) \mid \text{rest}, Y \propto \exp\{- g_1 (h(x_i),(a,b), (l,j,k)) \} \] 

   Metropolis-Hastings Step

iv. \[ (\lambda, \phi, \kappa) \mid \text{rest}, Y \propto \exp\{- g_2 (h(x_i),(a,b), (l,j,k)) \} \] 

   Metropolis-Hastings Step
3. Results

Here we present the marginal posterior distribution of the quantities of interest, namely $\alpha$ and $\beta$, the calibration parameters of the Rossiter model. The mean, the Monte Carlo standard deviation, and the quartiles are provided along with 95% posterior intervals. Overall, convergence was reached by the 2000th iteration. There were varying degrees of mixing, but generally the quality was acceptable.

3.1 Posterior Distribution of Calibration Parameters

The order of presentation is according to the prior distribution considered. In general, we observe that the posterior distributions are not invariant to the choice of prior mode. The location of the posterior distribution is apparently influenced by the location of the prior. In the uniform or non-informative case, the posterior was not substantially different, that is, it remained flat given the data. The amount of prior information $\nu$ did not play a crucial role in the posterior mean, but certainly, in the amount of uncertainty reduced as more prior information leads to less variability. Hence, the distance between the quartiles and the posterior intervals are shorter. If the “wrong” mode were taken for the prior, the posterior would not stray far from it.

<table>
<thead>
<tr>
<th>Prior $\nu$</th>
<th>Parameter</th>
<th>Mean</th>
<th>Std Dev</th>
<th>Q1</th>
<th>Median</th>
<th>Q3</th>
<th>L95%</th>
<th>U95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu = 2$</td>
<td>$\alpha$</td>
<td>0.48</td>
<td>0.30</td>
<td>0.22</td>
<td>0.47</td>
<td>0.75</td>
<td>0.02</td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>0.51</td>
<td>0.30</td>
<td>0.26</td>
<td>0.52</td>
<td>0.76</td>
<td>0.02</td>
<td>0.98</td>
</tr>
<tr>
<td>$\nu = 10$</td>
<td>$\alpha$</td>
<td>0.27</td>
<td>0.10</td>
<td>0.20</td>
<td>0.27</td>
<td>0.34</td>
<td>0.20</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>0.59</td>
<td>0.11</td>
<td>0.52</td>
<td>0.59</td>
<td>0.66</td>
<td>0.37</td>
<td>0.80</td>
</tr>
<tr>
<td>$\nu = 10$</td>
<td>$\alpha$</td>
<td>0.78</td>
<td>0.10</td>
<td>0.20</td>
<td>0.27</td>
<td>0.35</td>
<td>0.11</td>
<td>0.49</td>
</tr>
<tr>
<td>$\alpha = 0.25$</td>
<td>$\beta$</td>
<td>0.58</td>
<td>0.11</td>
<td>0.50</td>
<td>0.58</td>
<td>0.66</td>
<td>0.34</td>
<td>0.80</td>
</tr>
<tr>
<td>$\beta = 0.60$</td>
<td>$\beta$</td>
<td>0.14</td>
<td>0.07</td>
<td>0.08</td>
<td>0.12</td>
<td>0.18</td>
<td>0.03</td>
<td>0.31</td>
</tr>
<tr>
<td>$\nu = 10$</td>
<td>$\alpha$</td>
<td>0.76</td>
<td>0.10</td>
<td>0.70</td>
<td>0.78</td>
<td>0.84</td>
<td>0.55</td>
<td>0.93</td>
</tr>
<tr>
<td>$\alpha = 0.80$</td>
<td>$\beta$</td>
<td>0.15</td>
<td>0.07</td>
<td>0.08</td>
<td>0.12</td>
<td>0.18</td>
<td>0.03</td>
<td>0.31</td>
</tr>
<tr>
<td>$\beta = 0.10$</td>
<td>$\beta$</td>
<td>0.26</td>
<td>0.06</td>
<td>0.22</td>
<td>0.26</td>
<td>0.29</td>
<td>0.15</td>
<td>0.40</td>
</tr>
<tr>
<td>$\nu = 30$</td>
<td>$\alpha$</td>
<td>0.60</td>
<td>0.06</td>
<td>0.55</td>
<td>0.60</td>
<td>0.64</td>
<td>0.48</td>
<td>0.72</td>
</tr>
<tr>
<td>$\alpha = 0.25$</td>
<td>$\beta$</td>
<td>0.78</td>
<td>0.05</td>
<td>0.76</td>
<td>0.79</td>
<td>0.83</td>
<td>0.69</td>
<td>0.88</td>
</tr>
<tr>
<td>$\beta = 0.10$</td>
<td>$\beta$</td>
<td>0.11</td>
<td>0.04</td>
<td>0.08</td>
<td>0.11</td>
<td>0.14</td>
<td>0.05</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Table 2. Summary Statistics of Marginal Posterior Distribution of $\alpha$ and $\beta$
We show the posterior distributions for $\alpha$ and $\beta$ in the case of the expert’s prior distribution.

3.2 Convergence
For our checks on convergence, we plot the values of the running means. We also examine the quality of mixing in the space of $(\alpha, \beta)$. Both are visually satisfactory to us. The first row of plots refer to the expert’s prior distribution, while the second to the non-informative
4. Discussion

From the above analysis, it is evident that our MCMC runs converge but the posterior distribution we get is influenced to a large extent by the prior mode. The data do not “overwhelm” the prior. We speculate that our formulation contained more parameters than what our data can provide information on.

In another setting, the same data set was analyzed using a simpler formulation derived from the literature of Computer Experiments. Still using a Bayesian paradigm to calibrate the model, the analyses were made assuming that the stochastic discrepancy term was nil, and that the measurement error variance was known apriori. Only uncertainty in the calibration parameters was assumed. For an in depth discussion, we refer the paper of Higdon, et al on Combining Field and Simulation Data for Calibration and Prediction. Here a simpler Metropolis algorithm run at 10,000 iterations converged to a stationary distribution that was invariant to the choice of the prior distribution. The prior distributions ranged from independent \( U[0,1] \), \( \alpha \sim N(0.25,100) \) and \( \beta \sim N(0.60,100) \), and two variants of the Beta distribution with \( (AA+AB)=(BA+BB)=c(10,20) \). We verified that \( \alpha \in [0.2464, 0.2468] \) and \( \beta \in [0.6423, 0.6425] \). The high number of iterations is justified in a calibration problem where high precision on the model inputs of the simulator is desired.

A Partial output of the runs is shown below.
Bibliography


Rossiter, RAE Technical Report 64037 1964

Interview