Exercise 12.11

Suppose that \(X_1, \cdots, X_n \overset{i.i.d}{\sim} \text{Exp}(\theta)\). The hypotheses are given by

\[
H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1 \quad \text{where } \theta_1 > \theta_0
\]

In this setting, since both hypotheses are simple, we can use the Neyman-Pearson (N-P) lemma to construct the most powerful critical region. Given a random sample of size \(n\), the likelihood values under the null and the alternative are:

\[
L_0 = \left(\frac{1}{\theta_0}\right)^n e^{-\sum x_i / \theta_0}, \quad \text{and} \quad L_1 = \left(\frac{1}{\theta_1}\right)^n e^{-\sum x_i / \theta_1}
\]

By N-P lemma (Theorem 12.1), a critical region of size \(\alpha\) is obtained by solving the following equation for \(k\)

\[
\alpha = P\left(\frac{L_0}{L_1} \leq k \mid \theta_0\right)
\]

Now, we need to simplify the inequality in the above probability statement. Note that,

\[
\frac{L_0}{L_1} = \left(\frac{\theta_1}{\theta_0}\right)^n e^{-\sum x_i (\frac{1}{\theta_0} - \frac{1}{\theta_1})}
\]

On taking the log on both sides of \(\frac{L_0}{L_1} \leq k\), and simplifying it as a function of the sufficient statistic, we have

\[
-\sum X_i \left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right) \leq k_1
\]

Using the fact that \(\theta_1 > \theta_0\) implies that \(\frac{1}{\theta_0} - \frac{1}{\theta_1} > 0\), the above inequality simplifies to \(\sum_{i=1}^{n} X_i \geq c\).

Thus the most powerful critical region is given by \(\sum_{i=1}^{n} X_i \geq c\) where \(c\) can be determined by making use of the fact that \(\sum X_i\) has the gamma distribution with \(\alpha = n\) and \(\beta = \theta_0\).[Use the result from Example 7.16].

That is, \(c\) is a constant satisfying

\[
\alpha = \int_{c}^{\infty} \frac{1}{\Gamma(n)\theta_0^n} t^{n-1} e^{-t / \theta_0} dt.
\]

Given the sample size \(n\), \(\theta_0, \alpha\), the value of \(c\) can be found by solving for \(c\) numerically.
Exercise 12.12

Suppose that \( X \sim \text{B}(n, \theta). \) The hypotheses are given by
\[
H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1 \quad \text{where} \, \theta_1 < \theta_0.
\]
Also the likelihood functions are
\[
L_0 = \binom{n}{x} \theta_0^x (1 - \theta_0)^{n-x}, \quad \text{and} \quad L_1 = \binom{n}{x} \theta_1^x (1 - \theta_1)^{n-x}
\]
By N-P lemma (Theorem 12.1), a critical region of size \( \alpha \) is obtained from
\[
\alpha = P \left( \frac{L_0}{L_1} = \left( \frac{\theta_0(1 - \theta_1)}{\theta_1(1 - \theta_0)} \right)^X \left( \frac{(1 - \theta_0)}{(1 - \theta_1)} \right)^n \leq k \mid \theta_0 \right)
\]
\[
= P \left( \sum X_i \leq c \mid \theta_0 \right)
\]
since \( \frac{L_0}{L_1} \) is increasing as \( X \) increases from the fact that \( \theta_1 < \theta_0 \) results in \( \frac{\theta_0(1 - \theta_1)}{\theta_1(1 - \theta_0)} > 1. \) Thus the most powerful critical region is \( X \leq c \) where \( c \) is a positive integer determined from Table I and satisfying
\[
\alpha = \sum_{x=0}^{c} \binom{n}{x} \theta_0^x (1 - \theta_0)^{n-x}.
\]

Exercise 12.14

Suppose that \( X \sim \text{Geometric}(\theta). \) The hypotheses are given by
\[
H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1 \quad \text{where} \, \theta_1 > \theta_0.
\]
Also the likelihood functions are
\[
L_0 = \theta_0(1 - \theta_0)^{x-1}, \quad \text{and} \quad L_1 = \theta_1(1 - \theta_1)^{x-1}
\]
By N-P lemma (Theorem 12.1), a critical region of size \( \alpha \) is obtained from
\[
\alpha = P \left( \frac{L_0}{L_1} = \frac{\theta_0}{\theta_1} \left( \frac{1 - \theta_0}{1 - \theta_1} \right)^{X-1} \leq k \mid \theta_0 \right)
\]
\[
= P \left( X \leq c \mid \theta_0 \right)
\]
since \( \frac{L_0}{L_1} \) is increasing as \( X \) increases from the fact that \( \theta_1 > \theta_0 \) results in \( \frac{1 - \theta_0}{1 - \theta_1} > 1. \) Thus the most powerful critical region is \( X \leq c \) where \( c \) is a positive integer determined by
\[
\alpha = \sum_{X=1}^{c} \theta_0(1 - \theta_0)^{X-1} = \frac{\theta_0(1 - (1 - \theta_0)^c)}{1 - (1 - \theta_0)} = 1 - (1 - \theta_0)^c \Rightarrow c = \frac{\ln(1 - \alpha)}{\ln(1 - \theta_0)}.
\]
That is, the best critical region of size \( \alpha \) is
\[
X \leq c'
\]
where \( c' \) is the maximum positive integer less than or equal to \( \frac{\ln(1 - \alpha)}{\ln(1 - \theta_0)}. \)
Exercise 12.21

Suppose that \(X_1, \ldots, X_n \overset{i.i.d}{\sim} \text{Exp}(\theta)\). The hypotheses are given by

\[ H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta \neq \theta_0. \]

(a) Note that

\[ L = \left( \frac{1}{\theta} \right)^n e^{-\frac{\sum x_i}{\theta}}, \quad \text{and} \quad L_0 = \left( \frac{1}{\theta_0} \right)^n e^{-\frac{\sum x_i}{\theta_0}}. \]

Find the maximum likelihood estimate of \(\theta\).

\[
\ln L(\theta) = -n \ln \theta - \frac{\sum x_i}{\theta}
\]

\[
\frac{\partial}{\partial \theta} \ln L = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} = 0
\]

\[ \Rightarrow \hat{\theta}_{\text{MLE}} = \bar{x} \]

The likelihood ratio statistic, \(\lambda\) is expressed as

\[ \lambda = \frac{\max L_0}{\max L} = \frac{L_0}{L(\theta = \bar{x})} = \left( \frac{1}{\theta_0} \right)^n e^{-n} \left( \frac{\bar{x}}{\theta_0} \right)^n e^{-n\left( \frac{\bar{x}}{\theta_0} - 1 \right)} \] \hspace{1cm} (1)

(b) The critical region of the LRT from (1) is written as by Definition 12.4

\[ \lambda \leq k \Leftrightarrow \left( \frac{\bar{x}}{\theta_0} \right)^n e^{-n\left( \frac{\bar{x}}{\theta_0} - 1 \right)} \leq k \Leftrightarrow \frac{\bar{x}}{\theta_0} e^{-\left( \frac{\bar{x}}{\theta_0} - 1 \right)} \leq \sqrt{k} \Leftrightarrow \bar{x} e^{-\frac{\bar{x}}{\theta_0}} \leq \frac{\theta_0}{e} \sqrt{k} = K \]

However, to get more insight into this critical region, note that this inequality can also be written as

\[ \frac{\bar{x}}{\theta_0} e^{-\frac{\bar{x}}{\theta_0}} \leq K_1. \]

Now, letting \(\frac{\bar{x}}{\theta_0} = y\), the critical region can be expressed as \(ye^{-y} \leq k\). The function on the l.h.s. of this expression is same as that of a Gamma density with \(\alpha = 1, \beta = 1\). The maxima of this function is at \(y = 1\), with maximum value \(e^{-1}\). Thus for any \(K < e^{-1}\), this region is two tailed.

Exercise 12.24

Suppose that \(X_1, \ldots, X_n \overset{i.i.d}{\sim} \text{N}(\mu, \sigma^2)\) and both are unknown. The hypotheses are given by

\[ H_0 : \sigma = \sigma_0 \quad \text{vs.} \quad H_1 : \sigma \neq \sigma_0. \]

We know that \(\hat{\mu}_{\text{MLE}} = \bar{x}\) and \(\hat{\sigma}_{\text{MLE}} = \frac{\sum (x_i - \bar{x})^2}{n}\) referring to the results of Example 10.18. From

\[
L(\mu, \sigma) = \left( \frac{1}{\sigma \sqrt{2\pi}} \right)^n \exp \left( -\frac{\sum (x_i - \mu)^2}{2\sigma^2} \right)
\]

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we can derive the following LRT statistic
\[ \lambda = \frac{\max L_0}{\max L} = \frac{L(\bar{X}, \sigma_0)}{L(\bar{X}, \hat{\sigma}_{MLE})} \]
\[ = \left( \frac{1/\sigma_0}{1/\sqrt{\sum (x_i - \bar{x})^2/n}} \right)^n \exp \left( -\frac{\sum (x_i - \bar{x})^2}{2\sigma_0^2} + \frac{\sum (x_i - \bar{x})^2}{2\sum (x_i - \bar{x})^2/n} \right) \]
\[ = \left( \frac{1}{\sigma_0} \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} \right)^n \exp \left( -\frac{1}{2} \left\{ \frac{\sum (x_i - \bar{x})^2}{\sigma_0^2} \right\} - n \right). \]

Letting \( u = \frac{\sum (x_i - \bar{x})^2}{n\sigma_0^2} \), one can express the critical region of the likelihood ratio test as as \( ue^{-u} \leq k \).

**Exercise 12.30**

(a) The manufacturer should use the alternative hypothesis \( \mu < 20 \) and make the modification only if the null hypothesis can be rejected.

(b) The manufacturer should use the alternative hypothesis \( \mu > 20 \) and make the modification unless the null hypothesis can be rejected.

**Exercise 12.40**

Suppose that \( X_1, \ldots, X_{64} \overset{i.i.d}{\sim} N(\mu, 16^2) \) and it implies that \( \bar{X} \sim N(\mu, 2^2) \). The hypotheses are given by

\[ H_0 : \mu \leq 40 \quad \text{vs.} \quad H_1 : \mu > 40. \]

The critical region is \( \bar{X} > 43.5 \).

(a) For \( \mu = 37 \),
\[ \alpha = P(\text{Type I error}) = P(\bar{X} > 43.5 \mid \mu = 37) = P \left( Z > \frac{43.5 - 37}{2} \right) = P(Z > 3.25) \approx 0. \]

As we do the same way,
when \( \mu = 38 \), \( \alpha = P(Z > 2.75) = .003 \)
when \( \mu = 39 \), \( \alpha = P(Z > 2.25) = .5 - .4878 = .0122 \)
when \( \mu = 40 \), \( \alpha = P(Z > 1.75) = .5 - .4599 = .0401 \)

(b) For \( \mu = 41 \),
\[ \beta = P(\text{Type II error}) = P(\bar{X} \leq 43.5 \mid \mu = 41) = P \left( Z \leq \frac{43.5 - 41}{2} \right) = P(Z \leq 1.25) = .5 + .3944 = .8944 \]
By repeating the same procedure,
when $\mu = 42$, $\beta = P(Z \leq .75) = .5 + .2732 = .7734$
when $\mu = 43$, $\beta = P(Z \leq .25) = .5 - .0987 = .5987$
when $\mu = 44$, $\beta = P(Z \leq -.25) = .5 - .0987 = .4013$
when $\mu = 45$, $\beta = P(Z \leq -.75) = .7266$
when $\mu = 46$, $\beta = P(Z \leq -1.25) = .1056$
when $\mu = 47$, $\beta = P(Z \leq -1.75) = .5 - .4599 = .0401$
when $\mu = 48$, $\beta = P(Z \leq -2.25) = .5 - .4878 = .0122$

**Exercise 12.44**

Suppose that $x_1, \cdots, x_{20} \overset{i.i.d.}{\sim} \text{Exp}(\theta)$. The hypotheses are given by

$$H_0 : \theta = 15 \quad \text{vs.} \quad H_1 : \theta \neq 15.$$ 

From the result of Exercise 12.21 for (1), the LRT statistic is

$$\lambda = \left( \frac{\bar{x}}{\theta_0} \right)^n e^{-n\left(\frac{\bar{x}}{\theta_0} - 1\right)}.$$ 

By substituting $\bar{x} = 26.45$, $\theta_0 = 15$, $n = 20$ from the data, we get

$$-2 \ln \lambda = -2 \cdot 20 \cdot (\ln 1.763 - 1.763 + 1) = -40(1.567 - 1.763) = 7.84$$

By Theorem 12.2, $H_0$ must be rejected, since $-2 \ln \lambda = 7.84 > \chi^2_{0.05,1} = 3.841$ from Table V.